International Journal of Maps in Mathematics
Volume (2), Issue (2), (2019), Pages:(187-206)
ISSN: 2636-7467 (Online)
www.journalmim.com

NULL HYPERSURFACES IN INDEFINITE NEARLY KAEHLERIAN FINSLER SPACE-FORMS

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Abstract. We study the geometry of null hypersurfaces, M, in indefinite nearly Kaehlerian Finsler space-forms $\mathbb{F}^{2n}$. We prove new inequalities involving the point-wise vertical sectional curvatures of $\mathbb{F}^{2n}$, based on two special vector fields on an umbilic hypersurface. Such inequalities generalize some known results on null hypersurfaces of Kaehlerian space forms. Furthermore, under some geometric conditions, we show that the null hypersurface (M, B), where B is the local second fundamental form of M, is locally isometric to the null product $M_D \times M_{D'}$, where M_D and $M_{D'}$ are the leaves of the distributions D and D' which constitutes the natural null-CR structure on M.

1. Introduction

A Finsler manifold is a manifold $\mathbb{F}$ where each tangent space is equipped with a Minkowski norm, that is, a norm that is not necessarily induced by an inner product. This norm also induces a canonical inner product. However, in sharp contrast to the Riemannian case, these Finsler-inner products are not parameterized by points of $\mathbb{F}$, but by directions in the tangent space of $\mathbb{F}$. Thus one can think of a Finsler manifold as a space where the inner product does not only depend on where you are, but also in which direction you are looking. For example,

Received:2019-02-06 *Revised:2019-05-23* *Accepted:2019-08-22*

*2010 Mathematics Subject Classification.*Primary 53C25, Secondary 53C40, 53C50.

Key words: Null hypersurfaces, Totally umbilic hypersurfaces, Indefinite Finsler spaces

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length, geodesics, curvature, connections, covariant derivative, and structure equations all generalize. However, normal coordinates do not [22]. More details on the basics of Finsler spaces can be found in [17, 22], and any other references cited therein. Subspaces of definite Finsler spaces have been investigated in details. For example, [12] has studied the geometry of CR-submanifolds of Kaehlerian Finsler spaces.

Indefinite Finsler spaces have also been studied by many researchers, for instance see [8, 9]. Null subspaces naturally exists in indefinite spaces, and in case of semi-Riemannian spaces, they have been investigated to a good depth by a lot of scholars like [1, 2, 7, 11, 14, 15, 16, 18, 19, 21]. Despite such numerous work on null subspaces of indefinite semi-Riemannian spaces, there is only one paper by A. Bejancu [4] which talks about null hypersurfaces of indefinite Finsler spaces. The paper lays out the geometric objects induced on such hypersurfaces and also discusses the structure equations involving the vertical curvature tensors. The aim of this paper is to extend his work by fully investing the geometry of null hypersurfaces (M, g) in indefinite nearly Kaehlerian Finsler space-forms. Several new classification results are proved on totally umbilic hypersurfaces, as well as the geometry of the null hypersurface (M, B), where M is the second fundamental form of M. Such a hypersurface has also been studied by Bejan and Duggal in [7]. The paper is arranged as follows; Section 2 focusses on the basic preliminaries on Finsler spaces as well null subspaces necessary for the rest of the paper. In Section 3, we discuss totally umbilic hypersurfaces in which we give conditions on the sectional curvature of $\mathbb{F}$ depending on two vector fields U and V on M, as well as its null sectional curvature (see Theorems 3.1 and 3.2). Finally, Section 4 is devoted to the geometry of (M, B) in which we show that its a product manifold under some geometric conditions (see Theorems 4.2 and 4.3).

2. Preliminaries

Let $\overline{M}^{2n}$ be a smooth $2n$-dimensional manifold and $T\overline{M}^{2n}$ be the tangent bundle of $\overline{M}$. Let $i : \overline{M}^{2n} \longrightarrow T\overline{M}^{2n}$ be the natural imbedding, i.e., $i(x) = 0_x \in T_x\overline{M}^{2n}$, for $x \in \overline{M}^{2n}$. Let us put $T\overline{M}' = T\overline{M}^{2n} \backslash i(\overline{M}^{2n})$. The coordinates of a point of $T\overline{M}^{2n}$ are denoted by (x^i, y^i), where (x^i) and (y^i) are the coordinates of a point $x \in \overline{M}^{2n}$ and the components of a vector $y \in T_x\overline{M}^{2n}$, respectively. Consider a continuous function $L(x, y)$, for $(x, y) \in T\overline{M}'$, defined on $T\overline{M}^{2n}$ and suppose that the following conditions are satisfied

(1) L is smooth on $T\overline{M}'$.

(2) $L(x, \lambda y) = \lambda L(x, y)$, for all $\lambda \in [0, \infty)$.

(3) The metric tensor

$$\overline{g}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \tag{2.1}$$

is positive definite.

Then, we say that $(\overline{M}^{2n}, L)$ is a Finsler manifold [22].

Let $\pi : T\overline{M}' \longrightarrow \overline{M}^{2n}$ be the natural projection and $\pi^{-1}T\overline{M}^{2n} \longrightarrow T\overline{M}'$ be the pullback bundle of $T\overline{M}^{2n}$ by π. A bundle morphism $\overline{J} : \pi^{-1}T\overline{M}^{2n} \longrightarrow \pi^{-1}T\overline{M}$, $\overline{J}^2 = -I$ is said to be a *Finslerian almost complex structure* [12] on $\overline{M}^{2n}$. Let $u \in T\overline{M}'$, then $\pi_u^{-1}T\overline{M}^{2n} = \{u\} \times T_x\overline{M}^{2n}$, $x = \pi(u)$, denotes the fiber over u in $\pi^{-1}T\overline{M}^{2n}$. Moreover, any ordinary almost complex structure $\overline{J} : T\overline{M}^{2n} \longrightarrow T\overline{M}^{2n}$, $\overline{J}^2 = -I$, admits a natural lift to the Finslerian almost complex structure $\tilde{J}$ given by $\tilde{J}_u X = (u, \overline{J}_x \hat{\pi} X)$, $x = \pi(u)$, $X \in \pi_u^{-1}T\overline{M}^{2n}$, $u \in T\overline{M}'$, where $\hat{\pi}$ denotes the projection onto the second factor of $T\overline{M}' \times T\overline{M}^{2n}$. Denote by $\mathcal{V}T\overline{M}'$ the *vertical vector bundle* over $T\overline{M}'$, that is, $\mathcal{V}T\overline{M}' = \ker d\pi$, where $d\pi$ is the differential of π. Then, any section of $\mathcal{V}T\overline{M}'$ is called a Finsler vector field. Also, any section of the dual vector bundle $\mathcal{V}^*T\overline{M}'$ is a Finsler 1-form.

Let $\mathbb{F}^{2n}$ be a Finsler space endowed with the Finslerian almost complex structure $\overline{J}$. Then, $\mathbb{F}^{2n}$ is said to be an *almost Hermitian Finsler space* [12] if

$$\overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X, Y), \tag{2.2}$$

for all $X, Y \in \Gamma(\mathcal{V}T\overline{M}')$. A connection $\overline{\nabla}$ in the induced bundle $(\pi^{-1}T\overline{M}, \overline{g})$ is called *metrical* [12] (resp. *almost complex*) if

$$\overline{\nabla}\overline{g} = 0, \quad \text{resp.} \quad \overline{\nabla}\,\overline{J} = 0. \tag{2.3}$$

A tangent vector Z on $T\overline{M}'$ is called horizontal if $\overline{\nabla}_Z v = 0$, where v is the *Liouville vector field*. Let $\overline{N}$ be the distribution of all horizontal vector fields of $T\overline{M}'$. It is referred to as the *horizontal distribution* [12] of $\overline{\nabla}$. Then $\overline{\nabla}$ is *regular* if its horizontal distribution is *nonlinear connection* on $T\overline{M}'$. The pair $(\overline{N}, \overline{\nabla})$ consisting of a connection in $\pi^{-1}T\overline{M}^{2n}$ and a nonlinear connection on $T\overline{M}'$ is called a Finsler connection on $\overline{M}^{2n}$. The fundamental theorem of Finsler geometry asserts that there exists a regular connection $\overline{\nabla}$ in the induced bundle $(\pi^{-1}T\overline{M}, \overline{g})$, called a *Cartan connection* [12] on $(\overline{M}^{2n}, L)$. Consequently, $(\overline{M}^{2n}, L, \overline{J})$ is called a *Kaehlerian Finsler space* if its Cartan connection is almost complex (see [12] for more details).

According to Bejancu [3], an a almost Hermitian manifold $\overline{M}$ is nearly Kaehlerian if its Levi-Civita connection $\overline{\nabla}$ satisfies the relation

$$(\overline{\nabla}_X \overline{J})Y + (\overline{\nabla}_Y \overline{J})X = 0, \tag{2.4}$$

for any $X, Y \in \Gamma(T\overline{M})$. It then follows directly that a Kaehlerian manifold is a nearly Kaehlerian manifold. Let $(\overline{M}^{2n}, L, \overline{J})$ be an almost Hermitian Finsler space. We say that $(\overline{M}^{2n}, L, \overline{J})$ is a *nearly Kaehlerian Finsler space* if its Cartan connection $\overline{\nabla}$ satisfies (2.4).

Let $\overline{\nabla}$ be the Cartan connection of the Finsler space $(\overline{M}^{2n}, L)$ and $\overline{R}^{\mathcal{V}}$ its vertical curvature tensor. If $u \in T\overline{M}'$ and p is a 2-dimensional real subspace of a fibre $\pi_u^{-1}\overline{M}^{2n}$, let $s(p) = \overline{R}_u^{\mathcal{V}}(X, Y, X, Y)$, for some $\overline{g}_u$-orthonormal basis $\{X, Y\}$ in p, be the *vertical sectional curvature* of $(\overline{M}^{2n}, L)$ [13, p. 97]. Let $\sigma : GF_2(\overline{M}^{2n}) \longrightarrow T\overline{M}'$ be the bundle of all 2-subspaces in fibres of the induced bundle $\pi^{-1}T\overline{M}^{2n}$ of $(\overline{M}^{2n}, L)$. Its standard fibre is the Grassmann manifold $G_{2,2n}(\mathbb{R})$ of all 2-planes in $\mathbb{R}^{2n}$. Note that the vertical sectional curvature is a function $s : GF_2(\overline{M}^{2n}) \longrightarrow \mathbb{R}$ rather than a function on $T\overline{M}'$. Let $(\overline{M}^{2n}, L, \overline{J})$ be a nearly Kaehlerian Finsler space; a *Finslerian 2-plane* $p \in GF_2(\overline{M}^{2n})$ is said to be *holomorphic* if $\overline{J}(p) = p$. The restriction of s to the holomorphic 2-planes is referred to as the *holomorphic* $\mathcal{V}$*-sectional curvature*. Then, $(\overline{M}^{2n}, L, \overline{J})$ is said to be a complex Finslerian $\mathcal{V}$-space-form [12] if there exists $c \in C^\infty(T\overline{M}')$ such that the following equality $s = c \circ \sigma$ holds on all holomorphic 2-planes $p \in GF_2(\overline{M}^{2n})$. If $(\overline{\nabla}, \overline{N})$ is a metrical Finsler connection $(\overline{M}, L, \overline{J})$ then the associated vertical curvature $\overline{R}^{\mathcal{V}}(X, Y, Z, W) := \overline{g}(\overline{R}^{\mathcal{V}}(X, Y)Z, W)$ is skew-symmetric in X, Y, respectively in Z, W, and thus the above procedure is easily generalized such as to yield a well defined conept of (holomophic) $\mathcal{V}$-sectional curvature. Moreover, if the holomorphic $\mathcal{V}$-sectional curvature s (constracted with respect to $(\overline{\nabla}, \overline{N})$), does not depend on the 2-places $p \in \pi_u^{-1}T\overline{M}^{2n}$ but only on the direction $u \in T\overline{M}'$, then $\overline{M}^{2n}$ is also referred to as a complex $\mathcal{V}$-space form with respect to $(\overline{\nabla}, \overline{N})$. Let $(\overline{M}^{2n}, L, \overline{J})$ be a nearly Kaehlerian Finsler space-form. The vertical curvature tensor $\overline{R}^{\mathcal{V}}(X, Y, Z, W)$ is given by

$$\begin{aligned}\overline{R}^{\mathcal{V}}(X, Y, Z, W) = \frac{c}{4}[&\overline{g}(X, W)\overline{g}(Y, Z) - \overline{g}(X, Z)\overline{g}(Y, W) + \overline{g}(X, \overline{J}W)\overline{g}(Y, \overline{J}Z)\\ &- \overline{g}(X, \overline{J}Z)\overline{g}(Y, \overline{J}W) - 2\overline{g}(X, \overline{J}Y)\overline{g}(Z, \overline{J}W)] + \frac{1}{4}[\overline{g}((\overline{\nabla}_X \overline{J})W, (\overline{\nabla}_Y \overline{J})Z)\\ &- \overline{g}((\overline{\nabla}_X \overline{J})Z, (\overline{\nabla}_Y \overline{J})W) - 2\overline{g}((\overline{\nabla}_X \overline{J})Y, (\overline{\nabla}_Z \overline{J})W)],\end{aligned} \tag{2.5}$$

for all Finslerian vector fields X, Y, Z, W of $\overline{M}^{2n}$ (see [23]). Suppose, instead that $\overline{g}$ is non-degenerate on $T\overline{M}^{2n}$, i.e., $\mathrm{rank}(\overline{g}) = 2n$ on any coordinate neighborhood of $T\overline{M}$. Clearly, at any point u of $T\overline{M}'$, $\overline{g}_u$ is a pseudo-Euclidean metric on the fibre $\mathcal{V}T\overline{M}'$ at u. Denote by q the index of $\overline{g}_u$, i.e., q is the dimension of the largest subspace of $\mathcal{V}T\overline{M}'$ on which $\overline{g}$ is negative definite. We further suppose $\overline{g}$ is of constant index, q, on $T\overline{M}$. In this case $\overline{g}$ is said to be an *indefinite Finsler metric* and $\mathbb{F}^{2n} = (\overline{M}^{2n}, L, \overline{g})$ is called an *indefinite Finsler space* [4, 8, 9]. Furthermore, if $\mathbb{F}^{2n}$ is of constant (holomorphic) $\mathcal{V}$-sectional curvature as described earlier, then we say that $\mathbb{F}^{2n}$ is an *indefinite Finsler space-form.*

Consider a hypersurface (M, g) of $\overline{M}^{2n}$. From now on, we assume that $\overline{g}$ is of index q, where $1 < q < 2n$. In this case g may be degenerate in some points of TM; suppose g is degenerate on TM of constant rank $(2n-1)$. Then we call M a null hypersurface of $\mathbb{F}^{2n}$. Consider, for each $p \in TM$, the vector space $\mathcal{V}TM_p^{\perp} = \{X_p \in \mathcal{V}T\overline{M}'_p : \overline{g}_p(X_p, Y_p) = 0, \forall\, Y_p \in \mathcal{V}TM_p\}$, and construct $\mathcal{V}TM^{\perp} = \cup_{p\in TM}\mathcal{V}TM_p^{\perp}$. Notice that $\mathcal{V}TM^{\perp}$ is a one dimensional vector subbundle of $\mathcal{V}T\overline{M}'|_{TM}$. Moreover, M is a null hypersurface of $\mathbb{F}^{2n}$ if and only if $\mathcal{V}TM^{\perp}$ is a vector subbundle of $\mathcal{V}TM$. Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of differentiabale sections of a vector bundle E over M. We also assume that all associated structures are smooth.

In the theory of non-degenerate submanifolds of a Finsler space $\mathcal{V}TM^{\perp}$ plays an important role in introducing main geometrical objects, such as second fundamental form, shape opera tor, induced connection, etc. Contrary to this situation in case of a null hypersurface, $\mathcal{V}TM^{\perp}$ fails to be complementary to $\mathcal{V}TM$ in $\mathcal{V}T\overline{M}'|_{TM}$. Motivated by the above, the author [4] (also see [15]) constructed a complementary (non-orthogonal) vector bundle to $\mathcal{V}TM$ in $\mathcal{V}T\overline{M}'|_{TM}$ which plays the role of $\mathcal{V}TM^{\perp}$. In fact, consider a complementary vector bundle $S(\mathcal{V}TM)$ of $\mathcal{V}TM^{\perp}$ in $\mathcal{V}TM$, i.e. we have $\mathcal{V}TM = S(\mathcal{V}TM) \perp \mathcal{V}TM^{\perp}$. It is easy to see that $S(\mathcal{V}TM)$ is a non-degenerate vector subbundle of $\mathcal{V}TM$, whose existence is secured by the paracompactness of M. $S(\mathcal{V}TM)$ is called the *screen distribution* [4] of M. Next, along to $S(\mathcal{V}TM)$ we have the decomposition $\mathcal{V}T\overline{M}'|_{TM} = S(\mathcal{V}TM) \perp S(\mathcal{V}TM)^{\perp}$, where $S(\mathcal{V}TM)^{\perp}$ is the complementary vector bundle to $S(\mathcal{V}TM)$ in $\mathcal{V}T\overline{M}'|_{TM}$.

Theorem 2.1 ([4])**.** *Let M be a null hypersurface of $\mathbb{F}^{2n}$ and $S(\mathcal{V}TM)$ be the screen distribution of M. Then there exists a unique cector bundle $tr(\mathcal{V}TM)$ of rank 1 over TM, such that for any non-zero section ξ of $\mathcal{V}TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset TM$, there exists a unique section N of $tr(\mathcal{V}TM)$ on $\mathcal{U}$ satisfying: $\overline{g}(N, \xi) = 1$, and $\overline{g}(N, N) = \overline{g}(N, X) = 0$, for all $X \in \Gamma(S(\mathcal{V}TM)|_{\mathcal{U}})$.*

The vector bundle $tr(\mathcal{V}TM)$ above is called the *null transversal* vector bundle of M with respect to $S(\mathcal{V}TM)$. Moreover, we have the following decomposition

$$\begin{aligned}\mathcal{V}T\overline{M}'|_{TM} &= S(\mathcal{V}TM) \perp \{\mathcal{V}TM^{\perp} \oplus tr(\mathcal{V}TM)\} \\ &= \mathcal{V}TM \oplus tr(\mathcal{V}TM). \end{aligned} \tag{2.6}$$

Let (M, g) be a null hypersurface of $\mathbb{F}^{2n}$. In view of (2.6), the author [4] proves (see Theorem 2.1) that there is a unique nonlinear connection $\mathcal{H}TM$ [4] on TM, which is a subbundle of $tr(\mathcal{V}TM)\oplus\mathcal{V}T\overline{M}^{2n}$. Accordingly, $\mathcal{H}TM$ is called the induced nonlinear connection on TM. Denote by $(\mathcal{H}TM, \nabla)$, the induced Finsler connection on M by $(\overline{N}, \overline{\nabla})$ on $\overline{M}^{2n}$. Locally, the Gauss and Weingarten equations of M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \;\text{ and }\; \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.7}$$

for all $X \in \Gamma(TTM)$, $Y \in \Gamma(\mathcal{V}TM)$ and $N \in \Gamma(tr(\mathcal{V}TM))$. Here, B is called the local second fundamental form of M, and A_N its shape operator. furthermore, τ is a differential 1-form on TM. It then follows that ∇ is a linear connection on TM. Denote by P the projection of $\mathcal{V}TM$ onto $S(\mathcal{V}TM)$, then the local Gauss and Weingarten formulae of $S(\mathcal{V}TM)$ are given by

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi \;\text{ and }\; \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.8}$$

for all $X \in \Gamma(TTM)$ and $\xi \in \Gamma(\mathcal{V}TM^{\perp})$. Moreover, C is the local second fundamental form of $S(\mathcal{V}TM)$, and ∇^* is a linear connection on it, which is a metric connection. In general, ∇ is not a metric connection. In fact, if η is a one form on $\mathcal{V}TM$ by $\eta(\cdot) = \overline{g}(\cdot, N)$, then ∇g is given by

$$(\nabla_X g)(Y, Z) = B(X, Z)\eta(Y) + B(X, Y)\eta(Z), \tag{2.9}$$

for all $X \in \Gamma(TTM)$ and $Y, Z \in \Gamma(\mathcal{V}TM)$. Notice that B is degenerate and in fact, $B(\cdot, E) = 0$. The shape operators A^*_ξ and A_N are screen-valued and relate to their shape local second fundamental forms according to the relations

$$B(X, Y) = g(A^*_\xi X, Y) \;\text{ and }\; C(X, PY) = g(A_N X, PY). \tag{2.10}$$

The null hypersurface M is said to be *totally umbilic* [15] if $B = \rho \otimes g$, where ρ is a smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. In case $\rho = 0$, we say that M is *totally geodesic*. In the same line, M is called *screen totally umbilic* if $C = \varrho \otimes g$, where ϱ is a smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. When $\varrho = 0$, we say that M is *screen totally geodesic*.

Denote by $\overline{R}^{\mathcal{V}}$ and R, the curvature tensors of $\overline{\nabla}$ and ∇. Let further, ∇° be Schouten-Van Kampen connection [5] on TM and T° its torsion. Then

$$\begin{aligned}\overline{R}^{\mathcal{V}}(X,Y)Z =& R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+ (\nabla^t_X h)(Y,Z) - (\nabla^t_Y h)(X,Z) - h(T^{\circ}(X,Y),Z), \end{aligned} \tag{2.11}$$

for any $X, Y \in \Gamma(TTM)$, and $Z \in \Gamma(\mathcal{V}TM)$. Here, $(\nabla^t_X h)(Y,Z) = \nabla^t_X h(Y,Z) - h(\nabla^{\circ}_X Y, Z) - h(Y, \nabla_X Z)$, where ∇^t is the tranversal (linear) connection and $h(X,Y) = B(X,Y)N$. Further details regarding the fundamental equations of null hypersfaces can be found in [4, 15, 16].

As seen above, let ξ and N the metric normal and the transversal sections, respectively. Since $(\overline{g}, \overline{J})$ is an almost Hermitian structure and $\overline{J}\xi$ is a null vector field, it follows that $\overline{J}N$ is null too. Moreover, $\overline{g}(\overline{J}\xi, \xi) = 0$ and, thus, $\overline{J}\xi$ is tangent to TM. Let us consider $S(\mathcal{V}TM)$ containing $\overline{J}\mathcal{V}TM^{\perp}$ as a vector subbundle. Consequently, N is orthogonal to $\overline{J}\xi$ and we have $\overline{g}(\overline{J}N, \xi) = -\overline{g}(N, \overline{J}\xi) = 0$ and $\overline{g}(\overline{J}N, N) = 0$. This means that $\overline{J}N$ is tangent to TM and in particular, it belongs to $S(\mathcal{V}TM)$. Thus, $\overline{J}tr(\mathcal{V}TM)$ is also a vector subbundle of $S(\mathcal{V}TM)$. In view of (2.2), we have $\overline{g}(\overline{J}\xi, \overline{J}N) = 1$. It is then easy to see that $\overline{J}\mathcal{V}TM^{\perp} \oplus \overline{J}tr(\mathcal{V}TM)$ is a non-degenerate vector subbbundle of $S(\mathcal{V}TM)$, with 2-dimensional fibers. Then there exists a non-degenerate distribution D_0 on TM such that $S(\mathcal{V}TM) = \{\overline{J}\mathcal{V}TM^{\perp} \oplus \overline{J}tr(\mathcal{V}TM)\} \perp D_0$. It is easy to check that D_0 is an almost complex distribution with respect to $\overline{J}$, i.e. $\overline{J}D_0 = D_0$. The decomposition of $\mathcal{V}TM$ becomes $\mathcal{V}TM = \{\overline{J}\mathcal{V}TM^{\perp} \oplus \overline{J}tr(\mathcal{V}TM)\} \perp D_0 \perp \mathcal{V}TM^{\perp}$. If we set $D := \mathcal{V}TM^{\perp} \perp \overline{J}\mathcal{V}TM^{\perp} \perp D_0$ and $D' = \overline{J}tr(\mathcal{V}TM)$, then $\mathcal{V}TM = D \oplus D'$. Here, D is an almost complex distribution and D' is carried by $\overline{J}$ just into the transversal bundle. Thus, we have a null CR submanifold as in [15, 16] for null hypersurfaces of semi-Riemannian manifolds. Finally, let us set

$$U := -\overline{J}N \quad \text{and} \quad V := -\overline{J}\xi. \tag{2.12}$$

3. Totally umbilic hypersurfaces

In this section, we prove several charateriazation results on umbilic hypersurfaces of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n} := (\overline{M}(c)^{2n}, L, \overline{g}, \overline{J})$. To that end, we have the following.

Theorem 3.1. *Let $\mathbb{F}^{2n}$ be an indefinite $2n$-dimensional nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g). Then, c satisfies $c > 0$, $c = 0$ or $c < 0$ if and only if the vector field $\nabla^*_U V$ is timelike, null (or identically zero), or spacelike, respectively. Moreover, the umbilicity factor ρ satisfies the differential equations $\xi\rho + \rho\tau(\xi) - \rho^2 = 0$ and $PX\rho + \rho\tau(PX) = 0$. for any $X \in \Gamma(\mathcal{V}TM)$.*

Proof. Setting $Y = W = \xi$ and $X = Z$ in (2.5) and using (2.12), we derive

$$\overline{R}^{\mathcal{V}}(Z, \xi, Z, \xi) = -\frac{3c}{4}g(Z, V)^2 - \frac{3}{4}\overline{g}((\overline{\nabla}_Z\overline{J})\xi, (\overline{\nabla}_Z\overline{J})\xi). \tag{3.13}$$

A straightforward calculation, while considering (2.7), (2.8) and (2.12), leads to

$$(\overline{\nabla}_Z\overline{J})\xi = -\nabla^*_Z V - C(Z, V)\xi - \rho g(Z, V)N + \rho\overline{J}Z - \tau(Z)V. \tag{3.14}$$

In view of (3.14), the second term on the right hand side of (3.13) becomes

$$\begin{aligned}-\frac{3}{4}\overline{g}((\overline{\nabla}_Z\overline{J})\xi, (\overline{\nabla}_Z\overline{J})\xi) &= -\frac{3}{4}g(\nabla^*_Z V, \nabla^*_Z V) + \frac{3}{2}\rho g(\nabla^*_Z V, \overline{J}Z)\\ &+ \frac{3\rho^2}{2}g(Z, U)g(Z, V) - \frac{3\rho^2}{4}g(Z, Z).\end{aligned} \tag{3.15}$$

On the other hand, setting $Y = \xi$ and $X = Z$ in (2.11) and taking the inner product of the resulting equation with ξ, we get

$$\begin{aligned}\overline{R}^{\mathcal{V}}(Z, \xi, Z, \xi) &= \overline{g}((\nabla^t_Z h)(\xi, Z) - (\nabla^t_\xi h)(Z, Z) - h(T^\circ(Z, \xi), Z), \xi)\\ &= (\rho^2 - \rho\tau(\xi) - \xi\rho)g(Z, Z).\end{aligned} \tag{3.16}$$

Using (3.13), (3.15) and (3.16), we get

$$\begin{aligned}(\rho^2 - \rho\tau(\xi) - \xi\rho)g(Z, Z) &= -\frac{3c}{4}g(Z, V)^2 - \frac{3}{4}g(\nabla^*_Z V, \nabla^*_Z V)\\ &+ \frac{3}{2}\rho g(\nabla^*_Z V, \overline{J}Z) + \frac{3\rho^2}{2}g(Z, U)g(Z, V) - \frac{3\rho^2}{4}g(Z, Z),\end{aligned} \tag{3.17}$$

in which we have used the fact that the adapted connection ∇° coincides with ∇. Setting $Z = U$ and $Z = U + V$ in (3.17) in turn, we get

$$c + g(\nabla^*_U V, \nabla^*_U V) = 0, \tag{3.18}$$

$$\text{and} \quad 2(\xi\rho + \rho\tau(\xi) - \rho^2) = -\frac{3c}{4} - \frac{3}{4}g(\nabla^*_{U+V}V, \nabla^*_{U+V}V). \tag{3.19}$$

By considering the facts $(\overline{\nabla}_V\overline{J})V = 0$ and that M is totally umbilic, we derive $\nabla^*_V V = [\rho - C(V, V)]\xi - \tau(V)V$. Thus, $g(\nabla^*_{U+V}V, \nabla^*_{U+V}V) = g(\nabla^*_U V, \nabla^*_U V)$ and (3.18) and (3.19) implies that $\xi\rho + \rho\tau(\xi) - \rho^2 = 0$, proving the first assertions of the theorem and the first differential differential equation of ρ.

Next, we prove the second differential equation of the theorem. To that end, (2.5) and (2.11) implies

$$\begin{aligned}[X\rho+\rho\tau(X)]g(Y,Z)-[Y\rho+\rho\tau(Y)]g(X,Z) &= \frac{c}{4}[-\overline{g}(X,V)\overline{g}(Y,\overline{J}Z)\\ &+\overline{g}(Y,V)\overline{g}(X,\overline{J}Z)+2\overline{g}(Z,V)\overline{g}(X,\overline{J}Y)]+\frac{1}{4}[\overline{g}((\overline{\nabla}_X\overline{J})\xi,(\overline{\nabla}_Y\overline{J})Z)\\ &-\overline{g}((\overline{\nabla}_Y\overline{J})\xi,(\overline{\nabla}_X\overline{J})Z)-2\overline{g}((\overline{\nabla}_Z\overline{J})\xi,(\overline{\nabla}_X\overline{J})Y)],\end{aligned} \tag{3.20}$$

for all $X,Y,Z\in\Gamma(S(\mathcal{V}TM))$. Setting $Y=Z=V$ in (3.20), we get

$$-[V\rho+\rho\tau(V)]\overline{g}(X,V)=-\frac{3}{4}\overline{g}((\overline{\nabla}_V\overline{J})\xi,(\overline{\nabla}_X\overline{J})V). \tag{3.21}$$

Using (2.4), we see that $(\overline{\nabla}_V\overline{J})V=0$, which helps to derive

$$-\overline{\nabla}_V V=-\rho\xi+\tau(V)V. \tag{3.22}$$

Thus, in view of (2.7), (2.8) and (2.12), we have

$$(\overline{\nabla}_V\overline{J})\xi=-\overline{\nabla}_V V+\overline{J}A^*_\xi V-\tau(V)V=-\overline{\nabla}_V V+\rho\xi-\tau(V)V. \tag{3.23}$$

Hence, considering (3.21), (3.22) and (3.23), we get

$$-[V\rho+\rho\tau(V)]\overline{g}(X,V)=0,\ \ \forall\, X\in\Gamma(S(\mathcal{V}TM)). \tag{3.24}$$

Setting $X=U$ in (3.24), we get

$$V\rho+\rho\tau(V)=0. \tag{3.25}$$

On the other hand, setting $X=V$ and $Y=Z=U$ in (3.20), we derive

$$-[U\rho+\rho\tau(U)]=\frac{3}{4}\overline{g}((\overline{\nabla}_U\overline{J})\xi,(\overline{\nabla}_U\overline{J})V). \tag{3.26}$$

A straightforward calculation, using (2.7), (2.8) and (2.12), we derive $(\overline{\nabla}_U\overline{J})\xi=-\nabla_U V-\tau(U)V$ and $(\overline{\nabla}_U\overline{J})V=-\overline{J}\nabla_U V-\tau(U)\xi$. Thus, (3.26) gives

$$U\rho+\rho\tau(U)=0. \tag{3.27}$$

Next, let $Y=Z=\overline{J}X$ in (3.20), for some $X\in\Gamma(D_0)$, we get

$$[X\rho+\rho\tau(X)]g(\overline{J}X,\overline{J}X)=-\frac{3}{4}\overline{g}((\overline{\nabla}_X\overline{J})\overline{J}X,(\overline{\nabla}_{\overline{J}X}\overline{J})\xi). \tag{3.28}$$

As $0=(\overline{\nabla}_X\overline{J})X=\overline{\nabla}_X\overline{J}X-\overline{J}\,\overline{\nabla}_X X$ by (2.4), we have $\overline{J}\,\overline{\nabla}_X\,\overline{J}X+\overline{\nabla}_X X=0$. Using this relation, we derive

$$(\overline{\nabla}_X\overline{J})\overline{J}X=\overline{\nabla}_X\overline{J}^2X-\overline{J}\,\overline{\nabla}_X\overline{J}X=-\overline{\nabla}_X X-\overline{J}\,\overline{\nabla}_X\,\overline{J}X=0. \tag{3.29}$$

Considering (3.28) and (3.29), we get

$$X\rho + \rho\tau(X) = 0, \quad \forall\, X \in \Gamma(D_0). \tag{3.30}$$

Hence, part (3) of the theorem follows from (3.25), (3.27) and (3.30), which completes the proof.

As consequence, we have the following results.

Corollary 3.1. *Let $\mathbb{F}^{2n}$ be an indefinite nearly Kaehlerian Finsler space-form, such that $\overline{J}\,\overline{\nabla} = 0$. If $\mathbb{F}^{2n}$ admits a totally umbilic null hypersurface (M, g), then $c = 0$.*

Proof. From (3.18), we have $c + g(\nabla^*_U V, \nabla^*_U V) = 0$. Also, from the assumption $\overline{J}\,\overline{\nabla} = 0$, we see that $(\overline{\nabla}_U \overline{J})V = 0$, which implies that $-\overline{J}\nabla^*_U V = \tau(U)\xi - C(U, V)V$, in which we have used (2.7) and (2.8). Thus, $g(\nabla^*_U V, \nabla^*_U V) = \overline{g}(\overline{J}\nabla^*_U V, \overline{J}\nabla^*_U V) = 0$, which gives $c = 0$.

Corollary 3.2. *Let $\mathbb{F}^{2n}$ be an indefinite $2n$-dimensional nearly Kaehlerian Finsler space-form. If $\mathbb{F}^{2n}$ admits a totally umbilic null hypersurface (M, g), such that V is parallel with respect to ∇^*, then $c = 0$.*

Proof. From (3.18), we have $c = 0$. By (3.17), we have $2\rho^2 g(Z, U)g(Z, V) - \rho^2 g(Z, Z) = 0$, for all $Z \in \Gamma(\mathcal{V}TM)$. Setting $Z = X \in \Gamma(D_0)$ in this relation and noticing that $g(X, U) = g(X, V) = 0$, we get $\rho = 0$. Thus, M is totally geodesic which completes the proof.

Corollary 3.3. *Let $\mathbb{F}^{2n}$ be an indefinite nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g). If M is also screen totally umbilic, then M is screen totally geodesic.*

Proof. By a straightforward calculation, we have $g(\nabla_U \xi, U) = \overline{g}(\overline{\nabla}_U \xi, U) = \overline{g}(\overline{J}\,\overline{\nabla}_U \xi, N)$. In view of (2.4), we have $\overline{J}\,\overline{\nabla}_U \xi = \overline{\nabla}_U \overline{J}\xi + (\overline{\nabla}_\xi \overline{J})U$, and the previous relation simplifies to

$$\begin{aligned} g(\nabla_U \xi, U) &= \overline{g}(\overline{\nabla}_U \overline{J}\xi, N) + \overline{g}((\overline{\nabla}_\xi \overline{J})U, N) \\ &= -\overline{g}(\overline{J}\xi, \overline{\nabla}_U N) + \overline{g}(\overline{\nabla}_\xi N, N) - \overline{g}(\overline{\nabla}_\xi U, U) = \varrho. \end{aligned} \tag{3.31}$$

But using (2.8), we see that $g(\nabla_U \xi, U) = -B(U, U) = -\rho g(U, U) = 0$. Thus, in view of (3.31), we get $\varrho = 0$, showing that M is screen totally geodesic which completes the proof.

Let $x \in M$ and ξ be a null vector of T_xM. A plane H of T_xM is called a null plane directed by ξ if it contains ξ, $g_x(\xi, W) = 0$ for any $W \in H$ and there exists $W_0 \in H$ such that $g_x(W_0, W_0) \neq 0$. Thus, the null section curvature of H with respective to ξ and the induced connection ∇ of M, is defined as a real number $K_\xi(H) = g_x(R(W,\xi)\xi, W)/g_x(W,W)$, where $W \neq 0$ is any vector in H independent with ξ (see [15] or [16] for more details). Moreover, the author in [20] proved that an n(where $n \geq 3$)-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero.

Theorem 3.2. *Let $\mathbb{F}^{2n}$ be an indefinite $2n$-dimensional nearly Kaehlerian Finsler space-form, admitting a totally umbilic and screen totally umbilic null hypersurface (M, g). Then, the null sectional curvature $K_\xi(H)$ of M vanishes if and only if V is parallel with respect to ∇^*.*

Proof. Considering (2.11) and Corollary 3.3, we have

$$g(R(X,Y)Z, PW) = \overline{g}(\overline{R}(X,Y)Z, PW), \tag{3.32}$$

for all $X, Y \in \Gamma(TTM)$ and $Z, W \in \Gamma(\mathcal{V}TM)$. Setting $X = PW = W$ and $Y = Y = \xi$ in (3.32) and using (2.5), we have

$$g(R(W,\xi)\xi, W) = \frac{3c}{4} g(W,V)^2 + \frac{3}{4}\overline{g}((\overline{\nabla}_W \overline{J})\xi, (\overline{\nabla}_W \overline{J})\xi). \tag{3.33}$$

But in view of (2.8) and (2.12), we have $(\overline{\nabla}_W \overline{J})\xi = -\overline{\nabla}_W V + \rho \overline{J} W - \tau(W)V$. Thus, from (3.33) , we have

$$\begin{aligned}\overline{g}((\overline{\nabla}_W \overline{J})\xi, (\overline{\nabla}_W \overline{J})\xi) &= \overline{g}(\overline{\nabla}_W V, \overline{\nabla}_W V) - 2\rho\overline{g}(\overline{\nabla}_W V, \overline{J}W) + \rho^2 g(W,W) \\ &= g(\nabla^*_W V, \nabla^*_W V) - 2\rho g(\nabla^*_W V, \overline{J}W) \\ &\quad - 2\rho^2 g(W,U)g(W,V) + \rho^2 g(W,W).\end{aligned} \tag{3.34}$$

Suppose that V is parallel with respect to ∇^*, then the term $g(\nabla^*_W V, \nabla^*_W V)$ vanishes. Moreover, $-2\rho g(\nabla^*_W V, \overline{J}W)$ vanishes too. Furthermore, by Corollary 3.2, we have $c = 0$ and $\rho = 0$. Then, (3.34) reduces to

$$\overline{g}((\overline{\nabla}_W \overline{J})\xi, (\overline{\nabla}_W \overline{J})\xi) = -2\rho^2 g(W,U)g(W,V) + \rho^2 g(W,W) = 0. \tag{3.35}$$

Hence, considering (3.35) in (3.33), we see that $K_\xi(H) = 0$. The converse is obvious, and the proof is complete.

The following result follows easily from Theorem 3.2.

Corollary 3.4. *Let $\mathbb{F}^{2n}$ be an indefinite $2n$-dimensional normal nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g). Then, the null sectional curvature $K_\xi(H)$ of M vanishes.*

4. Geometry of (M, g) from the distributions D and D'

In this section, we give new results on the null hypersurface (M, g) based on the nature of D and D' with respect to the second fundamental form B of M. Denote by Q the projection morphism of TM onto D. Then, in view of (2.12), any $X \in \Gamma(\mathcal{V}TM)$ can be written as $X = QX + u(X)U$, where u is a 1-form locally defined on M by $u(\cdot) = g(\cdot, V)$. Applying $\overline{J}$ to this relation we have

$$\overline{J}X = FX + u(X)N, \tag{4.36}$$

where F is a (1,1)-tensor globally defined on M by $F = \overline{J} \circ Q$. Moreover, it is easy to see that (F, u, U) is a local almost contact structure on M, satisfying

$$F^2 = -I + u \otimes U, \quad u(U) = 1. \tag{4.37}$$

Notice that (F, u, U) is never an almost contact metric structure on with respect to degenerate metric g. Using (2.4), (2.7) and (2.8), we derive

$$(\nabla_X F)Y + (\nabla_Y F)X = u(Y)A_N X + u(X)A_N Y - 2B(X, Y)U, \tag{4.38}$$

and

$$\begin{aligned} B(X, FY) + B(Y, FX) &= -B(X, V)\eta(Y) - B(Y, V)\eta(X) \\ &\quad - g(\nabla_X V, Y) - g(\nabla_Y V, X) - u(X)\tau(Y) - u(Y)\tau(X), \end{aligned} \tag{4.39}$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$.

Theorem 4.1. *Let $\mathbb{F}^{2n}$ be an indefinite $2n$-dimensional normal nearly Kaehlerian Finsler space-form, admitting a totally umbilic and screen totally umbilic null hypersurface (M, g). If F is parallel then $c = 0$.*

Proof. Suppose that F is parallel. Then, (4.38) and the facts M is totally umbilic and screen totally umbilic, gives $\varrho u(Y)X + \varrho u(X)Y = 2\rho g(X, Y)U$. In view of Corollary 3.3, M is screen totally geodesic and thus, we have $2\rho g(X, Y)U = 0$. Hence, $\rho = 0$ and M is totally geodesic. On the other hand, $(\nabla_X F)\xi = 0$ and (2.8) gives $\nabla_X V = -\tau(X)V$, which together with (3.18) gives $c = 0$. This completes the proof.

In [6], the authors introduced the concept of *mixed geodesic* non-degenerate CR-submanifolds of a space form. More precisely, a CR-submanifold is called mixed geodesic if its second fundamental h satisfies $h(X,Y) = 0$, where $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$. Here, D is a $\overline{J}$-invariant distribution on M and $D^{\perp}$ is an anti-invariant distribution which is orthogonal and complementary to D in M. As we have already secured a CR structure on a null hypersurface (M,g) (see Section 2), in which the invariant distribution D is given by $D = \mathcal{V}TM^{\perp} \perp \overline{J}\mathcal{V}TM^{\perp} \perp D_0$ and its complementary (but not orthogonal) distribution D' by $D' = \overline{J}tr(\mathcal{V}TM)$, we can define the concept of mixed geodesic for M as follows.

Definition 4.1. Let (M,g) be a null hypersurface of a complex space. Then, M is said to be mixed totally geodesic if $B(X,Y) = 0$, for $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

It follows that any totally geodesic null hypersurface (M,g) of $\mathbb{F}^{2n}$ is trivially mixed geodesic.

Notice that the distribution D' in integrable while D is generally non-integrable. In fact, it is easy to show, using (2.4) and (2.7), that D is integrable if and only if

$$B(X,\overline{J}Y) - B(Y,\overline{J}X) = 2\overline{g}((\overline{\nabla}_X\overline{J})Y,\xi), \tag{4.40}$$

for all $X,Y \in \Gamma(D)$. A null hypersurface (M,g) will be called *mixed foliate* if M is mixed totally geodesic and (4.40) holds, i.e. D is integrable. Since D and D' are not g-orthogonal distributions, one may not be able to describe the nature of (M,g) depending on the leaves M_D and $M_{D'}$ of D, assumed integrable, and D', respectively. However, when M is mixed foliate, we know that $D \perp_B D'$. This prompt us to consider the null hypersurface (M,B), that is; the null hypersurface M endowed with its local second fundamental form B, instead of its natural degenerate metric g. Notice that (M,B) is also degenerate since the second fundamental form B is degenerate. More precisely, $B(\xi,\cdot) = 0$. It is easy to see that the radical distribution of (M,B) is $\ker A^*_{\xi}$. Such hypersurfaces were also studied by C. L. Bejan and K. L. Duggal [7]. A distribution $\mathcal{D}$ on M will be call B-totally null if B vanishes on $\mathcal{D}$. It follows that a totally geodesic M is B-totally null hypersurface.

The following lemma is fundamental to our study of (M,g) and (M,B).

Lemma 4.1. *Let (M,g) be a mixed foliate null hypersurface of $\mathbb{F}^{2n}$. Then,*

$$\begin{aligned}2cg(Y,Y) &= 4B(\overline{J}Y,\nabla_Y U) - 4B(Y,\nabla_{\overline{J}Y}U)\\ &\quad + \overline{g}((\overline{\nabla}_{\overline{J}Y}\overline{J})U,(\overline{\nabla}_Y\overline{J})\xi) - \overline{g}((\overline{\nabla}_Y\overline{J})U,(\overline{\nabla}_{\overline{J}Y}\overline{J})\xi),\end{aligned}$$

for all $Y \in \Gamma(D_0)$.

Proof. For any $X, Y \in \Gamma(D)$, we have

$$(\nabla_X B)(Y, U) = -B(\nabla_X Y, U) - B(Y, \nabla_X U), \tag{4.41}$$

where we have used the fact that M is mixed totally geodesic. Then, interchanging X and Y in (4.41) and subtracting the new relation from (4.41), we get

$$\begin{aligned}(\nabla_X B)(Y, U) &- (\nabla_Y B)(X, U)\\ &= -B([X, Y], U) + B(X, \nabla_Y U) - B(Y, \nabla_X U)\\ &= B(X, \nabla_Y U) - B(Y, \nabla_X U).\end{aligned} \tag{4.42}$$

In view of (4.42) and (2.11), we derive

$$\overline{g}(\overline{R}^{\mathcal{V}}(X, Y)U, \xi) = B(X, \nabla_Y U) - B(Y, \nabla_X U), \tag{4.43}$$

for any $X, Y \in \Gamma(D)$. On the other hand, (2.5) gives

$$\begin{aligned}\overline{R}^{\mathcal{V}}(X, Y, U, \xi) = \frac{c}{2} g(X, \overline{J}Y) + \frac{1}{4}[&\overline{g}((\overline{\nabla}_X \overline{J})\xi, (\overline{\nabla}_Y \overline{J})U)\\ &- \overline{g}((\overline{\nabla}_X \overline{J})U, (\overline{\nabla}_Y \overline{J})\xi) - 2\overline{g}((\overline{\nabla}_X \overline{J})Y, (\overline{\nabla}_U \overline{J})\xi)].\end{aligned} \tag{4.44}$$

Notice that $(\overline{\nabla}_{\overline{J}Y} \overline{J})Y = 0$, for any $Y \in \Gamma(D_0)$. In fact, by (2.4) we have $(\overline{\nabla}_Y \overline{J})Y = 0$, which implies that $\overline{\nabla}_Y \overline{J}Y = \overline{J}\, \overline{\nabla}_Y Y$. Then, using this relation, we derive

$$(\overline{\nabla}_{\overline{J}Y} \overline{J})Y = -(\overline{\nabla}_Y \overline{J})\overline{J}Y = \overline{\nabla}_Y Y + \overline{J}^2 \overline{\nabla}_Y Y = 0. \tag{4.45}$$

Setting $X = \overline{J}Y$, where $Y \in \Gamma(D_0)$, in (4.43) and (4.44), and considering (4.45), we get the lemma. Hence, the proof.

As a consequence of Lemma 4.1, we have the following result.

Theorem 4.2. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n}$, such that F is parallel. Then $c = 0$. Moreover, (M, B) is locally a null product manifold $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D, which is a B-totally null manifold and $M_{D'}$ is a curve of the anti-invariant D'.*

Proof. The assumption F is parallel implies that

$$\nabla_X U = u(\nabla_X U)U, \quad \forall\, X \in \Gamma(\mathcal{V}TM). \tag{4.46}$$

Using (4.46) and the fact that M is mixed foliate, we derive

$$B(\overline{J}Y, \nabla_Y U) = u(\nabla_Y U)B(\overline{J}Y, U) = 0, \tag{4.47}$$

$$\text{and} \quad B(Y, \nabla_{\overline{J}Y} U) = u(\nabla_{\overline{J}Y} U)B(Y, U) = 0, \tag{4.48}$$

for any $Y \in \Gamma(D_0)$. On the other hand, applying (2.7), (2.8) and the assumption F is parallel, we have

$$\begin{aligned}(\overline{\nabla}_X \overline{J})Y = -u(Y)A_N X + B(X, Y)U + [B(X, FY)\\ + Xu(Y) + u(Y)\tau(X) - u(\nabla_X Y)]N,\end{aligned} \tag{4.49}$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$. Replacing Y by U and ξ in (4.49), in turns, we get

$$(\overline{\nabla}_X \overline{J})U = -A_N X + [\tau(X) - u(\nabla_X U)]N, \tag{4.50}$$

$$\text{and} \quad (\overline{\nabla}_X \overline{J})\xi = -[B(X, V) + u(\nabla_X \xi)]N = 0, \tag{4.51}$$

respectively, for any $X \in \Gamma(\mathcal{V}TM)$. Considering (4.47), (4.48), (4.50) and (4.51) in Lemma 4.1, we get $2cg(Y, Y) = 0$ for any $Y \in \Gamma(D_0)$, which implies that $c = 0$ as D_0 is non-degenerate. Next, let us consider the manifold (M, B). As F is parallel, (4.38) implies that

$$2B(X, Y)U = u(Y)A_N X + u(X)A_N Y, \tag{4.52}$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$. Since u vanishes on D, (4.52) implies $B(X, Y) = 0$, for any $X, Y \in \Gamma(D)$. Thus, the distribution D is totally B-degenerate. Furthermore, using (4.52), we derive $B(U, U) = C(U, V)$ and $A_N V = 0$. From (4.46), D' is parallel. Notice that FX has no component in D' for any $X \in \Gamma(\mathcal{V}TM)$. In fact, by (4.36) and (2.12), we have $g(FX, V) = \overline{g}(\overline{J}X, V) = -\overline{g}(\overline{J}X, \overline{J}\xi) = -g(X, \xi) = 0$, i.e. $FX \in \Gamma(D)$ for all $X \in \Gamma(\mathcal{V}TM)$. Using this fact and the assumption that F is parallel, we have $\nabla_X FY = F\nabla_X Y \in \Gamma(D)$, for any $X \in \Gamma(\mathcal{V}TM)$ and $Y \in \Gamma(D)$, hence, D is parallel too. Consequently, since $D \perp_B D'$ by the mixed geodesic assumption and that D and D' are integrable distributions, then by the arguments originally used by de Rham [10], (M, B) is locally a semi-Riemannian product $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D, which is a totally null manifold with respect to B, and $M_{D'}$ is a curve of the anti-invariant distribution D', which completes the proof.

The following results follows immediately from Theorem 4.2.

Corollary 4.1. *There exist no mixed foliate null hypersurface (M, g) of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n}$, such that F is parallel and $c \neq 0$.*

Corollary 4.2. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n}$, such that F is parallel and $C(U, V) = 0$. Then, the following holds;*

(1) $c = 0$,

(2) *(M, B) is locally a null product manifold $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D, which is a B-totally null manifold, and $M_{D'}$ is a B-null curve of the anti-invariant D',*

(3) *each leaf M_D carries a Kaehlerian structure $(F, g_{|_D})$.*

In case $\overline{J}$ is parallel, such that both D' and $\overline{J}(\mathcal{V}TM^{\perp})$ are parallel distributions on M, we have the following result.

Theorem 4.3. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n}$, such that $\overline{\nabla}\,\overline{J} = 0$ and the distribution D' is parallel. Then, $c = 0$. Moreover, if $\overline{J}(\mathcal{V}TM^{\perp})$ is also parallel, then the following holds;*

(1) *if $\tau = 0$, the type numbers of M and $S(\mathcal{V}TM)$ satisfies $t_M(x) \leq 1$ and $t_{S(\mathcal{V}TM)}(x) \leq 1$, for any $x \in M$,*

(2) *(M, B) is locally a null product $M_D \times M_{D'}$, where M_D is a leaf of D, which is a B-totally degenerate manifold of complex dimension, and $M_{D'}$ is a B-null curve of D',*

(3) *(M, g) is totally geodesic,*

(4) *$(F, g_{|_D})$ is a Kaehlerian structure on the leaf M_D.*

Proof. As $\overline{\nabla}\,\overline{J} = 0$, we have

$$(\nabla_X F)Y = u(Y)A_N X - B(X, Y)U, \tag{4.53}$$

$$\text{and} \quad (\nabla_X u)Y = -B(X, FY) - u(Y)\tau(X), \quad \forall\, X, Y \in \Gamma(\mathcal{V}TM), \tag{4.54}$$

in which we have used (2.7), (2.8) and (4.36). Setting $Y = U$ in (4.53) and (4.54), we, respectively, get

$$\nabla_X U = F A_N X + \tau(X)U \quad \text{and} \quad \tau(X) = u(\nabla_X U). \tag{4.55}$$

On the other hand, setting $Y = \xi$ in (4.53) and taking the second relation of (4.55), we get

$$\nabla_X V = F A^*_\xi X - \tau(X)V. \tag{4.56}$$

First, suppose that D' is parallel, then by (4.55) we have

$$F A_N X = 0 \quad \text{and} \quad \nabla_X U = \tau(X)U, \quad \forall\, X \in \Gamma(\mathcal{V}TM). \tag{4.57}$$

Considering (4.57) in Lemma 4.1, we get

$$cg(Y,Y) = 2\tau(Y)B(\overline{J}Y,U) - 2\tau(\overline{J}Y)B(Y,U) = 0, \tag{4.58}$$

for all $Y \in \Gamma(D_0)$, in which we have used the fact that M is mixed geodesic. As D_0 is non-degenerate, (4.58) gives $c = 0$.

Now, suppose that $\overline{J}(\mathcal{V}TM^\perp)$ is parallel, then (4.56) implies $FA^*_\xi X = 0$ and $\nabla_X V = -\tau(X)V$, for any $X \in \Gamma(\mathcal{V}TM)$. Equivalently, we have

$$A^*_\xi X = B(X,V)U \quad \text{and} \quad \nabla_X V = -\tau(X)V. \tag{4.59}$$

Notice that, as D' and $\overline{J}(\mathcal{V}TM^\perp)$ are parallel distributions on M, we see, from 4.57) and (4.59), that the vector fields U and V are parallel with respect to ∇ if and only if $\tau = 0$. Hence, part (1) follows easily from Corollary 1 of [14, p. 184]. Furthermore, it is easy to show that D is also parallel. In fact, as D is integrable, (4.40) and $\overline{\nabla}\,\overline{J} = 0$ implies that $B(X,FY) = B(Y,FX)$, for any $X,Y \in \Gamma(D)$. Setting $Y = \xi$ in this relation and using the facts $-V = F\xi$ and $B(\xi,FX) = 0$, we get $B(X,V) = 0$, for any $X \in \Gamma(D)$. Thus, the first relation of (4.59) gives

$$A^*_\xi X = 0, \quad \forall\, X \in \Gamma(D). \tag{4.60}$$

In view of (4.60) and (4.53), we have

$$(\nabla_X F)Y = 0, \quad \forall X, Y \in \Gamma(D). \tag{4.61}$$

Using (4.61) together with the fact $FX \in \Gamma(D)$ for any $X \in \Gamma(\mathcal{V}TM)$, we get $\nabla_X FY = F\nabla_X Y \in \Gamma(D)$. Hence, D is parallel. Since $D \perp_B D'$ by the assumption of mixed geodesic, and that D and D' are integrable, we see that (M,B) is locally a product manifold $M_D \times M_{D'}$, in which we have considered de Rham's [10] arguments on the existence of product manifolds. Here, M_D is a B-totally null leaf of D as $B = 0$ on D (see (4.60)), and $M_{D'}$ is a B-null curve of D', as $B = 0$ on D' by (4.59) and the fact M is mixed geodesic. Furthermore, M_D has complex dimension since D is an invariant distribution. We have also seen that $B = 0$ on both D and D', which means that M is totally geodesic, proving part (3). Finally, part (4) follows from (4.37) and (4.61), which completes the proof.

Corollary 4.3. *The only mixed foliate null hypersurfaces of an indefinite nearly Kaehlerian Finsler space-form* $\mathbb{F}^{2n}$, *such that* $\overline{\nabla}\,\overline{J} = 0$ *and the distributions* D' *and* $\overline{J}(\mathcal{V}TM)$ *parallel are the geodesic ones.*

By definition of Lie derivative along vector fields, we have

$$\begin{aligned}(\mathcal{L}_V g)(X,Y) &= B(V,Y)\eta(X) + B(V,X)\eta(Y)\\ &\quad + g(\nabla_X V, Y) + g(X, \nabla_Y V), \quad \forall\, X,Y \in \Gamma(\mathcal{V}TM),\end{aligned} \tag{4.62}$$

in which we have used (2.9). In view of (4.39), we can rewrite (4.62) as

$$(\mathcal{L}_V g)(X,Y) = -B(X,FY) - B(Y,FX) - u(X)\tau(Y) - u(Y)\tau(X). \tag{4.63}$$

In case the distribution $\overline{J}(\mathcal{V}TM^{\perp})$ is killing, (4.63) becomes

$$B(X,FY) + B(Y,FX) + u(X)\tau(Y) + u(Y)\tau(X) = 0. \tag{4.64}$$

In particular, if $X,Y \in \Gamma(D)$, (4.64) gives $B(X,FY)+B(Y,FX) = 0$. Moreover, if $\overline{J}\,\overline{\nabla} = 0$ and D is integrable, then the last relation and (4.40) and (4.37) gives $B(X,Y) = 0$, for all $X,Y \in \Gamma(D)$. Furthermore, if D' is parallel, we see from (4.53) that $B(X,U) = C(X,V)$, for all $X \in \Gamma(\mathcal{V}TM)$. Also note, from (4.53) and the facts $B = 0$ on D and $FX \in \Gamma(D)$ for any $X \in \Gamma(\mathcal{V}TM)$, that the distribution D is parallel. Thus, putting all the above together, and considering Theorem 4.3, we have the following result.

Corollary 4.4. *Under the assumptions of Theorem 4.3, if instead the distribution* $\overline{J}(\mathcal{V}TM^{\perp})$ *is killing then* (M,B) *is locally a null product* $M_D \times M_{D'}$, *where* M_D *is a leaf of* D, *which is a* B*-totally degenerate manifold of complex dimension, and* $M_{D'}$ *is a* B*-non-null curve of* D', *unless* $C(U,V) = 0$.

Acknowledgments

Part of this work was done while the author was still at the University of KwaZulu-Natal. The author wishes to thank the referee for his/her valuable comments and suggestions which helped in shaping this paper.

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