



RIGIDITY OF YAMABE SOLITONS ON $N(k)$ -CONTACT METRIC MANIFOLDS

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Abstract. We study Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic and infinitesimal D -conformal deformations. We obtain rigidity results for Yamabe solitons under non-trivial D -homothetic deformation and show that an infinitesimal D -conformal deformation reduces to a homothetic deformation with constant factor. Consequently, the associated deformation vector field is affine, and no genuinely nontrivial Yamabe solitons arise under these natural geometric constraints.

Keywords: Infinitesimal D -conformal deformation, Yamabe soliton, $N(k)$ -manifold, D -homothetic deformation.

MSC (2020): 53C15, 53D10

1. INTRODUCTION

The study of conformal geometric evolution equations relies heavily on Yamabe solitons, which are self-similar solutions of the Yamabe flow. Yamabe solitons are more delicate and flexible in their classification because they rely solely on scalar curvature, in contrast to Ricci solitons. The notion of Yamabe flow was introduced by Hamilton [8] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold (M^n, g) , $n \geq 3$. The Yamabe flow is an evolution equation for metrics on a Riemannian manifold, given by

$$\frac{\partial}{\partial t}g = -rg,$$

where r is the scalar curvature corresponding to the metric g . In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$, the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric, whereas the Ricci flow does not in general. A Yamabe soliton on a Riemannian manifold (M, g) is a triplet (g, V, ρ) satisfying

$$\mathcal{L}_V g = 2(r - \rho)g, \tag{1.1}$$

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where $\mathcal{L}_V$ denotes the Lie derivative of g in the direction of the vector field V and ρ is a constant. The Yamabe soliton is said to be shrinking, steady, or expanding according as $\rho < 0$, $\rho = 0$, or $\rho > 0$, respectively.

If ρ is allowed to be a smooth function on M , then the metric satisfying (1.1) is called an almost Yamabe soliton. The rigidity of Ricci solitons and Ricci nearly solitons on K-contact and Sasakian manifolds has been studied in detail in the subject of contact metric geometry, especially in the context of symmetry assumptions like projective vector fields or D -homothetic invariance. Beyond the Sasakian context, however, very little is known about Yamabe solitons. A natural setting for extending these rigidity results is the class of $N(k)$ -contact metric manifolds. The study of curvature conditions involving the Reeb vector field naturally gives rise to these manifolds, which include Sasakian manifolds as the specific case ($k = 1$).

In [6], Daskalopoulos and Sesum showed that a compact gradient Yamabe soliton has constant scalar curvature. In [10], Sharma studied Yamabe solitons in contact Riemannian geometry, and proved that a 3-dimensional Sasakian manifold whose metric is Sasakian has constant scalar curvature and the soliton vector field V is Killing. Nagaraja and Sharma [7] established that a complete K -contact Ricci almost soliton with projective potential vector field is compact Sasakian and shrinking and that divergence is invariant under D -homothetic deformation. The purpose of this paper is to establish a unified rigidity theory for Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic deformation.

2. PRELIMINARIES

A $(2n + 1)$ -dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ , and a 1-form η satisfying [1, 3]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \quad (2.2)$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$ is integrable, where X is tangent to M , t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.

Let g be a compatible Riemannian metric with the almost contact structure (ϕ, ξ, η) , that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

for all vector fields X, Y on M .

Then M becomes an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g) . In an almost contact metric manifold the following hold.

$$\eta(X) = g(X, \xi), \quad g(\phi X, Y) = -g(X, \phi Y), \quad (2.4)$$

for all vector fields X, Y on M . An almost contact metric structure (ϕ, ξ, η, g) is said to be a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y), \quad (2.5)$$

for all vector fields X, Y on M . A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.6)$$

for all $X, Y \in TM$, where ∇ denotes the Levi-Civita connection of the Riemannian metric g . A contact metric manifold (M, ϕ, ξ, η, g) for which ξ is a Killing vector field is said to be a K -contact manifold. Every Sasakian manifold is K -contact, but the converse is not true. However, a 3-dimensional K -contact manifold is Sasakian [9]. Blair, Koufogiorgos and Papantoniou [5] introduced the (k, μ) -nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ) -nullity distribution $N(k, \mu)$ of a contact metric manifold M is defined by

$$N(k, \mu) : p \longmapsto N_p(k, \mu) = \{W \in T_p M : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ) -contact metric manifold. If $\mu = 0$, the (k, μ) -nullity distribution reduces to the k -nullity distribution [11].

The k -nullity distribution $N(k)$ of a Riemannian manifold is defined by [11]

$$N(k) : p \longmapsto N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

where k is a constant. If the characteristic vector field $\xi \in N(k)$, then the contact metric manifold is called an $N(k)$ -contact metric manifold [4]. If $k = 1$, then the manifold is Sasakian, and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [2].

However, for an $N(k)$ -contact metric manifold M of dimension $(2n + 1)$, we have [4]

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (2.7)$$

$$(\nabla_X \eta)(Y) = -g(\phi X + \phi hX, Y), \quad (2.8)$$

$$\nabla_X \xi = -\phi X - \phi hX, \quad (2.9)$$

where $h = \frac{1}{2}\mathcal{L}_\xi \phi$. Moreover, $h^2 = (k - 1)\phi^2$. In an $N(k)$ -contact metric manifold the following hold.

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad (2.10)$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \quad (2.11)$$

$$\text{Ric}(Y, \xi) = 2nk\eta(Y), \quad (2.12)$$

$$r = 2n(2n - 2 + k), \quad (2.13)$$

where r denotes the scalar curvature and Ric is the Ricci tensor.

A contact metric manifold (M, ϕ, ξ, η, g) is said to be Einstein if its Ricci tensor Ric satisfies $\text{Ric}(X, Y) = \lambda g(X, Y)$, where λ is a constant. It is said to be η -Einstein if

$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$, where α and β are smooth functions on M .

Definition 2.1. A vector field V on a Riemannian manifold (M, g) is called an affine vector field if its local flow preserves the Levi-Civita connection ∇ . Equivalently,

$$\mathcal{L}_V \nabla = 0. \quad (2.14)$$

Definition 2.2. A D -homothetic deformation of a contact metric structure is given by [12]

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = a g + a(a-1)\eta \otimes \eta, \quad (2.15)$$

where a is a positive constant. If (M, ϕ, ξ, η, g) is an almost contact metric manifold with contact form η , then $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric manifold.

3. RIGIDITY UNDER D -HOMOTHEIC DEFORMATION

In this section we study Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic deformation. The quantities with bar denote those with respect to D -homothetically deformed space.

Theorem 3.1. Let (M, ϕ, ξ, η, g) be an $N(k)$ -contact metric manifold and let $(M, \bar{g})$ be obtained from (M, g) by a non-trivial D -homothetic deformation. If (M, g, V, ρ) and $(M, \bar{g}, V, \bar{\rho})$ are both Yamabe solitons with the same soliton vector field V . Then the following assertions hold:

- (i) V is an infinitesimal contact transformation, and is strict if and only if $\bar{r} - \bar{\rho} = r - \rho$.
- (ii) V transforms η , ξ and ϕ according to

$$\mathcal{L}_V \eta = c\eta, \quad \mathcal{L}_V \xi = \sigma\xi, \quad \mathcal{L}_V \phi = \sigma\phi,$$

where

$$c = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1}, \quad \sigma = \frac{2a(\bar{r} - \bar{\rho}) - (3a-1)(r - \rho)}{a-1}.$$

Proof. Let (M, g) be an $N(k)$ -contact metric manifold and let $(M, \bar{g})$ be obtained from (M, g) by a D -homothetic deformation. Suppose that (M, g, V, ρ) and $(M, \bar{g}, V, \bar{\rho})$ are Yamabe solitons. Then

$$\mathcal{L}_V g = 2(r - \rho)g, \quad (3.16)$$

$$\mathcal{L}_V \bar{g} = 2(\bar{r} - \bar{\rho})\bar{g}. \quad (3.17)$$

Taking the Lie derivative of the D -homothetically deformed metric (2.15) along V , we obtain

$$\mathcal{L}_V \bar{g} = a\mathcal{L}_V g + a(a-1)((\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)). \quad (3.18)$$

Using (3.16) and (3.17), equation (3.18) becomes

$$2(\bar{r} - \bar{\rho})\bar{g} = 2a(r - \rho)g + a(a-1)[(\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)]. \quad (3.19)$$

Evaluating (3.19) on $(X, \bar{\xi})$, and using $\bar{g}(X, \bar{\xi}) = \bar{\eta}(X)$, $\bar{\xi} = \frac{1}{a}\xi$, and $\bar{\eta} = a\eta$, we obtain

$$2(\bar{r} - \bar{\rho})a\eta(X) = 2(r - \rho)\eta(X) + (a-1)[(\mathcal{L}_V \eta)(X) + \eta(X)(\mathcal{L}_V \eta)(\xi)]. \quad (3.20)$$

Lie differentiating $\eta(\xi) = 1$ along V gives

$$(\mathcal{L}_V \eta)(\xi) = -\eta(\mathcal{L}_V \xi). \quad (3.21)$$

Also, Lie differentiating $g(\xi, \xi) = 1$ gives

$$(\mathcal{L}_V g)(\xi, \xi) + 2\eta(\mathcal{L}_V \xi) = 0. \quad (3.22)$$

Using (3.16), this yields

$$\eta(\mathcal{L}_V \xi) = -(r - \rho), \quad (3.23)$$

and hence

$$(\mathcal{L}_V \eta)(\xi) = r - \rho. \quad (3.24)$$

Substitution of (3.24) into (3.20) gives

$$(\mathcal{L}_V \eta)(X) = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1} \eta(X). \quad (3.25)$$

Thus

$$\mathcal{L}_V \eta = c\eta, \quad c = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1}. \quad (3.26)$$

Hence V is an infinitesimal contact transformation. Moreover, V is strict precisely when $c = 0$.

Next, using

$$(\mathcal{L}_V g)(X, \xi) = (\mathcal{L}_V \eta)(X) - g(X, \mathcal{L}_V \xi)$$

and (3.16), we obtain

$$\mathcal{L}_V \xi = \sigma \xi, \quad (3.27)$$

where

$$\sigma = \frac{2a(\bar{r} - \bar{\rho}) - (3a-1)(r - \rho)}{a-1}. \quad (3.28)$$

Finally, since $\mathcal{L}_V$ commutes with d , from $d\eta(X, Y) = g(X, \phi Y)$ we have

$$d(\mathcal{L}_V \eta)(X, Y) = (\mathcal{L}_V g)(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y). \quad (3.29)$$

Using $\mathcal{L}_V \eta = c\eta$ and (3.16), equation (3.29) becomes

$$c g(X, \phi Y) = 2(r - \rho)g(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y). \quad (3.30)$$

Therefore

$$\mathcal{L}_V \phi = (c - 2(r - \rho))\phi = \sigma \phi. \quad (3.31)$$

This proves (i) – (ii). □

4. RIGIDITY OF YAMABE SOLITONS UNDER INFINITESIMAL D -CONFORMAL DEFORMATIONS

We begin with the notion of an infinitesimal D -conformal transformation in the contact metric setting.

Definition 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. A vector field V is said to generate an infinitesimal D -conformal deformation if*

$$\mathcal{L}_V g = 2\alpha g + 2\beta \eta \otimes \eta, \quad (4.32)$$

for smooth functions α, β on M .

Remark 4.1. When α is constant and $\beta = \alpha(\alpha - 1)$, the above definition corresponds to the infinitesimal generator of a global D -homothetic deformation.

Theorem 4.1. Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$ -contact metric manifold admitting a Yamabe soliton (g, V, ρ) . If V generates an infinitesimal D -conformal deformation, then

$$\mathcal{L}_V g = 2(r - \rho)g,$$

where $r = 2n(2n - 2 + k)$ is constant. Consequently, V is a homothetic and affine vector field.

Proof. Since (M, g, V, ρ) is a Yamabe soliton, we have

$$\mathcal{L}_V g = 2(r - \rho)g. \quad (4.33)$$

On the other hand, since V generates an infinitesimal D -conformal deformation, we have

$$\mathcal{L}_V g = 2\alpha g + 2\beta \eta \otimes \eta. \quad (4.34)$$

Comparing (4.33) and (4.34), we obtain

$$(r - \rho)g = \alpha g + \beta \eta \otimes \eta. \quad (4.35)$$

Let $X \in \ker \eta$, so that $\eta(X) = 0$. Evaluating (4.35) on (X, X) gives

$$(r - \rho)g(X, X) = \alpha g(X, X).$$

Therefore,

$$\alpha = r - \rho. \quad (4.36)$$

Substituting (4.36) into (4.35), we obtain

$$\beta \eta \otimes \eta = 0,$$

and hence

$$\beta = 0. \quad (4.37)$$

Thus the infinitesimal D -conformal deformation reduces to the infinitesimal homothetic deformation

$$\mathcal{L}_V g = 2(r - \rho)g. \quad (4.38)$$

For an $N(k)$ -contact metric manifold, the scalar curvature is

$$r = 2n(2n - 2 + k), \quad (4.39)$$

and hence r is constant. Since ρ is also constant, $r - \rho$ is constant. Therefore V is a homothetic vector field with constant homothety factor. Consequently,

$$\mathcal{L}_V g = 2cg, \quad c = r - \rho = \text{constant}. \quad (4.40)$$

Thus, V is a homothetic vector field. Using the standard formula for the Lie derivative of the Levi-Civita connection, we have

$$\begin{aligned} 2g((\mathcal{L}_V \nabla)(X, Y), Z) &= (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(X, Z) \\ &\quad - (\nabla_Z \mathcal{L}_V g)(X, Y). \end{aligned} \quad (4.41)$$

Since c is constant and $\nabla g = 0$, it follows that

$$\nabla_X(\mathcal{L}_V g) = \nabla_X(2cg) = 0.$$

Similarly,

$$\nabla_Y(\mathcal{L}_V g) = 0, \quad \nabla_Z(\mathcal{L}_V g) = 0.$$

Consequently,

$$\mathcal{L}_V \nabla = 0.$$

Hence, V is an affine vector field.

□

5. CONCLUDING REMARKS

The results demonstrate that Yamabe solitons on $N(k)$ -contact metric manifolds exhibit strong rigidity under D -homothetic and infinitesimal D -conformal deformations. In particular, an infinitesimal D -conformal deformation reduces to a homothetic deformation with constant factor, and the corresponding vector field is affine.

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REFERENCES

- [1] Blair, D. E. (1976). *Contact manifolds in Riemannian geometry*. Springer-Verlag.
- [2] Blair, D. E. (1977). Two remarks on contact metric structures. *Tohoku Mathematical Journal, Second Series*, 29(3), 319–324.
- [3] Blair, D. E. (2002). *Riemannian geometry of contact and symplectic manifolds*. Birkhäuser Boston.
- [4] Blair, D. E., Kim, J. S., & Tripathy, M. (2005). On the concircular curvature tensor of a contact metric manifold. *Journal of the Korean Mathematical Society*, 42(5), 883–892.
- [5] Blair, D. E., Koufogiorgos, T., & Papantoniou, B. J. (1995). Contact metric manifolds satisfying a nullity condition. *Israel Journal of Mathematics*, 91(1), 189–214.
- [6] Daskalopoulos, P., & Sesum, N. (2013). The classification of locally conformally flat Yamabe solitons. *Advances in Mathematics*, 240, 346–369.
- [7] Gangadharappa, N. H., & Sharma, R. (2020). D-homothetically deformed K-contact Ricci almost solitons. *Results in Mathematics*, 75(3), Article 124.
- [8] Hamilton, R. S. (1988). The Ricci flow on surfaces. In *Mathematics and general relativity* (pp. 237–262). Amer. Math. Soc.
- [9] Jun, J. B., Kim, I. B., & Kim, U. K. (1994). On 3-dimensional almost contact metric manifolds. *Kyungpook Math. J.*, 34(2), 293–301.
- [10] Sharma, R. (2012). A 3-dimensional Sasakian metric as a Yamabe soliton. *Int. J. Geom. Methods Mod. Phys.*, 9(4), Article 1220003.
- [11] Tanno, S. (1988). Ricci curvature of contact Riemannian manifolds. *Tohoku Math. J.*, 40, 441–448.
- [12] Tanno, S. (1968). The topology of contact Riemannian manifolds. *Illinois J. Math.*, 12, 700–717.
- [13] Yano, K. (1970). *Integral formulas in Riemannian geometry*. Marcel Dekker.

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