# ALMOST G-CONTACT METRIC MANIFOLD

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ABSTRACT. In this paper, starting from only a global basis of vector fields, we construct a class of almost contact metric manifolds and we give concrete example. Next, we study some essential types belonging to this class on dimension 3 and we construct several examples.

## 1. INTRODUCTION

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold. The notion of global (and local) frame plays an important technical role.

It should be mentioned however that a global basis of $\mathfrak{X}(M)$ ( the Lie algebra of smooth vector fields on a manifold $M$) i.e., $n$ vector fields that are linearly independent over $\mathcal{F}(M)$ and span $\mathfrak{X}(M)$, does not exist in general.

Manifolds that do admit such a global basis for $\mathfrak{X}(M)$ are called parallelizable. it is straightforward to show that a finite-dimensional manifold is parallelizable if and only if its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$).

As an illustration, we can prove that the tangent bundle, $TS^1$, of the circle, is trivial. Indeed, we can find a section that is everywhere nonzero, i.e. a non-vanishing vector field, namely

$$X(cos\theta, sin\theta) = (-sin\theta, cos\theta).$$

The reader should try proving that $TS^3$ is also trivial (use the quaternions). However, $TS^2$ is nontrivial, although this not so easy to prove.

More generally, it can be shown that $TS^n$ is nontrivial for all even $n \geq 2$. It can even be shown that $S^1, S^3$ and $S^7$ are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Here, starting from a Global frame we construct a class of almost contact metric structures, specifically, many well-known almost contact metric structures ( Sasakian, cosymplectic, Kenmotsu ) in dimension three and we confirm the construction each time with a concrete example showing that the case is non-vacuous.

This work is organized in the following way:

**Section 2** is devoted to the background of the structures which will be used in the sequel.

In **Section 3**, we give the necessary techniques to construct an almost contact metric structure from a global frame of vector fields and we give an example. In **Section 4**, we focus on the case of three-dimensional and we show how to construct some basic structures with concrete examples.

## 2. Review of needed notions

An odd-dimensional Riemannian manifold $(M^{2n+1}, g)$ is said to be an almost contact metric manifold if there exist on $M$ a $(1,1)$-tensor field $\varphi$, a vector field $\xi$ (called the structure vector field) and a 1-form $\eta$ such that

$$\begin{cases} (1) : \eta(\xi) = 1, \\ (2) : \varphi^2(X) = -X + \eta(X)\xi, \\ (3) : g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \end{cases} \tag{2.1}$$

for any vector fields $X$,$Y$ on $M$. In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad and \quad \eta \circ \varphi = 0. \tag{2.2}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Omega$, where $\Omega(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of $M$. If, in addition, $\xi$ is a Killing vector field,

then $M$ is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field $X$ on $M$.

On the other hand, the almost contact metric structure of $M$ is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta\ (X,Y)\xi = 0, \tag{2.3}$$

for any $X$, $Y$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of $\varphi$, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

An almost contact metric structures $(\varphi, \xi, \eta, g)$ on $M$ is said to be:

$$
\begin{cases}
(a): Sasaki \iff \Omega = \mathrm{d}\eta \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\
(b): Cosymplectic \iff \mathrm{d}\Omega = \mathrm{d}\eta = 0 \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\
(c): Kenmotsu \iff \mathrm{d}\eta = 0,\ \mathrm{d}\Omega = 2\eta \wedge \Omega \text{ and } (\varphi, \xi, \eta) \text{ is normal.}
\end{cases}
\tag{2.4}
$$

where d denotes the exterior derivative.

These manifolds can be characterized through their Levi-Civita connection, by requiring

$$
\begin{cases}
(1): Sasaki \iff (\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \\
(2): Cosymplectic \iff \nabla\varphi = 0, \\
(3): Kenmotsu \iff (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X.
\end{cases}
\tag{2.5}
$$

For more background on almost contact metric manifolds, we recommend the reference [1],[2], [3] and [5].

## 3. Almost G-contact metric manifold

Let $\{e_0, e_i\}_{1 \leq i \leq 2n}$ be the global frame of vector fields and $\{\theta^0, \theta^i\}_{1 \leq i \leq 2n}$ be the dual frame of differential 1-forms on a (2n+1)-dimensional smooth manifold $M$. Define a $(1,1)$-tensor field $\varphi$ on $M$ by

$$\varphi = \sum_{i=1}^n e_{2i} \wedge e_{2i-1} = \sum_{i=1}^n \left(\theta^{2i-1} \otimes e_{2i} - \theta^{2i} \otimes e_{2i-1}\right), \tag{3.6}$$

i.e. for all vector field $X$ on $M$, we have

$$
\begin{aligned}
\varphi X &= \sum_{i=1}^n (e_{2i} \wedge e_{2i-1})X \\
&= \sum_{i=1}^n \left(g(e_{2i-1}, X)e_{2i} - g(e_{2i}, X)e_{2i-1}\right) \\
&= \sum_{i=1}^n \theta^{2i-1}(X)e_{2i} - \theta^{2i}(X)e_{2i-1},
\end{aligned}
$$

and a Riemannian metric $g$ on $M$ wich $\{e_i\}_{0 \leq i \leq 2n+1}$ is an orthonormal frame, so that

$$g = \sum_{i=0}^{2n} \theta^i \otimes \theta^i. \tag{3.7}$$

With these identities, we state the following:

**Theorem 3.1.** *The manifold* $(M, \varphi, e_0, \theta^0, g)$ *defined as above is an almost contact metric manifold.*

**Proof.** According to the conditions (2.1), the data $(M, g, \varphi, e_0, \theta^0, g)$ is an almost contact metric manifold if only two conditions are satisfied

$$\varphi^2 X = -X + \theta^0(X)e_0 \qquad and \qquad g(\varphi X, \varphi Y) = g(X, Y) - \theta^0(X)\theta^0(Y).$$

Using formula (3.6) we get

$$\varphi e_{2i} = -e_{2i-1} \qquad and \qquad \varphi e_{2i-1} = e_{2i}.$$

To prove the first condition, we have for all $X$ vectors field on $M$

$$\begin{aligned}
\varphi^2 X &= \sum_{i=1}^{n} \left( \theta_{2i-1}(X)\varphi e_{2i} - \theta^{2i}(X)\varphi e_{2i-1} \right) \\
&= -\sum_{i=1}^{n} \left( \theta^{2i-1}(X)e_{2i-1} + \theta^{2i}(X)e_{2i} \right) \\
&= -\sum_{i=1}^{2n} \theta^i(X)e_i \\
&= -X + \theta^0(X)e_0.
\end{aligned}$$

For the second condition, for all $X$ and $Y$ vectors fields on $M$ we have

$$\begin{aligned}
g(\varphi X, \varphi Y) &= \sum_{i=1}^{n} \left( \theta^{2i-1}(X)\theta^{2i-1}(Y) + \theta^{2i}(X)\theta^{2i}(Y) \right) \\
&= \sum_{i=1}^{2n} \theta^i(X)\theta^i(Y) \\
&= g\left( X, \sum_{i=1}^{2n} \theta^i(Y)e_i \right) \\
&= g\left( X, Y - \theta^0(Y)e_0 \right) \\
&= g(X, Y) - \theta^0(X)\theta_0(Y),
\end{aligned}$$

which completes the proof.

We refer to this construction as **almost G-contact metric manifold**.

**Example 3.1.** *Let $(x_i)$ be the Cartesian coordinates in $\mathbb{R}^5$ and $\partial_i = \frac{\partial}{\partial x_i}$. Define a global frame of vector fields on $\mathbb{R}^5$ by:*

$$e_0 = \partial_5, \quad e_1 = \partial_1 + f\partial_5, \quad e_2 = \partial_2, \quad e_3 = \partial_3 + h\partial_5, \quad e_4 = \partial_4,$$

*where $f, h$ are two strictly positive functions on $\mathbb{R}^5$ and let $g$ be the Riemannian metric defined by*

$$g(e_i, e_j) = \delta_{ij} \qquad \forall i, j \in \{0, ..., 5\},$$

*that is, the form of the metric becomes*

$$g = \begin{pmatrix} 1+f^2 & 0 & fh & 0 & -f \\ 0 & 1 & 0 & 0 & 0 \\ fh & 0 & 1+h^2 & 0 & -h \\ 0 & 0 & 0 & 1 & 0 \\ -f & 0 & -h & 0 & 1 \end{pmatrix},$$

*and the 1-form corresponding to $e_0$ is $\theta^0 = -f\,dx_1 - h\,dx_3 + dx_5$.*

*To define $\varphi$, let us use the formula*

$$
\begin{aligned}
\varphi &= \sum_{i=1}^{2} e_{2i} \wedge e_{2i-1} \\
&= e_2 \wedge e_1 + e_4 \wedge e_3 \\
&= \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3,
\end{aligned}
$$

*we get*

$$\varphi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -f & 0 & -h & 0 \end{pmatrix},$$

*where we can check that $(\mathbb{R}^5, \varphi, e_0, \theta^0, g)$ is an almost G-contact metric manifold.*

**Remark 3.1.** *Any almost G-contact metric manifold is an almost contact metric manifold, the converse is not true in general.*

While this is an area of possible future research we mention briefly that one easily has the following:

The fundamental 2-form $\Phi$ of $(\varphi, e_0, \theta^0, g)$ is :

$$
\begin{aligned}
\Phi(X, Y) &= g(X, \varphi Y) \\
&= g\Big(X, \sum_{i=1}^{n}(e_{2i} \wedge e_{2i-1})Y\Big) \\
&= g\Big(X, \sum_{i=1}^{n} \theta^{2i-1}(Y)e_{2i} - \theta^{2i}(Y)e_{2i-1}\Big) \\
&= \sum_{i=1}^{n} \Big(\theta^{2i-1}(Y)\theta^{2i}(X) - \theta^{2i}(Y)\theta^{2i-1}(X)\Big) \\
&= 2\sum_{i=1}^{n}(\theta^{2i} \wedge \theta^{2i-1})(X, Y),
\end{aligned}
$$

we can check that is very simply as follows:

$$
\Phi = 2\, \theta^{2i} \wedge \theta^{2i-1}. \tag{3.8}
$$

**Proposition 3.1.** *Let $(M, \Phi, e_0, \theta^0, g)$ be an almost G-contact manifold. Then, we have*

$$
\mathrm{d}\Phi = \frac{\alpha}{n}\, \theta^0 \wedge \Phi, \tag{3.9}
$$

*where* $\mathrm{d}$ *denote the exterior derivative and*

$$
\alpha = div e_0 + \sum_{i<j}\Big((\mathcal{L}_{e_0}g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0}g)(e_{2i}, e_{2j})\Big).
$$

**Proof.**      Let $U = \sum_{i=1}^{n} e_{2i-1}$ and $V = \sum_{i=1}^{n} e_{2i}$ two vectors fields on $M$. Putting $\mathrm{d}\Phi = \sigma\, \theta^0 \wedge \Phi$ for a certain functions $\sigma$ on $M$. Then, we get

$$
3\,(\theta^0 \wedge \Phi)(e_0, U, V) = n,
$$

$$
\begin{aligned}
3\mathrm{d}\Phi(e_0, U, V) &= (\nabla_{e_0}\Phi)(U, V) + (\nabla_U\Phi)(V, \xi) + (\nabla_V\Phi)(\xi, U) \\
&= -\Phi(V, \nabla_U\theta^0) - \Phi(\nabla_V\theta^0, U) \\
&= \sum_{i=1}^{n}\big(\theta^{2i-1}(\nabla_U\theta^0) + \theta^{2i}(\nabla_V\theta^0)\big) \\
&= g(\nabla_U\theta^0, U) + g(\nabla_V\theta^0, V) \\
&= div\xi + \sum_{i<j}\Big((\mathcal{L}_{e_0}g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0}g)(e_{2i}, e_{2j})\Big) \\
&= \alpha,
\end{aligned}
$$

which implies $\sigma = \frac{\alpha}{n}$.

## 4. 3-DIMENSIONAL ALMOST G-CONTACT METRIC MANIFOLD

Let $\{e_0, e_1, e_2\}$ be the global frame of vector fields and $\{\theta^0, \theta^1, \theta^2\}$ be the dual frame of differential 1-forms on a 3-dimensional smooth manifold $M^3$. Define a $(1, 1)$-tensor field $\varphi$ on $M$ by

$$\varphi = e_2 \wedge e_1 = \theta^1 \otimes e_2 - \theta^2 \otimes e_1 \tag{4.10}$$

and a Riemannian metric $g$ on $M$ wich $\{e_i\}_{0 \leq i \leq 2}$ is an orthonormal frame, so that

$$g = \sum_{i=0}^{2} \theta^i \otimes \theta^i. \tag{4.11}$$

According to the theorem 3.1, $(M^3, \varphi, e_0, \theta^0, g)$ is an almost G-contact metric manifold.

Through the rest of this paper, we are mainly interested in dimension three. Below we recall certain results concerning this case.

For an arbitrary 3-dimensional almost contact metric manifold $(M^3, \xi, \eta, g)$, we have

$$d\Phi = 2\alpha\eta \wedge \Phi. \tag{4.12}$$

A 3-dimensional almost contact metric manifold $M$ is normal if and only if for all $X$ vectors field on $M$([4], Prop. 2)

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi, \tag{4.13}$$

or, equivalently,

$$\nabla_X\xi = -\alpha\varphi^2 X - \beta\varphi X, \tag{4.14}$$

and for a normal almost contact metric manifold $M$ we have ([4], Corollary 1)

$$\nabla_\xi\xi = 0 \quad and \quad d\eta = \beta\Phi. \tag{4.15}$$

where $\alpha$ and $\beta$ are the functions defined by $2\alpha = div\xi$ and $2\beta = tr(\varphi\nabla\xi)$ and $\nabla$ is the Levi-Civita connection on $M$.

From formulas (2.4) and (4.12)-(4.15), one can easily proof that

$$\begin{cases} (a): Sasaki \iff \alpha = 0, \ \beta = 1 \ and \ (\varphi, \xi, \eta) \ is \ normal, \\ (b): Cosymplectic \iff \alpha = \beta = 0 \ and \ (\varphi, \xi, \eta) \ is \ normal, \\ (c): Kenmotsu \iff \alpha = 1, \ \beta = 0 \ and \ (\varphi, \xi, \eta) \ is \ normal. \end{cases} \tag{4.16}$$

As a consequence of the above formulas (2.5), we immediately obtain the following result:

**Theorem 4.1.** *A 3-dimensional almost G-contact metric manifold is:*

$$\begin{cases} (1): G-Sasaki \iff \nabla_X e_0 = -\varphi X, \\ (2): G-cosymplectic \iff \nabla_X e_0 = 0, \\ (3): G-Kenmotsu \iff \nabla_X e_0 = -\varphi^2 X. \end{cases} \tag{4.17}$$

*for all vectors field $X$ on $M$.*

**Proof.** According to the cases given in formulas (2.5) we have:

(1): An almost G-contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)e_0 - \theta^0(Y)X, \tag{4.18}$$

taking $Y = e_0$ with $\theta^0(\nabla_X e_0) = 0$, we obtain

$$(\nabla_X \varphi)e_0 = \theta^0(X)e_0 - X \iff -\varphi\nabla_X e_0 = \theta^0(X)e_0 - X$$

$$\iff -\varphi^2 \nabla_X e_0 = -\varphi X$$

$$\iff \nabla_X e_0 = -\varphi X,$$

we proved that if $M$ is G-Sasakian then $\nabla_X e_0 = -\varphi X$. Conversely, suppose that

$$\nabla_X e_0 = -\varphi X. \tag{4.19}$$

It is easy to see that $\nabla_{\varphi X} e_0 = \varphi\nabla_X e_0$, then the structure $(\varphi, e_0, \theta^0)$ is normal and also we have ( see [4], Prop. 2)

$$\nabla_X e_0 = -\alpha\varphi^2 X - \beta\varphi X. \tag{4.20}$$

From formulas (4.19) and (4.20), we get

$$\alpha = 0 \quad and \quad \beta = 1,$$

following formulas (4.16), $M$ is a G-Sasakian manifold.

(2): An almost G-contact metric manifold is cosymplectic if and only if

$$(\nabla_X \varphi)Y = 0, \tag{4.21}$$

taking $Y = e_0$, we obtain $\nabla_X e_0 = 0$.

Conversely, suppose that

$$\nabla_X e_0 = 0. \tag{4.22}$$

It is easy to see that $\nabla_{\varphi X} e_0 = \varphi \nabla_X e_0 = 0$, then the structure $(\varphi, e_0, \theta^0)$ is normal, using formulas (4.14) , we get

$$\alpha = \beta = 0,$$

following formulas (4.16), $M$ is a G-cosymplectic manifold.

(3): An almost G-contact metric manifold is Kenmotsu if and only if

$$(\nabla_X \varphi) Y = g(\varphi X, Y) e_0 - \theta^0(Y) \varphi X, \tag{4.23}$$

taking $Y = e_0$, we get $\nabla_X e_0 = -\varphi^2 X$. we proved that if $M$ is G-Kenmotsu then $\nabla_X e_0 = -\varphi^2 X$. Conversely, suppose that

$$\nabla_X e_0 = -\varphi^2 X. \tag{4.24}$$

we obtain

$$\nabla_{\varphi X} e_0 = -\varphi^3 X = \varphi \nabla_X e_0,$$

therefore, the structure $(\varphi, e_0, \theta^0)$ is normal and also we have ( see [4], Prop. 2)

$$\nabla_X e_0 = -\alpha \varphi^2 X - \beta \varphi X. \tag{4.25}$$

From formulas (4.24) and (4.25), we get

$$\alpha = 1 \qquad and \qquad \beta = 0,$$

following formulas (4.16), $M$ is a G-Kenmotsu manifold.

## 5. EXAMPLES

Let $(x, y, z)$ denote the Cartesian coordinates in $\mathbb{R}^3$. We denote the global frame of vector fields on $\mathbb{R}^3$ by $(e_0, e_1, e_2)$ and the dual frame of differential 1-forms by $(\theta^0, \theta^1, \theta^2)$ such that $\theta^i(e_j) = \delta_{ij}$ for all $i, j \in \{0, 1, 2\}$.

**Example 5.1.** *(G-Sasakian manifold)*

*Consider*

$$\theta^0 = dx + 2z dy, \quad \theta^1 = dy, \quad \theta^2 = dz,$$

*and*

$$e_0 = \frac{\partial}{\partial x}, \quad e_1 = -2z \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z}.$$

*For the non-zero Lie brackets of $(e_i)$, we have:*

$$[e_1, e_2] = 2e_0.$$

*Define an almost contact structure $(\varphi, e_0, \theta^0)$ on $M$ by assuming*

$$\varphi e_0 = 0 \,, \; \varphi e_1 = e_2 \,, \; \varphi e_2 = -e_1.$$

*Let $g$ be the Riemannian metric on $M$ for which $(e_i)$ is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, e_0, \theta^0, g)$ is an almost contact metric structure on $\mathbb{R}^3$. For the Levi-Civita connection corresponding to $g$, we have*

$$\nabla_{e_0} e_0 = \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0, \qquad \nabla_{e_0} e_1 = \nabla_{e_1} e_0 = -e_2,$$

$$\nabla_{e_0} e_2 = \nabla_{e_2} e_0 = e_1, \qquad \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = e_0.$$

*We can easily check that $i \in \{0, 1, 2\}$*

$$\nabla_{e_i} e_0 = -\varphi e_i.$$

*Knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$ for all $X$ and $Y$ vectors fields on $M$, one can check that*

$$(\nabla_{e_i} \varphi)e_j = \delta_{ij} e_0 - \theta^0(e_j)e_i,$$

*for all $i, j \in \{0, 1, 2\}$. Therefore, $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G-Sasakian manifold.*

**Example 5.2.** *(G-cosymplectic manifold)*

*For the global frame*

$$e_0 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad e_1 = \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z},$$

*we define a Riemannian metric $g$ by*

$$g(e_0, e_1) = g(e_0, e_2) = g(e_1, e_2) = 0,$$

$$g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1$$

*that is, the form of the metric becomes*

$$g = \begin{pmatrix} 1 + x^2 & 0 & -x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{pmatrix},$$

*and the corresponding 1-forms are*

$$\theta^0 = dx, \quad \theta^1 = dy, \quad \theta^2 = -xdx + dz,$$

*To define $\varphi$, let's use the formula $\varphi = e_2 \wedge e_1$, we get*

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

*where we can check that $(\varphi, \xi, \eta, g)$ is an almost G-contact metric structure on $E^3$.*
*It is easy to see that for all $i, j \in \{0, 1, 2\}$,*

$$[e_i, e_j] = 0,$$

*therefore, all components of the Levi-Civita connection are zero. Then, for all $i \in \{0, 1, 2\}$*
*we obtain*

$$\nabla_{e_i} e_0 = 0,$$

*which shows that $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G-cosymplectic manifold. One can verify this result by*
*classical reasoning, using formulas (4.16).*

**Example 5.3.** *(G-Kenmotsu manifold)*
*Consider*

$$\theta^0 = -x\,dx + dz, \quad \theta^1 = e^z dx, \quad \theta^2 = e^z dy,$$

*and*

$$e_0 = \frac{\partial}{\partial z}, \quad e_1 = e^{-z}\left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right), \quad e_2 = e^{-z}\frac{\partial}{\partial y}.$$

*For the non-zero Lie brackets of $(e_i)$, we have:*

$$[e_0, e_1] = -e_1, \quad [e_0, e_2] = -e_2, \quad [e_1, e_2] = -xe^{-z}e_2.$$

*Define an almost contact structure $(\varphi, e_0, \theta^0)$ on M by assuming*

$$\varphi e_0 = 0\,, \ \varphi e_1 = e_2\,, \ \varphi e_2 = -e_1.$$

*Let $g$ be the Riemannian metric on M for which $(e_i)$ is an orthonormal frame, so that*
*$g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, e_0, \theta^0, g)$ is an almost contact metric structure on $\mathbb{R}^3$.*
*For the Levi-Civita connection corresponding to $g$, we have*

$$\nabla_{e_0} e_0 = \nabla_{e_0} e_1 = \nabla_{e_0} e_2 = \nabla_{e_1} e_2 = 0,$$

$$\nabla_{e_1} e_0 = e_1, \qquad \nabla_{e_2} e_0 = e_2, \qquad \nabla_{e_1} e_1 = e_0$$

$$\nabla_{e_2} e_1 = xe^{-z}e_2, \qquad \nabla_{e_2} e_2 = -e_0 - xe^{-z}e_1.$$

*We can see that for all $i \in \{0, 1, 2\}$*

$$\nabla_{e_i} e_0 = -\varphi^2 e_i.$$

*Knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$, one can check that*

$$(\nabla_X \varphi)Y = g(\varphi X, Y)e_0 - \theta^0(Y)\varphi X,$$

*for all $X, Y \in \{e_0, e_1, e_2\}$. Therefore, $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G-Kenmotsu manifold.*

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