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ON A GENERALIZED SUBCLASS OF MEROMORPHIC p-VALENT CLOSE TO CONVEX FUNCTIONS IN q-ANALOGUE.

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Abstract. In this article, we define a new subclass of meromorphic multivalent close to convex functions involving in q-calculus associated with janowski functions. We investigate some useful geometric properties such as sufficiency criteria, distortion problem, growth theorem, radii of starlikeness and convexity and coefficient estimates for this class.

1. Introduction

The q-calculus has motivated the researchers in the recent past due to its numerous physical and mathematical applications. The generalization of derivative and integral in q-calculus which are known as q-analogue of derivative and integral were introduced and studied by Jackson [11, 12]. Aral and Gupta [5, 6] used some what similar concept and defined q-Baskakov Durrmeyer operator by using q-beta function. Similarly the author's in [3, 7] generalized some complex operators, which are known as q-Picard and q-Gauss-Weierstrass singular integral operators.Later, Srivastava and Bansal [20, pp. 62] used the q-analogue of derivative in Geometric function theory by introducing the q-generalization of starlike functions for the first time, see also [19, pp. 347 *et seq.*].

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In 2014, the q-analogue of Ruscheweyh operator were studied by Kanas and Răducanu [14], and they investigated some of its properties as will. The applications of this differential operator were further studied by Mohammed and Darus [2] and Mahmood and Sokół [15]. In this article we introduce a subclass of meromorphic multivalent functions in association with janowski functions and studty its geometric properties like sufficiency criteria, inclusion property, coefficient bounds, radii problem and distortion theorem.

2. Preliminaries and Definitions

Let $\mathfrak{A}_p$ denote the class of all meromorphic multivalent functions $f(z)$ that are analytic in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and satisfying the normalization

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \;\; (z \in \mathbb{D}). \tag{2.1}$$

The q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \;\; (z \neq 0), \tag{2.2}$$

where $0 < q < 1$. It can easily be seen that for $n \in \mathbb{N}$ and $z \in \mathbb{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] \, a_n z^{n-1}, \tag{2.3}$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n} q^l, \quad [0, q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, \;\; n = 0, \\ [1, q]\,[2, q]\,[3, q] \cdots [n, q], \;\; n \in \mathbb{N}. \end{cases}$$

The Subordination concept has been utilized in the introduction of our new class which can be defined as

Definition 2.1. *If $h_1(z)$ and $h_2(z)$ are two functions both analytic in E, then $h_1(z) \prec h_2(z)$, and we say that $h_1(z)$ is subordinated to $h_2(z)$, while there is an analytic function $w(z)$ which is known as Schwarz function and satisfy the conditions $|w(z)| < 1$ and $w(0) = 0$ $(z \in E)$, imply that $h_1(z) = h_2(w(z))$. Especially, for a univalent function $h_2(z)$ this subordination is equivalent to $h_1(E) \subseteq h_2(E)$ and $h_1(0) = h_2(0)$.*

Motivated from the work discussed above and studied in [8, 10, 13, 17, 18, 21, 23], we now define a new subclass $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ of $\mathfrak{A}$ as follows;

Definition 2.2. *Let* $-1 \leq B < A \leq 1$ *and* $0 < q < 1$. *Then a function* $f \in \mathfrak{A}$ *is in the class* $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, *if it satisfies*

$$-\frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z)\, g(tz)} \prec \frac{p + [pB + (p-\alpha)(A-B)]z}{p(1+Bz)}. \tag{2.4}$$

where $g(z)$ *is in the class* $\mathcal{MS}_p^*(1/2)$.

$$F_\delta(z) = \frac{(1-\delta)[p,q]f(z) - \delta z \partial_q f(z)}{[p,q]}$$

and the notation "$\prec$*" denotes the familiar subordinations.*

We note that

(1) For $A = 1$, $B = -1, \delta = 0$ and $q \to 1^-$ we get $\mathcal{MK}_p(\alpha)$ the class of meromorphic multivalent close to convex functions order α.
(2) For $A = 1$, $B = -1, \delta = 0, \alpha = 0$ and $q \to 1^-$ we get $\mathcal{MK}_p$ the class of meromorphic multivalent close to convex functions.
(3) For $A = 1, B = -1, \delta = 0, p = 1$ and $q \to 1^-$ we get $\mathcal{MK}$ the class of meromorphic close to convex functions of order α.

Equivalently a function $f(z) \in \mathfrak{A}$ is in the class $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if and only if

$$\left| \frac{\frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z) g(tz)} + 1}{B + (1-\frac{\alpha}{p})(A-B) + B\frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z) g(tz)}} \right| < 1. \tag{2.5}$$

For our main reults we will need the following.

Lemma 2.1. [22] *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \prec k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$$

in $\mathbb{D}$. *If* $k(z)$ *is univalent in* $\mathbb{D}$ *and* $k(\mathbb{D})$ *is convex, then*

$$|d_n| \leq |k_1|, \ \textit{for } n \geq 1.$$

Theorem 2.1. [4] *Let* $g_i(z) \in \mathcal{MS}_p^*(\alpha_i)$ *with* $i = 1, 2$. *Then*

$$t_1^p t_2^p z^p g_1(t_1 z) g_2(t_2 z) \in \mathcal{MS}_p^*(\gamma),$$

where $\gamma = \alpha_1 + \alpha_2 - 1$ *and* $0 < |t_i| \leq 1$.

Now for $t_1 = 1, t_2 = t$ and $g_1(z) = g_2(z) = g(z)$ we get

Corollary 2.1. *If* $g(z) \in \mathcal{MS}_p^*(1/2)$ *then* $G(z) = t^p z^p g(z) g(tz) \in \mathcal{MS}_p^*(0) = \mathcal{MS}_p^*$.

3. Main Results

In this Section we start with sufficiency criteria for this class in the following theorem.

Theorem 3.1. *Let $f \in \mathfrak{A}$ be of the form (2.1). Then the function $f \in \mathcal{MK}_{p,q}(\alpha,\delta,A,B)$, if and only if the following inequality holds*

$$\sum_{n=1}^{\infty}\left(\left(\tfrac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)(1+B)[n+p,q]\left|a_{n+p}\right|+(1+B\right.$$

$$\left.+(1-\tfrac{\alpha}{p})(A-B)\tfrac{2p[p,q]}{p+n}\right)\leq \quad (1-\tfrac{\alpha}{p})(A-B)[p,q]. \tag{3.6}$$

Proof. Let us suppose that the first inequality (3.6) holds. Then to show that $f \in \mathcal{MK}_{p,q}(\alpha,\delta,A,B)$, we only need to prove the inequality (2.5). For this consider

$$\left|\frac{\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}+1}{B+(1-\frac{\alpha}{p})(A-B)+B\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}}\right| \;=\; \left|\frac{z\partial_q F_\delta(z)+[p,q]G(z)}{\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]G(z)+Bz\partial_q F_\delta(z)}\right|.$$

Now with the help of (2.2), (2.3), (2.1) and

$$G(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}b_{k+p}z^{k+p},\quad (z\in\mathbb{D}), \tag{3.7}$$

we have

$$= \left|\frac{-\frac{[p,q]}{z^p}+\sum_{n=1}^{\infty}\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}z^{n+p}+\frac{[p,q]}{z^p}+[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}}{\left(B+(1-\frac{\alpha}{p})(A-B)\right)\left(\frac{[p,q]}{z^p}+[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\right)+B\left(-\frac{[p,q]}{z^p}+\sum_{n=1}^{\infty}\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}z^{n+p}\right)}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+[p,q]b_{n+p}\right)z^{n+p}}{\frac{(1-\frac{\alpha}{p})(A-B)[p,q]}{z^p}+\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]b_{n+p}\right)z^{n+p}}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+[p,q]b_{n+p}\right)z^{n+2p}}{(1-\frac{\alpha}{p})(A-B)[p,q]+\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]b_{n+p}\right)z^{n+2p}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]|a_{n+p}|+[p,q]|b_{n+p}|\right)}{(1-\frac{\alpha}{p})(A-B)[p,q]-\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]|a_{n+p}|+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]|b_{n+p}|\right)}$$

As $g(z)\in\mathcal{MS}_p^*(1/2)$ then by corrolary 2.1 $G(z)$ is in the class $\mathcal{MS}_p^*$ with representation (3.7) then by [24]

$$|b_{p+n}|\leq\frac{2p}{p+n} \tag{3.8}$$

we get

$$\leq \frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]|a_{n+p}|+\frac{2p[p,q]}{p+n}\right)}{(1-\frac{\alpha}{p})(A-B)[p,q]-\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]|a_{n+p}|+\left(B+(1-\frac{\alpha}{p})(A-B)\right)\frac{2p[p,q]}{p+n}\right)}$$

$$<1$$

where we have used the inequality (3.6) and this completes the direct part.

Conversely, let $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ be given by (2.1). Then from (2.5), we have for $z \in \mathbb{D}$,

$$\left|\frac{\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}+1}{B+(1-\frac{\alpha}{p})(A-B)+B\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+[p,q]b_{n+p}\right)z^{n+2p}}{(1-\frac{\alpha}{p})(A-B)[p,q]+\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]b_{n+p}\right)z^{n+2p}}\right|$$

Since $|\mathfrak{Re}z| \leq |z|$, we have

$$\mathfrak{Re}\left\{\frac{\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+[p,q]b_{n+p}\right)z^{n+2p}}{(1-\frac{\alpha}{p})(A-B)[p,q]+\sum_{n=1}^{\infty}\left(B\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)[n+p,q]a_{n+p}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]b_{n+p}\right)z^{n+2p}}\right\}$$

$$< 1 \tag{3.9}$$

Now choose values of z on the real axis so that $\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}$ is real. Upon clearing the denominator in (3.9) and letting $z \to 1^-$ through real values, we obtain (3.6).

Taking $q \to 1^-$ we get the result.

Corollary 3.1. [4] *Let $f \in \mathfrak{A}$ be of the form* (2.1). *Then the function $f \in \lim_{q\to 1^-} \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if and only if the following inequality holds*

$$\sum_{n=1}^{\infty}\left(\left(\frac{(1-\delta)p-\delta(p+n)}{p}\right)(1+B)(p+n)|a_{n+p}| + (1+B\right.$$
$$\left. +(1-\frac{\alpha}{p})(A-B)\frac{2p^2}{p+n}\right) \leq \quad (p-\alpha)(A-B).$$

Now we calculate the coefficients estimates for this newly defined class.

Theorem 3.2. *Let $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ and be of the form* (2.1). *Then*

$$|a_{p+n}| \quad \leq \quad \frac{[p,q]^2}{[p+n,q]((1-\delta)[p,q]-\delta[p+n,q])}\left(\frac{2p}{p+n} + 2(p-\alpha)(A-B)\sum_{i=2}^{n-1}\frac{1}{p+i}\right).$$

Proof. For $f \in \mathfrak{A}$ is in the class $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if it satisfies

$$\frac{-z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z)\, g(tz)} \prec \frac{1+[B+(1-\frac{\alpha}{p})(A-B)]z}{1+Bz}.$$

Now if

$$G(z) = t^p z^p g(z) g(tz)$$

and

$$h(z) = \frac{-z\partial_q F_\delta(z)}{[p,q]G(z)}, \tag{3.10}$$

and it will be of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

Since

$$h(z) \prec \frac{1+[B+(1-\frac{\alpha}{p})(A-B)]z}{1+Bz} = 1 + \frac{(p-\alpha)(A-B)}{p} z + \ldots.$$

Then by Lemma 2.1 we get

$$|d_n| \leq \frac{(p-\alpha)(A-B)}{p} \tag{3.11}$$

Now putting the series expansions of $h(z), G(z)$ and $f(z)$ in (3.10), simplifying and comparing the coefficients of z^{p+n} on both sides

$$-\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]^2}[p+n,q]\, a_{p+n} = b_{p+n} + b_{p+n-1}d_1 + b_{p+n-2}d_2 + \ldots + b_{p+1}d_{n-1}.$$

Taking absolute on both sides, using the triangle inequility and then using (3.11) and (3.8) we obtain

$$\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]^2}[p+n,q]\,|a_{p+n}| \leq \frac{2p}{n+p} + \frac{(p-\alpha)(A-B)}{p}\sum_{i=2}^{n-1}\frac{2p}{p+i},$$

which implies that

$$|a_{p+n}| \leq \frac{[p,q]^2}{[p+n,q]((1-\delta)[p,q]-\delta[p+n,q])}\left(\frac{2p}{p+n} + 2(p-\alpha)(A-B)\sum_{i=2}^{n-1}\frac{1}{p+i}\right).$$

where $|a_1| = 1$ and we get the desired proof.

Taking $q \to 1^-$ we get the coefficient estimates for the class which was studied by Arif et. al. [4].

Corollary 3.2. *Let $f \in \mathfrak{A}$ be of the form* (2.1), *and $f \in \lim_{q\to 1^-} \mathcal{MK}_{p,q}(\alpha,\delta,A,B)$, then*

$$|a_{p+n}| \leq \frac{p^2}{(p+n)((1-\delta)p-\delta(p+n))}\left(\frac{2p}{p+n} + 2(p-\alpha)(A-B)\sum_{i=2}^{n-1}\frac{1}{p+i}\right).$$

The next result is about the distortion theorem for this class of functions.

Theorem 3.3. *If $f \in \mathcal{MK}_{p,q}(\alpha,\delta,A,B)$ and has the form* (2.1). *Then for $|z| = r$*

$$\frac{[p,q](1-Cr)(1-r)^{p+1}}{r^{p+1}(1-Br)} \leq |\partial_q F_\delta(z)| \leq \frac{[p,q](1+Cr)(1+r)^{p+1}}{r^{p+1}(1+Br)}$$

where $C = B + (1-\frac{\alpha}{p})(A-B)$.

Proof. Suppose that $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$. Then we can write

$$\frac{-z^{1}\partial_{q}F_{\delta}(z)}{[p,q]G(z)} \prec \frac{1+Cz}{1+Bz}$$

then with $|z| = r$ and

$$\left|\frac{-z^{1}\partial_{q}F_{\delta}(z)}{[p,q]G(z)} - \frac{1-CBr^{2}}{1-B^{2}r^{2}}\right| \leq \frac{(C-B)r}{1-B^{2}r^{2}}.$$

simplification gives us

$$\frac{1-Cr}{1-Br} \leq \left|\frac{-z\partial_{q}F_{\delta}(z)}{[p,q]G(z)}\right| \leq \frac{1+Cr}{1+Br}. \tag{3.12}$$

Now since $G(z) \in \mathcal{MS}_{p}^{*}$, thus we have

$$\frac{(1-r)^{p+1}}{r^{p}} \leq |G(z)| \leq \frac{(1+r)^{p+1}}{r^{p}}. \tag{3.13}$$

Now by using (3.13) in (3.12), we obtain the required result.

In the following we give the growth theorem for this class.

Theorem 3.4. *Let $f \in \mathcal{MK}_{q}^{*}(p, \mu, A, B)$ and has the form (2.1). Then for $|z| = r$*

$$\frac{1}{r^{p}} - \tau_{1}r^{p} \leq |f(z)| \leq \frac{1}{r^{p}} + \tau_{1}r^{p},$$

where

$$\tau_{1} = \frac{[p,q]^{2}\left((p-\alpha)(A-B) - (p(1+B) + (p-\alpha)(A-B))\right)}{(p+1)(1+B)[p+1,q]\left((1-\delta)[p,q] - \delta[p+1,q]\right)}.$$

Proof. Consider

$$\begin{aligned} |f(z)| &= \left|\frac{1}{z^{p}} + \sum_{n=1}^{\infty} a_{n+p}\ z^{n+p}\right|, \\ &\leq \frac{1}{|z^{p}|} + \sum_{n=1}^{\infty} |a_{n+p}|\ \ |z|^{n+p} \\ &= \frac{1}{r^{p}} + \sum_{n=1}^{\infty} |a_{n+p}|\ \ r^{n+p} \end{aligned}$$

As $|z| = r < 1$ so $r^{n+p} < r^{p}$ and

$$|f(z)| \leq \frac{1}{r^{p}} + r^{p}\sum_{n=1}^{\infty} |a_{n+p}| \tag{3.14}$$

Similarly

$$|f(z)| \geq \frac{1}{r^{p}} - r^{p}\sum_{n=1}^{\infty} |a_{n+p}| \tag{3.15}$$

Since (3.6) implies that

$$\sum_{n=1}^{\infty}\left(\left(\tfrac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)(1+B)\,[n+p,q]\,|a_{n+p}|+\left(1+B+(1-\tfrac{\alpha}{p})(A-B)\right)\tfrac{2p[p,q]}{p+n}\right) \leq (1-\tfrac{\alpha}{p})\,(A-B)\,[p,q].$$

But

$$(p\,(1+B)+(p-\alpha)(A-B))\,\tfrac{2[p,q]}{p+1}+\tfrac{((1-\delta)[p,q]-\delta[p+n,q])[p+1,q](1+B)}{[p,q]}\sum_{n=1}^{\infty}|a_{n+p}|$$

$$\leq \sum_{n=1}^{\infty}\left(\left(\tfrac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)(1+B)\,[n+p,q]\,|a_{n+p}|+.\left(1+B+(1-\tfrac{\alpha}{p})(A-B)\right)\tfrac{2p[p,q]}{p+n}\right).$$

Hence

$$(p\,(1+B)+(p-\alpha)(A-B))\,\tfrac{2[p,q]}{p+1}+\tfrac{((1-\delta)[p,q]-\delta[p+1,q])[p+1,q](1+B)}{[p,q]}\sum_{n=1}^{\infty}|a_{n+p}|$$

$$\leq (1-\tfrac{\alpha}{p})\,(A-B)\,[p,q],$$

which gives

$$\sum_{n=1}^{\infty}|a_{n+p}| \leq \tfrac{[p,q]^2((p-\alpha)(A-B)-(p(1+B)+(p-\alpha)(A-B)))}{(p+1)(1+B)[p+1,q]((1-\delta)[p,q]-\delta[p+1,q])}$$

Now by putting this value in (3.14) and (3.15) we get the required result.

In the next two results we determine the radii of convexity and starlikeness of order σ.

Theorem 3.5. *Let $f\in\mathcal{MK}_q^*(p,\mu,A,B)$. Then $f\in\mathcal{MC}_p(\sigma)$ for $|z|<r_1$, where*

$$r_1 = \left(\tfrac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))}\right)^{\frac{1}{n+2p}}.$$

Proof. Let $f\in\mathcal{MK}_q^*(p,\mu,A,B)$. To prove $f\in\mathcal{MC}_p(\sigma)$, we only need to show

$$\left|\frac{zf''(z)+(p+1)\,f'(z)}{zf''(z)+(1+2\sigma-p)\,f'(z)}\right|<1.$$

Using (2.1) along with some simple computation yields

$$\sum_{n=1}^{\infty}\frac{(p+n)\,(n+p+\sigma)}{p\,(p-\sigma)}\,|a_{n+p}|\,|z|^{n+2p}<1. \tag{3.16}$$

From (3.6), we can easily obtain that

$$\sum_{n=1}^{\infty}\left(\tfrac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]}\right)(1+B)\,[n+p,q]\,|a_{n+p}|$$

$$\leq \tfrac{[p,q]((p-\alpha)(A-B)-2(p(1+B)+(p-\alpha)(A-B))p)}{p(p+1)}.$$

$$\Rightarrow \sum_{n=1}^{\infty}\tfrac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))}\,|a_{n+p}| < 1.$$

Now inequality (3.16) will be true, if the following holds

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p+\sigma)}{p(p-\sigma)} |a_{n+p}| |z|^{n+2p} < \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}|,$$

which implies that

$$|z|^{n+2p} < \frac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))},$$

and so

$$|z| < \left(\frac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))}\right)^{\frac{1}{n+2p}}, = r_1.$$

we get the required condition.

Theorem 3.6. *Let $f \in \mathcal{MK}_q^*(p,\mu,A,B)$. Then $f \in \mathcal{MS}_p^*(\sigma)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))}\right)^{\frac{1}{n+2p}},$$

Proof. We know that $f \in \mathcal{MS}_p^*(\sigma)$, if and only if

$$\left|\frac{zf'(z)+pf(z)}{zf'(z)-(p-2\sigma)f(z)}\right| \leq 1.$$

Using (2.1) and upon simplification yields

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\sigma}{p-\sigma}\right) |a_{n+p}| |z|^{n+2p} < 1. \tag{3.17}$$

Now from (3.6) we can easily obtain

$$\Rightarrow \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}| < 1.$$

For inequality (3.17) to be true it will be enough if

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\sigma}{p-\sigma}\right) |a_{n+p}| |z|^{n+2p} < \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}|.$$

This gives

$$|z|^{n+2p} < \frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))},$$

and hence

$$|z| < \left(\frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))}\right)^{\frac{1}{n+2p}} = r_2,$$

Thus we obtain the required result.

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