



SYMMETRY VECTOR FIELDS ON THE 4-DIMENSIONAL RIEMANNIAN AND LORENTZIAN LIE GROUP Sol_0^4

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Abstract. In this paper, we investigate various symmetry vector fields on the 4-dimensional Riemannian and Lorentzian Lie group Sol_0^4 . Using Lie derivatives, we characterize affine vector fields with respect to the Levi-Civita connection. We further obtain Ricci collineations vector fields, curvature collineations vector fields, and matter collineations vector fields by computing the Lie derivatives of the Ricci tensor, Riemann curvature tensor, and energy-momentum tensor, respectively. These results contribute to the understanding of geometric and physical symmetries on Sol_0^4 , relevant in both mathematical physics and differential geometry.

Keywords: Lorentzian Sol_0^4 group, affine vector fields, Ricci collineations, curvature collineations, matter collineations.

MSC (2020): 53C50, 53B30.

1. INTRODUCTION

The primary objective of all research in gravitational physics is to construct gravitational potentials that comply with Einstein's field equations, which is frequently accomplished by applying symmetries to the geometry that align with the dynamics of the selected matter distribution. The symmetries of space-time, from a geometric perspective, are expressible through the nullification of the Lie derivative of certain tensors in relation to a vector. The concept of a symmetry of Tensor T involves a one-parameter group of diffeomorphisms that keeps T unchanged. According to this formulation, a vector field X that satisfies $L_X T = 0$, where L is the Lie derivative, corresponds to each symmetry.

Thurston geometries constitute a fascinating subset of Riemannian homogeneous spaces, which have been thoroughly investigated in three dimensions owing to Thurston's geometrization of three-manifolds. This profound result enumerates the potential Riemannian structures of compact orientable three-manifolds, closely resembling the uniformization theorem applicable to compact orientable surfaces. In simple terms, any three-manifold can be partitioned into segments, each of which possesses a Riemannian metric that is locally isometric to one of eight three-dimensional model spaces: the Thurston geometries, i.e., $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$,

Received: 2026.02.18

Accepted: 2026.03.31

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$S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{SL}(2, \mathbb{R})$, Nil_3 and Sol_3 . As of now, no analogous geometrization has been established in four dimensions.

Based on four-dimensional Thurston geometries, Sol_0^4 is one of them. As stated by Filipkiewicz [13], there exist 19 model spaces in four dimensions i.e., $\mathbb{R}^4$, $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{S}^3 \times \mathbb{R}^1$, $\mathbb{H}^3 \times \mathbb{R}$, Sol_0^4 , Sol_1^4 , $\mathbb{F}^4$, $\mathbb{CP}^2$, $\mathbb{CH}^2$, $\mathbb{R}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $Nil_3 \times \mathbb{R}$, $\tilde{SL}(2, \mathbb{R}) \times \mathbb{R}$, Nil^4 , $Sol_{m,n}^4$. Ishihara [15] demonstrated homogeneous Riemannian 4-spaces are either locally symmetric or Lie groups that possess a left invariant metric. Furthermore, Wall [19] provided a comprehensive list of all 4-dimensional geometries that allow for a compatible complex structure, which are all the abovementioned 19 model spaces, except $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{H}^3 \times \mathbb{R}$, Nil^4 , $Sol_{m,n}^4$. In this paper, symmetries of Sol_0^4 will be investigated.

Some well-known symmetries are the Levi-Civita connection ∇ symmetries that match the affine vector fields; the Ricci tensor Ric symmetries that match the Ricci collineations; and the curvature tensor R symmetries that match the curvature collineations. Research on curvature collineations was initiated by Katzin et al. in 1969 [16].

Matter symmetries offer a distinct and more physically substantiated perspective on the symmetries of spacetimes. A Riemannian manifold (M, g) with matter collineation is represented by a vector field X , which corresponds to a symmetry in the energy-momentum tensor $T = Ric - (\frac{1}{2})\tau g$, where τ is the scalar curvature [11, 12]. Extensive research has been conducted on these symmetries, emphasizing several of them in the past decade that are pertinent to this paper [6, 7, 8, 9, 10, 14, 18]. Batat et al. worked on the symmetries of the three-dimensional Heisenberg group [2] in 2017 and in 2020, Belarbi investigated symmetries on Sol_3 Lie group [4] and symmetries of five-dimensional solvable Lie group [17]. In the same year, he investigated symmetries of three-dimensional generalised symmetric spaces [5] and also on the symmetries of Lorentzian four-dimensional generalised symmetric spaces of type C [3]. Recently, in 2023, Atashpeykar et al. discussed the symmetries of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ [1].

Main Results:

We investigate the Lie derivatives of the Levi-Civita connection ∇ to characterize affine vector fields on the Riemannian Lie group Sol_0^4 .

Theorem 1.1. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

– *X is an affine vector field if and only if*

$$F_1 = \gamma_1 x e^{-t}, \quad F_2 = \gamma_1 y e^{-t}, \quad F_3 = -2\gamma_1 z e^{2t}, \quad F_4 = \gamma_1, \quad (1.1)$$

where $\gamma_1 \in \mathbb{R}$.

Next, we compute the Lie derivatives of the Ricci tensor Ric and the Riemann curvature tensor R to identify Ricci collineations and curvature collineations on the Riemannian Lie group Sol_0^4 .

Theorem 1.2. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

(a) X is a Ricci collineation if and only if

$$F_1 = g_1(x, y, z), \quad F_2 = g_2(x, y, z), \quad F_3 = g_3(x, y, z), \quad F_4 = a_2, \quad (1.2)$$

where g_1, g_2, g_3 are smooth functions on Sol_0^4 , and $a_2 \in \mathbb{R}$.

(b) X is a curvature collineation vector field if and only if

$$\begin{cases} F_1 = (a_3x + b_2y + c_4)e^{-t}, & F_2 = (a_3y - b_2x + c_5)e^{-t}, \\ F_3 = (-2a_3z + c_6)e^{2t}, & F_4 = a_3, \end{cases} \quad (1.3)$$

where $a_3, b_2, c_4, c_5, c_6 \in \mathbb{R}$.

We now turn to the study of matter collineations, which are derived from the Lie derivative of the energy-momentum tensor T .

Theorem 1.3. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

– X is a matter collineation vector field if and only if

$$\begin{cases} F_1 = (a_4x + b_3y + c_7)e^{-t}, & F_2 = (a_4y - b_3x + c_8)e^{-t}, \\ F_3 = (-2a_4z + c_9)e^{2t}, & F_4 = a_4, \end{cases} \quad (1.4)$$

where $a_4, b_3, c_7, c_8, c_9 \in \mathbb{R}$.

We now shift our focus to the Lorentzian structure on the Lie group Sol_0^4 , beginning with affine vector fields under the Lorentzian metric.

Theorem 1.4. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

– X is an affine vector field if and only if

$$F_1 = \gamma_2xe^{-t}, \quad F_2 = \gamma_2ye^{-t}, \quad F_3 = -2\gamma_2ze^{2t}, \quad F_4 = \gamma_2, \quad (1.5)$$

where $\gamma_2 \in \mathbb{R}$.

Continuing in the Lorentzian setting, we analyze Ricci and curvature collineations using the Lie derivatives of the Ricci and Riemann curvature tensors.

Theorem 1.5. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

(a) X is a Ricci collineation if and only if

$$F_1 = g_4(x, y, z), \quad F_2 = g_5(x, y, z), \quad F_3 = g_6(x, y, z), \quad F_4 = a_6, \quad (1.6)$$

where g_4, g_5, g_6 are smooth functions on Sol_0^4 , and $a_6 \in \mathbb{R}$.

(b) X is a curvature collineation vector field if and only if

$$\begin{cases} F_1 = (a_7x + b_5y + c_{13})e^{-t}, & F_2 = (a_7y - b_5x + c_{14})e^{-t}, \\ F_3 = (-2a_7z + c_{15})e^{2t}, & F_4 = a_7, \end{cases} \quad (1.7)$$

where $a_7, b_5, c_{13}, c_{14}, c_{15} \in \mathbb{R}$.

Finally, we consider the Lie derivative of the energy-momentum tensor T on the Lorentzian Lie group Sol_0^4 to determine matter collineations.

Theorem 1.6. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

– X is a matter collineation vector field if and only if

$$\begin{cases} F_1 = (a_8x + b_6y + c_{15})e^{-t}, & F_2 = (a_8y - b_6x + c_{16})e^{-t}, \\ F_3 = (-2a_8z + c_{17})e^{2t}, & F_4 = a_8, \end{cases} \quad (1.8)$$

where $a_8, b_6, c_{15}, c_{16}, c_{17} \in \mathbb{R}$.

2. PRELIMINARIES

The group operation associated with the Sol_0^4 Lie group is given by:

$$(x, y, z, t) \star (x', y', z', t') = (x + e^t x', y + e^t y', z + e^{-2t} z', t + t'). \quad (2.9)$$

Let g be the left-invariant Riemannian metric, ∇ the Levi-Civita connection, and R the curvature tensor on the four-dimensional solvable Lie group Sol_0^4 . Then the curvature tensor is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.10)$$

where $X, Y, Z \in \chi(M)$, and the Ricci tensor is given by:

$$\text{Ric}(X, Y) = \sum_{i=1}^4 g(e_i, e_i) g(R(e_i, X)Y, e_i), \quad (2.11)$$

where $\{e_i\}_{i=1}^4$ is an orthonormal basis.

The Riemannian Sol_0^4 Group. The left-invariant Riemannian metric on the Lie group Sol_0^4 is:

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2, \quad (2.12)$$

where (x, y, z, t) are the standard Cartesian coordinates on $\mathbb{R}^4$.

The Riemannian Sol_0^4 Lie group possesses the left-invariant basis vector fields $\{e_1, e_2, e_3, e_4\}$ given by:

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \quad (2.13)$$

The Levi-Civita connection ∇ of the Riemannian Lie group Sol_0^4 satisfies:

$$\begin{aligned} \nabla_{e_1} e_1 &= e_4, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= -e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= e_4, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= -e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= -2e_4, & \nabla_{e_3} e_4 &= 2e_3, \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= 0. \end{aligned} \quad (2.14)$$

The Lie brackets of the basis vector fields are:

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_2, e_3] = 0, \\ [e_4, e_1] &= e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3. \end{aligned} \quad (2.15)$$

Using (2.10), (2.14) and (2.15), the non-zero components of the curvature tensor R are:

$$\begin{aligned} R(e_1, e_2)e_2 &= -e_1, & R(e_1, e_3)e_3 &= 2e_1, & R(e_1, e_4)e_4 &= -e_1, \\ R(e_2, e_3)e_3 &= 2e_2, & R(e_2, e_4)e_4 &= -e_2, & R(e_3, e_4)e_4 &= -4e_3, \\ R(e_1, e_2)e_1 &= e_2, & R(e_1, e_3)e_1 &= -2e_3, & R(e_1, e_4)e_1 &= e_4, \\ R(e_2, e_3)e_2 &= -2e_3, & R(e_2, e_4)e_2 &= e_4, & R(e_3, e_4)e_3 &= 4e_4. \end{aligned} \quad (2.16)$$

Using (2.11), (2.12) and (2.16), the components of the Ricci curvature tensor Ric_{ij} are:

$$\begin{aligned} \text{Ric}_{11} &= \text{Ric}_{12} = \text{Ric}_{13} = \text{Ric}_{14} = 0, \\ \text{Ric}_{22} &= \text{Ric}_{23} = \text{Ric}_{24} = \text{Ric}_{33} = \text{Ric}_{34} = 0, \\ \text{Ric}_{44} &= -6. \end{aligned} \quad (2.17)$$

Finally, using (2.12), and (2.17), the scalar curvature τ of the Riemannian Sol_0^4 Lie group is:

$$\tau = \text{trRic} = \sum_{i=1}^4 g(e_i, e_i) \text{Ric}(e_i, e_i) = -6. \quad (2.18)$$

The Lorentzian Sol_0^4 Group. The left-invariant Lorentzian metric of the Lorentzian Sol_0^4 Lie group is given by

$$g = e^{-2t}(dx^2 + dy^2) - e^{4t}dz^2 + dt^2, \quad (2.19)$$

where (x, y, z, t) represents the standard Cartesian coordinates on $\mathbb{R}^4$.

The Lorentzian Sol_0^4 Lie group possesses a left-invariant basis of vector fields $\{e_1, e_2, e_3, e_4\}$ given by:

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \quad (2.20)$$

The Levi-Civita connection ∇ on the Lorentzian Sol_0^4 Lie group satisfies:

$$\begin{aligned} \nabla_{e_1} e_1 &= e_4, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= -e_1 \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= e_4, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= -e_2 \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 2e_4, & \nabla_{e_3} e_4 &= 2e_3 \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= 0. \end{aligned} \quad (2.21)$$

The Lie brackets of the basis vector fields are:

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_2, e_3] = 0, \\ [e_4, e_1] &= e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3. \end{aligned} \quad (2.22)$$

Using (2.10), (2.21) and (2.22), the non-zero components of the curvature tensor R are:

$$\begin{aligned} R(e_1, e_2)e_2 &= -e_1, & R(e_1, e_3)e_3 &= -2e_1, & R(e_1, e_4)e_4 &= -e_1 \\ R(e_2, e_3)e_3 &= -2e_2, & R(e_2, e_4)e_4 &= -e_2, & R(e_3, e_4)e_4 &= -4e_3 \\ R(e_1, e_2)e_1 &= e_2, & R(e_1, e_3)e_1 &= -2e_3, & R(e_1, e_4)e_1 &= e_4 \\ R(e_2, e_3)e_2 &= -2e_3, & R(e_2, e_4)e_2 &= e_4, & R(e_3, e_4)e_3 &= -4e_4. \end{aligned} \quad (2.23)$$

Using (2.11), (2.19) and (2.23), the components of the Ricci curvature tensor Ric_{ij} are:

$$\begin{aligned} \text{Ric}_{11} &= \text{Ric}_{12} = \text{Ric}_{13} = \text{Ric}_{14} = \text{Ric}_{22} = 0, \\ \text{Ric}_{23} &= \text{Ric}_{24} = \text{Ric}_{33} = \text{Ric}_{34} = 0, \quad \text{Ric}_{44} = -6. \end{aligned} \quad (2.24)$$

Using (2.19), (2.24), the scalar curvature τ associated with the Lorentzian Sol_0^4 Lie group is:

$$\tau = \text{tr}(\text{Ric}) = \sum_{i=1}^4 g(e_i, e_i) \text{Ric}(e_i, e_i) = -6. \quad (2.25)$$

3. PROOF OF THEOREMS

Throughout the paper, we denote the Cartesian basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\}$ by $\{\partial_x, \partial_y, \partial_z, \partial_t\}$.

Proof of Theorem 1.1. Let $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

We compute the Lie derivative of the Levi-Civita connection ∇ in order to determine the affine vector fields. Using (2.14) and (2.15), we obtain the following equations:

$$\begin{aligned} (L_X \nabla)(e_1, e_1) &= \left(-(\partial_t F_1 + F_1) - 2e^t \partial_x F_4 + e^{2t} \partial_x^2 F_1 \right) e_1 \\ &\quad + \left(-(F_2 + \partial_t F_2) + e^{2t} \partial_x^2 F_2 \right) e_2 \\ &\quad + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 \right) e_3 \\ &\quad + \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 \right) e_4, \end{aligned} \quad (3.26)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_2) &= \left(-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 \right) e_1 + \left(-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 \right) e_2 \\ &\quad + \left(e^{2t} \partial_x \partial_y F_3 \right) e_3 + \left(e^t (\partial_y F_1 + \partial_x F_2) + e^{2t} \partial_x \partial_y F_4 \right) e_4, \end{aligned} \quad (3.27)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_4) &= \left(e^t (\partial_x \partial_t F_2 + \partial_x F_2) \right) e_2 + \left(e^t (\partial_x \partial_t F_3 + \partial_x F_3) \right) e_3 \\ &\quad + \left(-\partial_t F_4 + e^t (\partial_x \partial_t F_1 + \partial_x F_1) \right) e_1 \\ &\quad + \left(F_1 + \partial_t F_1 + e^t (\partial_x \partial_t F_4 + \partial_x F_4) \right) e_4, \end{aligned} \quad (3.28)$$

$$\begin{aligned}
(L_X \nabla)(e_2, e_2) &= \left(-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 \right) e_1 + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_y^2 F_3 \right) e_3 \\
&+ \left(-(F_2 + \partial_t F_2) - 2e^t \partial_y F_4 + e^{2t} \partial_y^2 F_2 \right) e_2 \\
&+ \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_y F_2 + e^{2t} \partial_y^2 F_4 \right) e_4,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
(L_X \nabla)(e_2, e_4) &= \left(e^t (\partial_y F_1 + \partial_y \partial_t F_1) \right) e_1 + \left(e^t (\partial_y F_3 + \partial_y \partial_t F_3) \right) e_3 \\
&+ \left(e^t (\partial_y F_4 + \partial_y \partial_t F_4) + F_2 + \partial_t F_2 \right) e_4 \\
&+ \left(e^t (\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 \right) e_2,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
(L_X \nabla)(e_3, e_3) &= \left(2(F_1 + \partial_t F_1) + e^{-4t} \partial_z^2 F_1 \right) e_1 + \left(2(F_2 + \partial_t F_2) + e^{-4t} \partial_z^2 F_2 \right) e_2 \\
&+ \left(2\partial_t F_4 - 4(e^{-2t} \partial_z F_3 + 2F_4) + e^{-4t} \partial_z^2 F_4 \right) e_4 \\
&+ \left(2(\partial_t F_3 - 2F_3) + e^{-4t} \partial_z^2 F_3 + 4e^{-2t} \partial_z F_4 \right) e_3,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
(L_X \nabla)(e_3, e_4) &= \left(e^{-2t} (\partial_z \partial_t F_1 - 2\partial_z F_1) \right) e_1 + \left(e^{-2t} (\partial_z \partial_t F_2 - 2\partial_z F_2) \right) e_2 \\
&+ \left(e^{-2t} (\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 \right) e_3 \\
&+ \left(e^{-2t} (\partial_z \partial_t F_4 - 2\partial_z F_4) - 2(\partial_t F_3 - 2F_3) \right) e_4,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
(L_X \nabla)(e_4, e_4) &= \left(\partial_t^2 F_1 - F_1 \right) e_1 + \left(\partial_t^2 F_2 - F_2 \right) e_2 \\
&+ \left(\partial_t^2 F_4 \right) e_4 + \left(\partial_t^2 F_3 - 4F_3 \right) e_3.
\end{aligned} \tag{3.33}$$

To find affine vector fields F , we solve above system of partial differential equations by setting all the coefficients of the Lie derivative to zero. Therefore, comparing the components of e_4 on both sides in (3.26), (3.29), (3.31) and (3.33) we get

$$-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 = 0, \tag{3.34}$$

$$-(\partial_t F_4 + 2F_4) + 2e^t \partial_y F_2 + e^{2t} \partial_y^2 F_4 = 0, \tag{3.35}$$

$$2\partial_t F_4 - 4(e^{-2t} \partial_z F_3 + 2F_4) + e^{-4t} \partial_z^2 F_4 = 0, \tag{3.36}$$

and

$$\partial_t^2 F_4 = 0, \tag{3.37}$$

respectively.

Similarly, comparing the components of e_1 on both sides in (3.27), (3.28) and (3.29), we get

$$-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 = 0, \tag{3.38}$$

$$-\partial_t F_4 + e^t(\partial_x \partial_t F_1 + \partial_x F_1) = 0, \quad (3.39)$$

and

$$-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 = 0, \quad (3.40)$$

respectively.

Similarly, comparing the components of e_2 on both sides in (3.27), (3.30), (3.31) and (3.32) we get

$$-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 = 0, \quad (3.41)$$

$$e^t(\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 = 0 \quad (3.42)$$

$$2(F_2 + \partial_t F_2) + e^{-4t} \partial_z^2 F_2 = 0 \quad (3.43)$$

and

$$e^{-2t}(\partial_z \partial_t F_2 - 2\partial_z F_2) = 0, \quad (3.44)$$

respectively.

Similarly, comparing the components of e_3 on both sides in (3.26), (3.28), (3.30) and (3.32) we get

$$-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 = 0, \quad (3.45)$$

$$e^t(\partial_x \partial_t F_3 + \partial_x F_3) = 0, \quad (3.46)$$

$$e^t(\partial_y F_3 + \partial_y \partial_t F_3) = 0, \quad (3.47)$$

and

$$e^{-2t}(\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 = 0, \quad (3.48)$$

respectively.

On solving (3.37), we get

$$F_4 = h_1(x, y, z).t + h_2(x, y, z). \quad (3.49)$$

where, $h_1 = h_1(x, y, z)$ and $h_2(x, y, z)$ are real-valued smooth functions on Sol_0^4 .

Using (3.49) in (3.34), we get

$$\partial_x F_1 = \frac{e^{-t} \left(h_1 + 2h_1.t + 2h_2 - e^{2t}(\partial_x^2 h_1.t + \partial_x^2 h_2) \right)}{2}, \quad (3.50)$$

and

$$\partial_x \partial_t F_1 = \frac{e^{-t} \left(h_1 - 2h_1.t - 2h_2 - e^{2t}(\partial_x^2 h_1.t + \partial_x^2 h_2 + \partial_x^2 h_1) \right)}{2}. \quad (3.51)$$

Using (3.49), (3.50) and (3.51) in (3.39), we obtain

$$\partial_x^2 h_1 = 0 = \partial_x^2 h_2. \quad (3.52)$$

Similarly, using (3.49) in (3.35), we get

$$\partial_y F_2 = \frac{e^{-t} \left(h_1 + 2h_1.t + 2h_2 - e^{2t} (\partial_y^2 h_1.t + \partial_y^2 h_2) \right)}{2}, \quad (3.53)$$

and

$$\partial_y \partial_t F_2 = \frac{e^{-t} \left(h_1 - 2h_1.t - 2h_2 - e^{2t} (\partial_y^2 h_1.t + \partial_y^2 h_2 + \partial_y^2 h_1) \right)}{2}. \quad (3.54)$$

Using (3.49), (3.53) and (3.54) in (3.42), we obtain

$$\partial_y^2 h_1 = 0 = \partial_y^2 h_2. \quad (3.55)$$

Similarly, using (3.49) in (3.36), we get

$$\partial_z F_3 = \frac{e^{2t} \left(2h_1 - 8h_1.t - 8h_2 + e^{-4t} (\partial_z^2 h_1.t + \partial_z^2 h_2) \right)}{4}, \quad (3.56)$$

and

$$\partial_z \partial_t F_3 = \frac{e^{2t} \left(-4h_1 - 16h_1.t - 16h_2 - e^{-4t} (2\partial_z^2 h_1.t + 2\partial_z^2 h_2 - \partial_z^2 h_1) \right)}{4}. \quad (3.57)$$

Using (3.49), (3.56) and (3.57) in (3.48), we obtain

$$\partial_z^2 h_1 = 0 = \partial_z^2 h_2. \quad (3.58)$$

On solving (3.52), (3.55) and (3.58), we get

$$h_1(x, y, z) = a_1 x + a_2 y + a_3 z + a_4 xy + a_5 yz + a_6 xz + a_7 xyz + a_8, \quad (3.59)$$

$$h_2(x, y, z) = b_1 x + b_2 y + b_3 z + b_4 xy + b_5 yz + b_6 xz + b_7 xyz + \gamma_1, \quad (3.60)$$

where $a_i, b_j, \gamma_1 \in \mathbb{R}$, $i=1, \dots, 8$ and $j=1, \dots, 7$.

Using (3.49), (3.50), (3.59) and (3.60) in (3.38), we get

$$a_2 = a_4 = a_5 = a_7 = 0. \quad (3.61)$$

Using (3.49), (3.53), (3.59) and (3.60) in (3.41), we get

$$a_1 = a_6 = 0. \quad (3.62)$$

Using (3.52), (3.59), (3.60), (3.61) and (3.62) in (3.50), we get

$$\begin{aligned} F_1 = \alpha_1 + \frac{e^{-t}}{2} & \left((1+2t)(a_3 z x + a_8 x) + (b_1 x^2 + b_4 x^2 y \right. \\ & \left. + b_6 x^2 z + b_7 x^2 y z) + 2(b_2 x y + b_3 x y + b_5 x y z + \gamma_1 x) \right), \end{aligned} \quad (3.63)$$

where $\alpha_1 \in \mathbb{R}$.

Using (3.55), (3.59), (3.60), (3.61) and (3.62) in (3.53), we get

$$\begin{aligned} F_2 = \alpha_2 + \frac{e^{-t}}{2} & \left((1+2t)(a_3 z y + a_8 y) + (b_2 y^2 + b_4 x y^2 \right. \\ & \left. + b_5 y^2 z + b_7 x y^2 z) + 2(b_1 x y + b_3 z y + b_6 x y z + \gamma_1 y) \right), \end{aligned} \quad (3.64)$$

where $\alpha_2 \in \mathbb{R}$.

Using (3.58), (3.59), (3.60), (3.61) and (3.62) in (3.56), we get

$$F_3 = \alpha_3 + \frac{e^{2t}}{4} \left((1-4t)(a_3z^2 + 2a_8z) - 4(b_3z^2 + b_5yz^2 + b_6xz^2 + b_7xyz^2) - 8(b_1xz + b_2yz + b_4xyz + \gamma_1z) \right), \quad (3.65)$$

where $\alpha_3 \in \mathbb{R}$.

Using (3.63) in (3.40), we get

$$\alpha_1 = a_3 = a_8 = 0. \quad (3.66)$$

Using (3.65) in (3.47), we get

$$b_5 = b_2 = b_4 = b_7 = 0. \quad (3.67)$$

Using (3.65) in (3.45), we get

$$\alpha_3 = 0. \quad (3.68)$$

Using (3.65) in (3.46), we get

$$b_1 = b_6 = 0. \quad (3.69)$$

Using (3.64) in (3.43) and (3.44), we get

$$b_3 = \alpha_2 = 0. \quad (3.70)$$

Using (3.61) $\sim$ (3.62) and (3.66) $\sim$ (3.70) in (3.59) $\sim$ (3.60), we find

$$h_1(x, y, z) = 0, \quad (3.71)$$

$$h_2(x, y, z) = \gamma_1. \quad (3.72)$$

Now, using (3.71) $\sim$ (3.72) in (3.49) and (3.66) $\sim$ (3.70) in (3.63) $\sim$ (3.65), we obtain (1.1). Thus the proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. Let $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

Using (2.15) and (2.17), we compute the Lie derivative of the Ricci tensor Ric with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\begin{cases} (L_X Ric)(e_1, e_1) = 0, (L_X Ric)(e_1, e_2) = 0, \\ (L_X Ric)(e_1, e_3) = 0, (L_X Ric)(e_1, e_4) = -6e^t \partial_x F_4, \\ (L_X Ric)(e_2, e_2) = 0, (L_X Ric)(e_2, e_3) = 0, \\ (L_X Ric)(e_2, e_4) = -6e^t \partial_y F_4, (L_X Ric)(e_3, e_3) = 0, \\ (L_X Ric)(e_3, e_4) = -6e^{-2t} \partial_z F_4, (L_X Ric)(e_4, e_4) = -12\partial_t F_4. \end{cases} \quad (3.73)$$

These relations (3.73) give the necessary and sufficient conditions for X to be a Ricci collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.2 yields the stated form of X given in (1.2).

Now, to prove Theorem 1.2(b), we compute the Lie derivative of the curvature tensor R in order to determine the curvature collineations. Using (2.15) and (2.16), we obtain the following equations:

$$(L_X R)(e_1, e_2, e_1) = \left(-e^t(\partial_y F_1 + \partial_x F_2) \right) e_1 - 3 \left(e^t \partial_y F_3 \right) e_3 - 2 \left(F_4 - e^t \partial_x F_1 \right) e_2, \quad (3.74)$$

$$(L_X R)(e_1, e_2, e_2) = \left(2(F_4 - e^t \partial_y F_2) \right) e_1 + 3 \left(e^t \partial_x F_3 \right) e_3 + \left(e^t(\partial_y F_1 + \partial_x F_2) \right) e_2, \quad (3.75)$$

$$(L_X R)(e_1, e_2, e_3) = \left(2e^t \partial_y F_3 - e^{-2t} \partial_z F_2 \right) e_1 + \left(e^{-2t} \partial_z F_1 - 2e^t \partial_x F_3 \right) e_2, \quad (3.76)$$

$$(L_X R)(e_1, e_3, e_1) = \left(2(e^{-2t} \partial_z F_1 + e^t \partial_x F_3) \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 + 4 \left(F_4 - e^t \partial_x F_1 \right) e_3 + 3 \left(e^{-2t} \partial_z F_4 \right) e_4, \quad (3.77)$$

$$(L_X R)(e_1, e_3, e_2) = \left(2e^t \partial_y F_3 - e^{-2t} \partial_z F_2 \right) e_1 - 2e^t \left(\partial_y F_1 + \partial_x F_2 \right) e_3, \quad (3.78)$$

$$(L_X R)(e_1, e_3, e_3) = 4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_1 - 6 \left(e^t \partial_x F_4 \right) e_4 - 2 \left(e^t \partial_x F_3 - e^{-2t} \partial_z F_1 \right) e_3, \quad (3.79)$$

$$(L_X R)(e_2, e_3, e_3) = 4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_2 - 6 \left(e^t \partial_y F_4 \right) e_4 - 2 \left(e^{-2t} \partial_z F_2 - e^t \partial_y F_3 \right) e_3, \quad (3.80)$$

$$(L_X R)(e_2, e_4, e_2) = - \left(\partial_t F_2 + F_2 + e^t \partial_y F_4 \right) e_2 - 3 \left(\partial_t F_3 - 2F_3 \right) e_3 - 2 \left(F_4 - e^t \partial_y F_2 \right) e_4, \quad (3.81)$$

$$(L_X R)(e_2, e_4, e_4) = -2 \left(\partial_t F_4 \right) e_2 - 3e^t \left(\partial_y F_3 \right) e_3 + \left(\partial_t F_2 + F_2 + e^t \partial_y F_4 \right) e_4, \quad (3.82)$$

$$(L_X R)(e_3, e_4, e_1) = 2 \left(\partial_t F_1 + F_1 - 2e^t \partial_x F_4 \right) e_3 + \left(4e^t \partial_x F_3 + e^{-2t} \partial_z F_1 \right) e_4, \quad (3.83)$$

$$(L_X R)(e_3, e_4, e_3) = -6 \left(\partial_t F_1 + F_1 \right) e_1 - 4 \left(\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4 \right) e_3 - 6 \left(F_2 + \partial_t F_2 \right) e_2 + \left(8(e^{-2t} \partial_z F_3 + 2F_4) \right) e_4, \quad (3.84)$$

$$(L_X R)(e_3, e_4, e_4) = 3 \left(e^{-2t} \partial_z F_1 \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 - 8 \left(\partial_t F_4 \right) e_3 + 4 \left(\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4 \right) e_4. \quad (3.85)$$

To find curvature collineation vector fields F , we solve above system of partial differential equations by setting all the coefficients of the Lie derivative to zero. Therefore, comparing the components of e_4 on both sides in (3.77), (3.79), (3.80), (3.82) and (3.84), we get

$$3e^{-2t}\partial_z F_4 = 0, \quad (3.86)$$

$$-6e^t\partial_x F_4 = 0, \quad (3.87)$$

$$-6e^t\partial_y F_4 = 0, \quad (3.88)$$

$$\partial_t F_2 + F_2 + e^t\partial_y F_4 = 0, \quad (3.89)$$

and

$$8(e^{-2t}\partial_z F_3 + 2F_4) = 0, \quad (3.90)$$

respectively.

Similarly, comparing the components of e_1 on both sides in (3.74), (3.75) and (3.78), we get

$$-e^t(\partial_y F_1 + \partial_x F_2) = 0, \quad (3.91)$$

$$2(F_4 - e^t\partial_y F_2) = 0, \quad (3.92)$$

and

$$2e^t\partial_y F_3 - e^{-2t}\partial_z F_2 = 0, \quad (3.93)$$

respectively.

Similarly, comparing the components of e_2 on both sides in (3.74) and (3.76), we get

$$-2(F_4 - e^t\partial_x F_1) = 0, \quad (3.94)$$

and

$$e^{-2t}\partial_z F_1 - 2e^t\partial_x F_3 = 0, \quad (3.95)$$

respectively.

Similarly, comparing the components of e_3 on both sides in (3.81), (3.83) and (3.85), we get

$$-3(\partial_t F_3 - 2F_3) = 0, \quad (3.96)$$

$$2(\partial_t F_1 + F_1 - 2e^t\partial_x F_4) = 0, \quad (3.97)$$

and

$$-8\partial_t F_4 = 0, \quad (3.98)$$

respectively.

On solving (3.86), (3.87), (3.88) and (3.98), we get

$$F_4 = a_3, \quad a_3 \in \mathbb{R}. \quad (3.99)$$

Using (3.99) in (3.94), we get

$$F_1 = a_3 x e^{-t} + h_3(y, z, t), \quad (3.100)$$

where $h_3(y, z, t)$ is real valued smooth function.

Using (3.99) in (3.92), we get

$$F_2 = a_3 y e^{-t} + h_4(x, z, t), \quad (3.101)$$

where $h_4(x, z, t)$ is real valued smooth function.

Using (3.99) in (3.90), we get

$$F_3 = -2a_3 z e^{2t} + h_5(x, y, t), \quad (3.102)$$

where $h_5(x, y, t)$ is real valued smooth function.

Using (3.100) in (3.97), we get

$$h_3(y, z, t) = h_6(y, z) e^{-t}, \quad (3.103)$$

where $h_6(y, z)$ is real valued smooth function.

Using (3.101) in (3.89), we get

$$h_4(x, z, t) = h_7(x, z) e^{-t}. \quad (3.104)$$

where $h_7(x, z)$ is real valued smooth function.

Using (3.102) in (3.96), we get

$$h_5(x, z, t) = h_8(x, y) e^{2t}, \quad (3.105)$$

where $h_8(x, y)$ is real valued smooth function.

Using (3.100), (3.101), (3.103) and (3.104) in (3.91) we get

$$h_6(y, z) = b_2 y + h_9(z), \quad (3.106)$$

where $h_9(z)$ is real valued smooth functions.

and

$$h_7(x, z) = -b_2 x + h_{10}(z), \quad (3.107)$$

where $h_{10}(z)$ is real valued smooth functions.

Using (3.100), (3.102), (3.103), (3.105) and (3.106) in (3.95), we get

$$h_9(z) = c_4, \quad c_4 \in \mathbb{R}, \quad (3.108)$$

$$\partial_x h_8(x, y) = 0. \quad (3.109)$$

Using (3.101), (3.102), (3.104), (3.105) and (3.107) in (3.93), we get

$$h_{10}(z) = c_5, \quad c_5 \in \mathbb{R}, \quad (3.110)$$

$$\partial_y h_8(x, y) = 0. \quad (3.111)$$

On solving (3.109) and (3.111), we get

$$h_8(x, y) = c_3, \quad c_3 \in \mathbb{R}. \quad (3.112)$$

Using (3.106) and (3.108) in (3.103), we get

$$h_3(y, z, t) = (b_2 y + c_4)e^{-t}, \quad \text{where } c_4 \in \mathbb{R}. \quad (3.113)$$

Using (3.107) and (3.110) in (3.104), we get

$$h_4(x, z, t) = (-b_2 x + c_5)e^{-t}, \quad \text{where } c_5 \in \mathbb{R}. \quad (3.114)$$

From (3.105) and (3.112), we get

$$h_5(x, y, t) = c_3 e^{2t}, \quad \text{where } c_4 \in \mathbb{R}. \quad (3.115)$$

Using (3.99) and (3.113) $\sim$ (3.115) in (3.100) $\sim$ (3.102), we obtain (1.3). Thus the proof of Theorem 1.2 is complete. $\square$

Proof of Theorem 1.3. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 . The Energy momentum tensor $T = Ric - (\frac{\tau}{2})g$ is:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Using (2.15), we compute the Lie derivative of the energy-momentum tensor T with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\left\{ \begin{array}{l} (L_X T)(e_1, e_1) = -6[F_4 - e^t \partial_x F_1], \\ (L_X T)(e_1, e_2) = 3e^t [\partial_x F_2 + \partial_y F_1], \\ (L_X T)(e_1, e_3) = 3[e^{-2t} \partial_z F_1 + e^t \partial_x F_3], \\ (L_X T)(e_1, e_4) = 3[\partial_t F_1 + F_1 - e^t \partial_x F_4], \\ (L_X T)(e_2, e_2) = -6[F_4 - e^t \partial_y F_2], \\ (L_X T)(e_2, e_3) = 3[e^t \partial_y F_3 + e^{-2t} \partial_z F_2], \\ (L_X T)(e_2, e_4) = 3[\partial_t F_2 + F_2 - e^t \partial_y F_4], \\ (L_X T)(e_3, e_3) = 6[e^{-2t} \partial_z F_3 + 2F_4], \\ (L_X T)(e_3, e_4) = 3[\partial_t F_3 - 2F_3 - e^{-2t} \partial_z F_4], \\ (L_X T)(e_4, e_4) = -6\partial_t F_4. \end{array} \right. \quad (3.116)$$

These relations (3.116) give the necessary and sufficient conditions for X to be a matter collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.3 yields the stated form of X given in (1.4). $\square$

Proof of Theorem 1.4. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

We compute the Lie derivative of the Levi-Civita connection ∇ in order to determine the affine vector fields. Using (2.15) and (2.21), we obtain the following equations:

$$\begin{aligned} (L_X \nabla)(e_1, e_1) &= \left(-(\partial_t F_1 + F_1) - 2e^t \partial_x F_4 + e^{2t} \partial_x^2 F_1 \right) e_1 \\ &+ \left(-(F_2 + \partial_t F_2) + e^{2t} \partial_x^2 F_2 \right) e_2 + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 \right) e_3 \\ &+ \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 \right) e_4, \end{aligned} \quad (3.117)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_2) &= \left(-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 \right) e_1 \\ &+ \left(-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 \right) e_2 + \left(e^{2t} \partial_x \partial_y F_3 \right) e_3 \\ &+ \left(e^t (\partial_y F_1 + \partial_x F_2) + e^{2t} \partial_x \partial_y F_4 \right) e_4, \end{aligned} \quad (3.118)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_4) &= \left(-\partial_t F_4 + e^t (\partial_x \partial_t F_1 + \partial_x F_1) \right) e_1 \\ &+ \left(e^t (\partial_x \partial_t F_3 + \partial_x F_3) \right) e_3 + \left(e^t (\partial_x \partial_t F_2 + \partial_x F_2) \right) e_2 \\ &+ \left(F_1 + \partial_t F_1 + e^t (\partial_x \partial_t F_4 + \partial_x F_4) \right) e_4, \end{aligned} \quad (3.119)$$

$$\begin{aligned} (L_X \nabla)(e_2, e_2) &= \left(-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 \right) e_1 + \left(-(\partial_t F_3 - 2F_3) \right. \\ &\quad \left. + e^{2t} \partial_y^2 F_3 \right) e_3 + \left(-(F_2 + \partial_t F_2) - 2e^t \partial_y F_4 + e^{2t} \partial_y^2 F_2 \right) e_2 \\ &\quad - \left((\partial_t F_4 + 2F_4) - 2e^t \partial_y F_2 - e^{2t} \partial_y^2 F_4 \right) e_4, \end{aligned} \quad (3.120)$$

$$\begin{aligned} (L_X \nabla)(e_2, e_4) &= \left(e^t (\partial_y F_1 + \partial_y \partial_t F_1) \right) e_1 + \left(e^t (\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 \right) e_2 \\ &+ \left(e^t (\partial_y F_3 + \partial_y \partial_t F_3) \right) e_3 + \left(e^t (\partial_y F_4 + \partial_y \partial_t F_4) + F_2 + \partial_t F_2 \right) e_4, \end{aligned} \quad (3.121)$$

$$\begin{aligned} (L_X \nabla)(e_3, e_3) &= \left(-2(F_1 + \partial_t F_1) + e^{-4t} \partial_z^2 F_1 \right) e_1 + \left(-2(F_2 + \partial_t F_2) \right. \\ &\quad \left. + e^{-4t} \partial_z^2 F_2 \right) e_2 + \left(-2\partial_t F_4 + 4(e^{-2t} \partial_z F_3 + 2F_4) \right. \\ &\quad \left. + e^{-4t} \partial_z^2 F_4 \right) e_4 + \left(-2(\partial_t F_3 - 2F_3) + e^{-4t} \partial_z^2 F_3 + 4e^{-2t} \partial_z F_4 \right) e_3, \end{aligned} \quad (3.122)$$

$$\begin{aligned} (L_X \nabla)(e_3, e_4) &= \left(e^{-2t} (\partial_z \partial_t F_1 - 2\partial_z F_1) \right) e_1 + \left(e^{-2t} (\partial_z \partial_t F_2 - 2\partial_z F_2) \right) e_2 \\ &\quad + \left(e^{-2t} (\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 \right) e_3 \\ &\quad + \left(e^{-2t} (\partial_z \partial_t F_4 - 2\partial_z F_4) + 2(\partial_t F_3 - 2F_3) \right) e_4, \end{aligned} \quad (3.123)$$

$$\begin{aligned} (L_X \nabla)(e_4, e_4) &= \left(\partial_t^2 F_1 - F_1 \right) e_1 + \left(\partial_t^2 F_2 - F_2 \right) e_2 \\ &\quad + \left(\partial_t^2 F_4 \right) e_4 + \left(\partial_t^2 F_3 - 4F_3 \right) e_3. \end{aligned} \quad (3.124)$$

Solving the above system of partial differential equations by setting all coefficients of the Lie derivative to zero, we find that the affine vector fields are given by (1.5). Thus the proof of Theorem 1.4 is complete. $\square$

Proof of Theorem 1.5. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

Using (2.12) and (2.15), we compute the Lie derivative of the Ricci tensor Ric with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\begin{cases} (L_X Ric)(e_1, e_1) = 0, (L_X Ric)(e_1, e_2) = 0, \\ (L_X Ric)(e_1, e_3) = 0, (L_X Ric)(e_1, e_4) = -6e^t \partial_x F_4, \\ (L_X Ric)(e_2, e_2) = 0, (L_X Ric)(e_2, e_3) = 0, \\ (L_X Ric)(e_2, e_4) = -6e^t \partial_y F_4, (L_X Ric)(e_3, e_3) = 0, \\ (L_X Ric)(e_3, e_4) = -6e^{-2t} \partial_z F_4, (L_X Ric)(e_4, e_4) = -12\partial_t F_4. \end{cases} \quad (3.125)$$

These relations (3.125) give the necessary and sufficient conditions for X to be a Ricci collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.5 yields the stated form of X given in (1.6).

Now, to prove Theorem 1.5(b), we compute the Lie derivative of the curvature tensor R in order to determine the curvature collineations. Using (2.15) and (2.23), we obtain the following equations:

$$\begin{aligned} (L_X R)(e_1, e_2, e_1) &= \left(-e^t (\partial_y F_1 + \partial_x F_2) \right) e_1 - 3 \left(e^t \partial_y F_3 \right) e_3 \\ &\quad - 2 \left(F_4 - e^t \partial_x F_1 \right) e_2, \end{aligned} \quad (3.126)$$

$$\begin{aligned} (L_X R)(e_1, e_2, e_2) &= \left(2(F_4 - e^t \partial_y F_2) \right) e_1 + 3 \left(e^t \partial_x F_3 \right) e_3 \\ &\quad + \left(e^t (\partial_y F_1 + \partial_x F_2) \right) e_2, \end{aligned} \quad (3.127)$$

$$(L_X R)(e_1, e_2, e_3) = - \left(2e^t \partial_y F_3 + e^{-2t} \partial_z F_2 \right) e_1 + \left(e^{-2t} \partial_z F_1 + 2e^t \partial_x F_3 \right) e_2, \quad (3.128)$$

$$\begin{aligned} (L_X R)(e_1, e_3, e_1) &= \left(2(e^{-2t} \partial_z F_1 - e^t \partial_x F_3) \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 \\ &\quad + 4 \left(F_4 - e^t \partial_x F_1 \right) e_3 + 3 \left(e^{-2t} \partial_z F_4 \right) e_4, \end{aligned} \quad (3.129)$$

$$(L_X R)(e_1, e_3, e_2) = - \left(2e^t \partial_y F_3 + e^{-2t} \partial_z F_2 \right) e_1 - 2e^t \left(\partial_y F_1 + \partial_x F_2 \right) e_3, \quad (3.130)$$

$$\begin{aligned} (L_X R)(e_1, e_3, e_3) &= -4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_1 + 6 \left(e^t \partial_x F_4 \right) e_4 \\ &\quad + 2 \left(e^t \partial_x F_3 - e^{-2t} \partial_z F_1 \right) e_3, \end{aligned} \quad (3.131)$$

$$(L_X R)(e_2, e_3, e_3) = -4(e^{-2t} \partial_z F_3 + 2F_4)e_2 + 6(e^t \partial_y F_4)e_4 - 2(e^{-2t} \partial_z F_2 - e^t \partial_y F_3)e_3, \quad (3.132)$$

$$(L_X R)(e_2, e_4, e_2) = -(\partial_t F_2 + F_2 + e^t \partial_y F_4)e_2 - 3(\partial_t F_3 - 2F_3)e_3 - 2(F_4 - e^t \partial_y F_2)e_4, \quad (3.133)$$

$$(L_X R)(e_2, e_4, e_4) = -2(\partial_t F_4)e_2 - 3e^t(\partial_y F_3)e_3 + (\partial_t F_2 + F_2 + e^t \partial_y F_4)e_4, \quad (3.134)$$

$$(L_X R)(e_3, e_4, e_1) = 2(\partial_t F_1 + F_1 - 2e^t \partial_x F_4)e_3 + (-4e^t \partial_x F_3 + e^{-2t} \partial_z F_1)e_4, \quad (3.135)$$

$$(L_X R)(e_3, e_4, e_3) = 6(\partial_t F_1 + F_1)e_1 + 4(\partial_t F_3 - 2F_3 - e^{-2t} \partial_z F_4)e_3 + 6(F_2 + \partial_t F_2)e_2 - (8(e^{-2t} \partial_z F_3 + 2F_4))e_4, \quad (3.136)$$

$$(L_X R)(e_3, e_4, e_4) = 3(e^{-2t} \partial_z F_1)e_1 + 3(e^{-2t} \partial_z F_2)e_2 - 8(\partial_t F_4)e_3 + 4(-(\partial_t F_3 - 2F_3) + e^{-2t} \partial_z F_4)e_4. \quad (3.137)$$

Solving the above system of partial differential equations by setting all coefficients of the Lie derivative to zero, we find that the curvature collineation are given by (1.7). Thus the proof of Theorem 1.5 is complete. $\square$

Proof of Theorem 1.6. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 . The Energy momentum tensor $T = Ric - (\frac{\tau}{2})g$ is:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Using (2.15), we compute the Lie derivative of the energy-momentum tensor T with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\left\{ \begin{array}{l} (L_X T)(e_1, e_1) = -6[F_4 - e^t \partial_x F_1], \\ (L_X T)(e_1, e_2) = 3e^t [\partial_x F_2 + \partial_y F_1], \\ (L_X T)(e_1, e_3) = 3[e^{-2t} \partial_z F_1 - e^t \partial_x F_3], \\ (L_X T)(e_1, e_4) = 3[\partial_t F_1 + F_1 - e^t \partial_x F_4], \\ (L_X T)(e_2, e_2) = -6[F_4 - e^t \partial_y F_2], \\ (L_X T)(e_2, e_3) = 3[-e^t \partial_y F_3 + e^{-2t} \partial_z F_2], \\ (L_X T)(e_2, e_4) = 3[\partial_t F_2 + F_2 - e^t \partial_y F_4], \\ (L_X T)(e_3, e_3) = -6[e^{-2t} \partial_z F_3 + 2F_4], \\ (L_X T)(e_3, e_4) = -3[\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4], \\ (L_X T)(e_4, e_4) = -6\partial_t F_4. \end{array} \right. \quad (3.138)$$

These relations (3.138) give the necessary and sufficient conditions for X to be a matter collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.6 yields the stated form of X given in (1.8). $\square$

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

Conflict of interest. The authors declare no potential conflict of interests.

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