



DOUBLE-DEPRESSED EUCLIDEAN-DESCARTES POLYNOMIALS AND ASSOCIATED CUBIC CURVES

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Abstract. The aim of this paper is to continue the study [5] of quartic real polynomials P having the same Euclidean norm as their Descartes resolvent $DR(P)$. The polynomial P is given in its reduced form P_d which means without a cubic term while double-depressed means the lack of the quadratic term. The class of these polynomials is found to consists in two 1-parameter families and the discriminant of cubic curve defined naturally by $DR(P_d)$ is computed. We consider also the double-depressed Descartes resolvent as a self-map of $\mathbb{R}^2$ and its fixed points are determined. Some special angles appear throughout this papers since the coefficients of P_d are trigonometrical functions of the real parameter t .

Keywords: Quartic real polynomial, Descartes resolvent, Euclidean-Descartes polynomial, Cubic curve, Discriminant.

MSC (2020): 12D05, 51N35, 65H04.

1. INTRODUCTION

The well-known Descartes method associates to a given monic (and reduced) quartic polynomial P_d its Descartes resolvent $DR(P_d)$ as a monic cubic polynomial. In the paper [5] we have considered both polynomials P_d and $DR(P_d)$ as vectors in $\mathbb{R}^3$ and hence we introduce and study the class of *Euclidean-Descartes polynomials* by asking the equality $\|P_d\| = \|DR(P_d)\|$ where $\|\cdot\|$ is the usual Euclidean norm of $\mathbb{R}^n$, $n \in \mathbb{N}^*$. The method of Lagrange is the subject of the paper [3].

The defining condition of isometry of the Descartes map $P_d \rightarrow DR(P_d)$ was expressed as an equation in terms of the coefficients (p, q, r) of P_d and to solve it we have fixed q in the set $\{0, \pm 1\}$. The present paper continues this line of research with another condition, namely $p = 0$; therefore P_d and $DR(P_d)$ belong now to $\mathbb{R}^2$. Since p appears in the coefficients of x^2 and z^2 of P_d respectively $DR(P_d)$ we call this case as being *double-depressed*. It follows a quadratic equation for the pair (q^2, r) which is naturally interpreted as the equation of an ellipse for which we provide the eccentricity e .

Since an ellipse is usually parametrised with the trigonometrical functions $(\cos t, \sin t)$ we obtain that $q = \cos t$ while r comes in pairs $r_{\pm} = \pm \frac{\sin 2t}{2\sqrt{15}}$ for the arbitrary real number t . It

follows two 1-parametric families of double-depressed quartic polynomials. A second interest in the cubic curves $y^2 = DR(P_d(t))(x)$ for which we compute the discriminant Δ ; on this way we continue the studies of [2], [4] and [6]. Hence, we discuss also the vanishing of Δ which gives a special angle $\simeq 80.96^\circ$.

The last section deals with the fixed points of the double-depressed Descartes map $ddDR : (q, r) \in \mathbb{R}^2 \rightarrow (-4r, -q^2) \in \mathbb{R}^2$. In addition to the obvious $(0, 0)$ we get the fixed point $(1/4, -1/16)$ implying a second remarkable angle, namely $\arccos(\frac{1}{4})$. We point out that the associated cubic curve is not an elliptic one over the field $\mathbb{R}$, see e.g. [7], since its discriminant Δ is strictly positive. Another important conclusion is the non-existence of double-depressed Euclidean-Descartes quartic polynomials with multiple roots.

2. DOUBLE DEPRESSED EUCLIDEAN-DISCARTES QUARTIC POLYNOMIALS

Fix a natural number $n \in \mathbb{N}^*$. The general framework of this paper is provided by the n -dimensional real linear space of monic polynomials of grade n :

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1x^{n-1} + \dots + a_n; a_1, \dots, a_n \in \mathbb{R}\}.$$

Since $\mathbb{R}^n$ is an Euclidean space with respect the usual scalar product $\langle \cdot, \cdot \rangle$ the polynomial P has the Euclidean norm:

$$\|P\| := (a_1^2 + \dots + a_n^2)^{\frac{1}{2}}. \quad (2.1)$$

Fix now the quartic polynomial $P \in \mathbb{R}_4^{monic}[x]$:

$$P(x) := y^4 + Ay^3 + By^2 + Cy + D$$

to which we apply the Tschirnhausen transformation $y = x - \frac{A}{4}$ have the *reduced* form (here the subscript d means *depressed*):

$$P_d(x) := x^4 + px^2 + qx + r. \quad (2.2)$$

The Descartes approach for P_d is one of (relative many) methods to deal with P_d ; see e.g. [8, p. 174] as well as [9] for an universal method to solve the quartic equation. More precisely, Descartes performs a product decomposition of P_d into two quadratic polynomials:

$$P_d(x) = (x^2 + ax + b)(x^2 + \tilde{a}x + \tilde{b}) \quad (2.3)$$

and therefore:

$$a + \tilde{a} = 0, \quad b + \tilde{b} + a\tilde{a} = p, \quad a\tilde{b} + \tilde{a}b = q, \quad b\tilde{b} = r. \quad (2.4)$$

It follows immediately:

$$4r = 4b\tilde{b} = (b + \tilde{b})^2 - (b - \tilde{b})^2 = (p + a^2)^2 - \left(\frac{q}{a}\right)^2 \quad (2.5)$$

and hence, denoting $a^2 = z$, it follows a cubic equation in u :

$$DR(P_d)(z) := z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0 \quad (2.6)$$

and let us call the polynomial $DR(P_d) \in \mathbb{R}_3^{monic}[z]$ as being the *Descartes resolvent* of P_d . From:

$$DR(P_d)(0) = -q^2 \leq 0 < +\infty = \lim_{z \rightarrow +\infty} DR(P_d)(z)$$

the Descartes resolvent have at least one positive root $z_0 = a^2$ which yields the whole data $(a, \tilde{a} = -a, b, \tilde{b})$ through the relations (2.4). In addition to the previous decomposition of P_d the Descartes resolvent is also useful in the following irreducibility characterization: suppose that $P_d \in \mathbb{Q}[x]$. Then P_d is irreducible over $\mathbb{Q}$ if and only if P_d does not have a rational root and $DR(P_d)$ does not have a root that is a square of a rational number.

In a previous study [5] of quartic polynomials we introduce:

Definition 2.1. *i) P (or equivalently P_d) is an Euclidean-Descartes polynomial if it preserves the Euclidean norm with respect to Descartes transformation DR :*

$$\|P_d\| = \|DR(P_d)\|. \quad (2.7)$$

So, the restriction of DR to the class of Euclidean-Descartes polynomials is an Euclidean isometry.

ii) The Descartes-cubic curve associated to (arbitrary) P_d is:

$$DC(P_d) : y^2 = DR(P_d)(x). \quad (2.8)$$

A straightforward computation yields:

Proposition 2.1. *P_d is an Euclidean-Descartes polynomial if and only if:*

$$p^4 + q^4 + 15r^2 - 8p^2r + 3p^2 - q^2 = 0. \quad (2.9)$$

With the 3D-transformation $(X := p^2, Y := q^2, Z = r)$ we have the equation of a two-sheets hyperboloid:

$$H_2(P_d) : X^2 + Y^2 + 15Z^2 - 8XZ + 3X - Y = 0. \quad (2.10)$$

Proof. The hypothesis means that:

$$\|P_d\|^2 = \|DR(P_d)\|^2 \rightarrow p^2 + q^2 + r^2 = 4p^2 + (p^2 - 4r)^2 + q^4$$

which is exactly (2.9). □

In [5] we point out an easy case of reduction namely $q \in \{0, \pm 1\}$ and hence P_d which such a coefficient was be called *reduced*. Now, we remark that the coefficient of x^2 in P_d is half the coefficient of z^2 in $DR(P_d)$ and then we introduce a new class:

Definition 2.2. *The depressed polynomial P_d is called double-depressed if $p = 0$.*

Remark 2.1. *i) If we look at the initial quartic polynomial P since $p = B - \frac{3A^2}{8}$ it follows that the double-depressed condition means:*

$$B = \frac{3A^2}{8}. \quad (2.11)$$

Concerning the Descartes coefficients $(a, \tilde{a} = -a, b, \tilde{b})$ the vanishing of p means: $b + \tilde{b} = a^2 = z_0$.

ii) The Descartes-cubic curve is:

$$DC(P_d) : y^2 = x^3 + Px + Q, \quad P = -4r, \quad Q = -q^2 \quad (2.12)$$

and we recall its discriminant: $\Delta := 4P^3 + 27Q^2$.

iii) The intersection of the hyperboloid $H_2(P_d)$ with the plane $X = 0$ is the ellipse:

$$E : Y^2 + 15Z^2 - Y = 0 \quad (2.13)$$

with the eccentricity:

$$e = \sqrt{\frac{14}{15}} \simeq 0.966092. \quad (2.14)$$

□

Since it is easy to give a parametrization of E it follows a complete characterization of double-depressed quartic polynomials:

Theorem 2.1. *The class of double-depressed quartic polynomials consists in two 1-parametric families with the coefficients:*

$$q(t) = \cos t, \quad r_{\pm}(t) = \pm \frac{\sin 2t}{2\sqrt{15}}, \quad t \in \mathbb{R}. \quad (2.15)$$

The discriminant of the Descartes-cubic curve $DC(P_d)(t)$ is:

$$\begin{cases} DC(P_d)(t) : y^2 = x^3 \mp \left(\frac{2\sin 2t}{\sqrt{15}} \right) x - (\cos t)^2, \\ \Delta_{\pm}(t) = (\cos t)^3 \left[27 \cos t \mp \frac{256}{15\sqrt{15}} (\sin t)^3 \right]. \end{cases} \quad (2.16)$$

Proof. The equation (2.9) with $p = 0$ means:

$$15r^2 = q^2(1 - q^2) \quad (2.17)$$

forcing q to be a cosine of a real parameter t . Hence, the claimed expressions (2.15) follow immediately. Also, the value of the discriminant being:

$$\Delta = 27q^4 - 16^2r^3 \quad (2.18)$$

the formulae (2.16) is directly. □

We are interested now in the vanishing of $\Delta_{\pm}$:

- i) $\Delta_- = 0$ implies $\cos t \in \{0, \simeq 0.1572\}$; this means $t \in \{90^\circ, \simeq 80.96^\circ\}$
- ii) $\Delta_+ = 0$ implies $\cos t \in \{0, \simeq -0.1572\}$.

3. THE FIXED POINTS OF THE DOUBLE-DEPRESSED DESCARTES MAP

A second way to find double-depressed Euclidean-Descartes polynomials is to find the fixed points of the *dd-Descartes map*:

$$ddDR : (q, r) \in \mathbb{R}^2 \rightarrow (-4r, -q^2) \in \mathbb{R}^2 \quad (3.1)$$

or, in a matrix form:

$$ddDR : \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2 \rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q^2 \\ r^2 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2. \quad (3.2)$$

By denoting Γ_1 and Γ_2 the 2×2 matrices above it results that $\Gamma_2 = 4(\Gamma_1)^T$. The Descartes map is a bijective map on its image $\mathbb{R}_-^2 := \{(A, B) \in \mathbb{R}^2; B \leq 0\}$ with:

$$(ddDR)^{-1}(A, B) = \left(\pm\sqrt{-B}, -\frac{A}{4} \right) \quad (3.3)$$

and the iteration of the map $ddDR$ produces the interesting quadratic map:

$$(q, r) \in \mathbb{R}^2 \rightarrow (4q^2, -16r^2) \in \mathbb{R}^2. \quad (3.4)$$

A direct computation yields:

Proposition 3.1. *The dd -Descartes map (3.1) has only two fixed points: $(0, 0)$ which corresponds via (2.15) to $t_1 = \frac{\pi}{2}$ and $(\frac{1}{4}, -\frac{1}{16})$ which corresponds to $r_-(t_2 = \arccos(0.25) \simeq 75.52^\circ)$.*

It results:

- a) $DC(P_d)(t_1) : y^2 = x^3$, which is the *semicubical parabola*, $\Delta(t_1) = 0$ for both signs $\pm$,
- b) $DR(P_d(t_2))(z) = z^3 + \frac{z}{4} - \frac{1}{16}$, $\|P_d(t_2)\| = \frac{\sqrt{17}}{16}$, $DC(P_d)(t_2) : y^2 = x^3 + \frac{x}{4} - \frac{1}{16}$, $\Delta_-(t_2) = \frac{43}{256} > 0$. $DR(P_d(t_2))$ has an unique real root $z_0 \simeq 0.2119$. We recall that a real polynomial P with all real roots is called *hyperbolic*, see e.g. [1]; so, $DR(P_d(t_2))$ is not a hyperbolic polynomial.

The fixed point $(\frac{1}{4}, -\frac{1}{16})$ corresponds to the point $(Y, Z) = (\frac{1}{16}, -\frac{1}{16})$ which is the intersection of the ellipse E with the second bisectrix $B_2 : y = -x$.

It is worth to point out $P_d(t_1)(x) = x^4$ have multiple roots. We finish this section with the problem of existence of double-depressed Euclidean-Descartes polynomials with multiple roots. In [5] we have obtained:

Proposition 3.2. *Excepting $P_d(x) = x^4$ there are only two Euclidean-Descartes polynomials with multiple roots:*

$$\begin{cases} P_d^\pm(x) = x^4 \pm 4\sqrt{3}x^2 + 12 = (x^2 \pm 2\sqrt{3})^2, \\ DR(P_d^\pm)(z) = z^2(z \pm 8\sqrt{3}), \quad \|P_d^\pm\| = 8\sqrt{3}. \end{cases} \quad (3.5)$$

The value $p = \pm 4\sqrt{3} \neq 0$ means that there are no double-depressed Euclidean-Descartes polynomials with multiple roots.

Acknowledgments. The author would like to thank the referee for several useful comments and very helpful suggestions that have improved the quality of this paper. Also I express my sincere gratitude to Prof. Dr. Bayram Sahin for all its support.

Conflict of interest. The author declares no potential conflict of interests.

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