

RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS SATISFYING SEMI-SYMMETRY CURVATURE CONDITIONS

S. SUNITHA DEVI , B. V. S. T. SAI , V. KAVITHA , AND K. L. SAI PRASAD *

Abstract. The main objective of this paper is to study Ricci solitons on a class of indefinite almost paracontact metric manifolds, namely indefinite para-Kenmotsu (also termed, (ϵ) -para Kenmotsu) manifolds satisfying semi-symmetry curvature conditions. In the conclusion, we develop an example of an indefinite para-Kenmotsu manifold with dimension of 3, which confirms the results described in this paper. A comparison table with the results obtained for (ϵ) -para Sasakian manifolds is included in the final version of the results.

Keywords: Indefinite para-Kenmotsu manifolds, Curvature tensor, Ricci solitons, Shrinking, Space-like vector field.

MSC (2020): 53C15, 53C50, 53C25.

1. INTRODUCTION

A Ricci soliton, defined on a Riemannian manifold, is a natural generalization of an Einstein metric, as well as a self-similar solution to Hamilton's Ricci flow presented by Hamilton [4] in 1988. For a manifold M_n , a Ricci soliton is a triplet (g, V, λ) if there is a smooth vector field V , a Riemannian metric g , and a real scalar λ such that

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1.1)$$

where L_V is the Lie-derivative of the metric g along the vector field V and S is the Ricci tensor of M_n . The Ricci soliton is stated to be shrinking, steady, and expanding, depending on whether λ is negative, zero, or positive. If the potential vector field V disappears uniformly, the Ricci soliton becomes trivial, and the manifold is an Einstein one.

Motivated by Hamilton's findings, many geometers have conducted extensive research on the generalizations of Ricci soliton. Various classes of Ricci solitons have been investigated and studied due to their numerous applications in both physics and Riemannian geometry [1, 2, 19, 18, 20].

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* Corresponding author

S. Sunitha Devi (sunithamallakula@yahoo.com) $\diamond$ ORCID:0000-0002-0398-0556

B. V. S. T. Sai (bvstsai@vvit.net) $\diamond$ ORCID:0009-0008-2482-6399

V. Kavitha (v.kavitha@vardhaman.org) $\diamond$ ORCID:0000-0002-2354-8138

K. L. Sai Prasad (klsprasad@gvpcew.ac.in) $\diamond$ ORCID:0000-0002-8447-008X

The index of the indefinite metric, on the other hand, creates diverse vector fields, including space-like, time-like, and light-like fields, rendering the examination of indefinite structures on manifolds exceedingly significant and stimulating. With an emphasis on Sasakian manifolds, in 1969, Takahashi [15] presented pseudo-Riemannian metrics on almost contact manifolds. In 2009, De and Sarkar [3] presented a (ϵ) -Kenmotsu manifold, which is the Kenmotsu manifold [5] that admits an indefinite metric. Several geometers have extensively studied these manifolds and have produced a number of intriguing findings [9, 12, 17].

In 1976, Sato [11] defined an almost paracontact structure, which is analogous to the almost contact structure. Tripathi *et al.*, [16] presented the notion of an indefinite almost paracontact metric structure, also referred as (ϵ) -almost paracontact structure, by linking an almost paracontact structure with a semi-Riemannian metric, which need not be the Lorentzian. In this instance, $(\epsilon = 1)$ or $(\epsilon = -1)$ indicates that the vector field ξ can be either space-like or time-like. In addition, they introduced and examined the characteristics of (ϵ) -para Sasakian [16] and (ϵ) -para Sasakian 3-manifolds [8].

Analogous to these, in 2025, Sunitha and Sai Prasad [14] introduced a novel category of indefinite almost paracontact metric manifolds known as (ϵ) -para Kenmotsu manifolds, focusing specifically on the curvature characteristics of (ϵ) -para Kenmotsu 3-manifolds. These manifold structures are derived from a category of almost paracontact metric manifolds, specifically para-Kenmotsu manifolds, which were delineated in 1995 by Sinha and Sai Prasad [13]. Further, Sai Prasad along with Sunitha and Satyanarayana have recently explored the Z-symmetries of three-dimensional indefinite para-Kenmotsu manifolds [10].

Inspired by the prior study, this current paper continues the study of indefinite para-Kenmotsu manifolds. Apart from a few requirements about indefinite para-Kenmotsu manifolds in section 2, subsequently in sections 3, 4 and 5, we examine Ricci solitons within the indefinite para-Kenmotsu manifolds admitting the following semi-symmetry curvature conditions:

$$R(\xi, X).R = 0, \quad (1.2)$$

$$P(\xi, X).R = 0, \quad (1.3)$$

$$S(\xi, X).P = 0, \quad (1.4)$$

where $R(X, Y)$ and $P(X, Y)$ denote the Riemann and projective curvature tensors, respectively. In order to validate the findings presented in this study, we also develop an example of a 3-dimensional (ϵ) -para Kenmotsu manifold. At the end, the results are provided in a more comprehensive form, including a comparison table with the results obtained for (ϵ) -para Sasakian manifolds [7].

2. PRELIMINARIES

Assuming that an n -dimensional differentiable manifold M_n is capable of accepting a vector field ξ , a $(1, 1)$ tensor field ϕ , and a 1-form η that satisfy the requirements, then it is considered to have a structure that is an almost paracontact:

$$\phi^2 X = X - \eta(X) \xi, \quad (2.5)$$

$$\eta(\xi) = 1, \quad (2.6)$$

$$\eta(\phi X) = 0, \quad \phi\xi = 0. \quad (2.7)$$

By taking into account the semi-Riemannian metric $g(X, Y)$, which has an index of $g(X, Y) = \nu$, it is possible to assert that

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad (2.8)$$

for any X, Y belongs to $\chi(M_n)$, which is a group of vector fields on M_n , and (ϵ) is either 1 or -1 . Then M_n is considered as a (ϵ) -almost paracontact metric manifold endowed with (ϵ) -almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$.

According to the information provided in reference [6], the manifold M_n is regarded to be a Lorentzian manifold that is almost paracontact if the index of the function $g(X, Y)$ is one. Due to the fact that the metric $g(X, Y)$ is positive definite, the manifold M_n is regarded as almost paracontact metric manifold [11]. A clear and convincing demonstration of the assertion is provided by equation (2.8) that

$$g(\phi X, Y) = g(X, \phi Y) \text{ and} \quad (2.9)$$

$$g(X, \xi) = \epsilon \eta(X). \quad (2.10)$$

In addition, the equations (2.5), (2.6), (2.7) and (2.9), give us the following:

$$g(\xi, \xi) = \epsilon. \quad (2.11)$$

The structure of the (ϵ) -paracontact metric is also known as the (ϵ) -para Kenmotsu metric if

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \epsilon \eta(Y) \phi X. \quad (2.12)$$

Moreover, the manifold M_n that contains the structure $(\phi, \xi, \eta, g, \epsilon)$ is understood to be the (ϵ) -para Kenmotsu manifold [14]. The symbol ∇ is used to denote the Levi-Civita connection for the indefinite metric $g(X, Y)$. In a particular instance, for the Riemannian metric $g(X, Y)$ and $\epsilon = 1$, the manifold is para Kenmotsu [13]. We may change the manifold M_n into a Lorentzian para-Kenmotsu manifold by substituting $-\xi$ for ξ and $\epsilon = -1$. This manifold is characterised by the Lorentzian metric $g(X, Y)$ [6].

Among the features that the curvature tensor $R(X, Y)$ possesses in an (ϵ) -para Kenmotsu manifold are as follows [14]:

$$\nabla_X \xi = \epsilon[X - \eta(X)\xi], \quad (2.13)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.14)$$

$$R(\xi, X)Y = \epsilon\eta(Y)X - g(X, Y)\xi, \quad (2.15)$$

$$R(\xi, X)\xi = \epsilon X - \epsilon\eta(X)\xi, \quad (2.16)$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \quad (2.17)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \quad (2.18)$$

for all X, Y , and $Z \in \chi(M_n)$.

We can get the following from the equation (2.18):

$$S(X, \phi Y) = S(\phi X, Y) \text{ and} \quad (2.19)$$

$$S(X, \xi) = -(n - 1)\eta(X). \quad (2.20)$$

Consider a Ricci soliton (ξ, g, λ) in a (ϵ) -para Kenmotsu manifold. Using (2.13), we get

$$(L_{\xi}g)(X, Y) = 2 \epsilon [g(X, Y) - \epsilon \eta(X)\eta(Y)]. \quad (2.21)$$

With reference to equations (1.1) and (2.21), we have

$$S(X, Y) = -(\epsilon + \lambda) g(X, Y) + \eta(X)\eta(Y). \quad (2.22)$$

This gives

$$S(\xi, Y) = -\lambda \epsilon \eta(Y), \quad (2.23)$$

$$QX = -(\epsilon + \lambda)X + \epsilon \eta(X)\xi, \text{ and} \quad (2.24)$$

$$r = -n(\epsilon + \lambda) + \epsilon. \quad (2.25)$$

The definition of the projective curvature tensor $P(X, Y)$ is

$$P(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [S(X, Z)Y - S(Y, Z)X]. \quad (2.26)$$

Equations (2.17), (2.23), and (2.25) allow $P(X, Y)$ to reduce to

$$\begin{aligned} P(\xi, Y)Z &= \left[1 - \frac{\lambda}{n-1}\right] [\epsilon \eta(Z)Y - g(Y, Z)\xi] \\ &\quad + \left[\frac{1}{n-1}\right] [\epsilon g(Y, Z) - \eta(Y)\eta(Z)]\xi. \end{aligned} \quad (2.27)$$

Additionally, by using the equations (2.6), (2.17), (2.23), and (2.24) and adding $X = \xi$ to (2.26), we get

$$\begin{aligned} P(Y, \xi)Z &= \left[-1 + \frac{\lambda}{n-1}\right] [\epsilon \eta(Z)Y - g(Y, Z)\xi] \\ &\quad - \left[\frac{1}{n-1}\right] [\epsilon g(Y, Z) - \eta(Y)\eta(Z)]\xi. \end{aligned} \quad (2.28)$$

Using the equations (2.6), (2.17), (2.21), and (2.24), for $Z = \xi$, we can obtain from (2.26)

$$P(X, Y)\xi = \left[1 - \frac{\lambda \epsilon}{n-1}\right] [\eta(X)Y - \eta(Y)X]. \quad (2.29)$$

3. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$R(\xi, X).R = 0$$

Assume that $R(\xi, X).R = 0$. This yields

$$\begin{aligned} R(\xi, X)R(Y, Z)U - R(Y, Z)R(\xi, X)U \\ - R(Y, R(\xi, X)Z)U - R(R(\xi, X)Y, Z)U = 0. \end{aligned} \quad (3.30)$$

Using equation (2.17), the given equation can be reduced to

$$\begin{aligned} \epsilon \eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi - \epsilon \eta(U)R(Y, Z)X \\ + g(X, Y)R(\xi, Z)U - \epsilon \eta(Z)R(Y, X)U + g(X, Z)R(Y, \xi)U \\ - \epsilon \eta(Y)R(X, Z)U + g(X, U)R(Y, Z)\xi = 0. \end{aligned} \quad (3.31)$$

By applying (2.6) and (2.7) to the inner product of the previous equation with ξ , we arrive at

$$\begin{aligned} & \eta(X) \eta(R(Y, Z) U) - \epsilon g(X, R(Y, Z) U) + g(X, Y) \eta(R(\xi, Z) U) \\ & - \epsilon \eta(Y) \eta(R(X, Z) U) + g(X, Z) \eta(R(Y, \xi) U) - \epsilon \eta(Z) \eta(R(Y, X) U) \\ & + \epsilon g(X, U) \eta(R(Y, Z) \xi) - \epsilon \eta(U) \eta(R(Y, Z) X) = 0. \end{aligned} \quad (3.32)$$

Using (2.22), the equation above takes the form:

$$\begin{aligned} \epsilon g(X, R(Y, Z) U) &= (1 - \epsilon)[g(Z, U) \eta(X) \eta(Y) - g(Y, U) \eta(X) \eta(Z)] \\ &+ \epsilon[g(X, Z) g(Y, U) - g(X, Y) g(Z, U)]. \end{aligned} \quad (3.33)$$

After entering $U = X = e_i$ in the equation above and calculating the sum over i ($1 \leq i \leq n$), we obtain

$$S(Y, Z) = \eta(Y) \eta(Z) - n \epsilon g(Y, Z). \quad (3.34)$$

Using $Z = \xi$ in (3.34), we get

$$S(Y, \xi) = -(n - 1) \eta(Y). \quad (3.35)$$

Then by using (2.23), the above equation gives $\lambda > 0$, showing that the λ is expanding. Based on this, we can conclude that:

Theorem 3.1. *A Ricci soliton (g, ξ, λ) on the indefinite para-Kenmotsu manifold under the semi-symmetry condition $R(\xi, X) \cdot R = 0$ are expanding, and ξ is a vector field that is space-like.*

4. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$P(\xi, X) \cdot R = 0$$

Assume that $P(\xi, X) \cdot R = 0$. This yields

$$\begin{aligned} & P(\xi, X) R(Y, Z) W - R(Y, Z) P(\xi, X) W \\ & - R(Y, P(\xi, X) Z) W - R(P(\xi, X) Y, Z) W = 0. \end{aligned} \quad (4.36)$$

Equation (2.28) allows the equation above to be reduced to

$$\begin{aligned} & \epsilon \eta(R(Y, Z) W) X - \epsilon \eta(Z) R(Y, X) W + \frac{\epsilon \lambda}{n - 1} \eta(Y) R(X, Z) W \\ & + \frac{\epsilon + \lambda}{n - 1} g(X, R(Y, Z) W) \xi - \frac{1}{n - 1} \eta(X) \eta(R(Y, Z) W) \xi \\ & - \epsilon \eta(Y) R(X, Z) W + g(X, Y) R(\xi, Z) W - \epsilon \frac{\lambda}{n - 1} \eta(R(Y, Z) W) X \\ & - \frac{\epsilon + \lambda}{n - 1} g(X, Y) R(\xi, Z) W + \frac{1}{n - 1} \eta(X) \eta(Y) R(\xi, Z) W \\ & - \frac{\epsilon + \lambda}{n - 1} g(X, Z) R(Y, \xi) W + \frac{1}{n - 1} \eta(X) \eta(Z) R(Y, \xi) W \\ & + \frac{\epsilon \lambda}{n - 1} \eta(Z) R(Y, X) W + g(X, Z) R(Y, \xi) W - g(X, R(Y, Z) W) \xi \\ & - R(Y, Z) R(\xi, X) W - \left[1 - \frac{\lambda}{n - 1}\right] \epsilon \eta(W) R(Y, Z) X \\ & + \left[1 - \frac{\lambda + \epsilon}{n - 1}\right] g(X, W) R(Y, Z) \xi + \frac{1}{n - 1} \eta(X) \eta(W) R(Y, Z) \xi = 0. \end{aligned} \quad (4.37)$$

Taking the inner product of (4.37) with ξ and applying (2.6) and (2.7) yield

$$\begin{aligned}
 & \eta(R(Y, Z) W) \eta(X) - \epsilon g(X, R(Y, Z) W) - \eta(Y) \eta(R(X, Z) W) \\
 & + \frac{\epsilon + \lambda}{n-1} g(X, R(Y, Z) W) \epsilon - \frac{1}{n-1} \eta(X) \eta(R(Y, Z) W) \epsilon \\
 & + \epsilon g(X, Y) \eta(R(\xi, Z) W) - \frac{\lambda}{n-1} \eta(R(Y, Z) W) \eta(X) \\
 & + \frac{\lambda}{n-1} \eta(Y) \eta(R(X, Z) W) - \frac{\epsilon + \lambda}{n-1} g(X, Y) \epsilon \eta(R(\xi, Z) W) \\
 & + \frac{1}{n-1} \eta(X) \eta(Y) \epsilon \eta(R(\xi, Z) W) - \eta(Z) \eta(R(Y, X) W) \\
 & + \epsilon g(X, Z) \eta(R(Y, \xi) W) + \frac{1}{n-1} \epsilon \eta(X) \eta(Z) \eta(R(Y, \xi) W) \\
 & - \epsilon \eta(R(Y, Z) R(\xi, X) W) - \left[1 - \frac{\lambda}{n-1}\right] \eta(W) \eta(R(Y, Z) X) \\
 & + \left[1 - \frac{\epsilon + \lambda}{n-1}\right] g(X, W) \epsilon \eta(R(Y, Z) \xi) \\
 & + \frac{1}{n-1} \eta(X) \eta(W) \epsilon \eta(R(Y, Z) \xi) = 0.
 \end{aligned} \tag{4.38}$$

Equations (2.10) and (2.17) allow us to derive from the equation above that

$$\left[\frac{\lambda}{n-1} - 1\right] [g(Y, X) \eta(Z) - g(Z, X) \eta(Y)] \epsilon \eta(W) = 0, \tag{4.39}$$

which gives λ as $n-1$ as the term $[g(Y, X) \eta(Z) - g(Z, X) \eta(Y)] \epsilon \eta(W) \neq 0$. This indicates that as λ is expanding as $\lambda > 0$. We can therefore assert the following.

Theorem 4.1. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds admitting the condition $P(\xi, X) \cdot R = 0$ are expanding, and ξ is a vector field that is space-like.*

5. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$S(\xi, X) \cdot P = 0$$

Let $S(X, \xi) \cdot P(Y, Z) W = 0$. Then

$$\begin{aligned}
 & S(X, \xi) P(Y, Z) W - P(Y, Z) S(\xi, X) W \\
 & - P(Y, S(X, \xi) Z) W - P(S(X, \xi) Y, Z) W = 0.
 \end{aligned} \tag{5.40}$$

Equation (2.17) allows the equation above to be reduced to

$$\begin{aligned}
 & S(X, \xi) R(Y, Z) W - R(Y, S(X, \xi) Z) W \\
 & - R(Y, Z) S(\xi, X) W - R(S(X, \xi) Y, Z) W \\
 & - \frac{1}{n-1} S(S(X, \xi) Y, W) Z + \frac{1}{n-1} S(Z, W) S(X, \xi) Y \\
 & + \frac{1}{n-1} S(X, \xi) S(Y, W) Z - \frac{1}{n-1} S(Y, W) S(X, \xi) Z \\
 & + \frac{1}{n-1} S(S(X, \xi) Z, W) Y - \frac{1}{n-1} S(X, \xi) S(Z, W) Y \\
 & - \frac{1}{n-1} S(Y, S(\xi, X) W) Z + \frac{1}{n-1} S(Z, S(\xi, X) W) Y = 0.
 \end{aligned} \tag{5.41}$$

Taking the above equation's inner product with ξ and applying equations (2.7) and (2.10), we obtain

$$\begin{aligned}
& S(X, \xi) \eta(R(Y, Z)W) + \frac{1}{n-1} S(X, \xi) S(Z, W) \epsilon \eta(Y) \\
& - \frac{1}{n-1} S(X, \xi) S(Y, W) \epsilon \eta(Z) - \eta(R(Y, S(X, \xi)Z)W) \\
& - \eta(R(S(X, \xi)Y, Z)W) - \eta(R(Y, Z) S(\xi, X)W) \\
& - \frac{1}{n-1} S(\xi, X)S(Y, W) \epsilon \eta(Z) + \frac{1}{n-1} S(Z, W) S(\xi, X) \epsilon \eta(Y) = 0,
\end{aligned} \tag{5.42}$$

which further reduces to

$$\frac{2\epsilon}{n-1} S(X, \xi)[S(Z, W) \eta(Y) - S(Y, W) \eta(Z)] - 2S(X, \xi) \eta(R(Y, Z)W) = 0. \tag{5.43}$$

Using (2.20) and (2.22), (5.43) gives

$$-\frac{\lambda \epsilon}{n-1} [-2 - \lambda \epsilon + n] [g(Y, W) \eta(Z) - g(Z, W) \eta(Y)] \eta(X) = 0. \tag{5.44}$$

Applying the equations (2.6), (2.10), and (2.11) to the preceding equation and substituting $Z = W = \xi$, we obtain

$$\lambda \epsilon = 0 \text{ (or) } -2 - \lambda \epsilon + n = 0, \tag{5.45}$$

which gives

$$\lambda = 0 \text{ (or) } \lambda = (n-2) \epsilon. \tag{5.46}$$

Now, let us suppose that ξ be a space-like vector field in indefinite para-Kenmotsu manifolds. From the equation above, we can find that either $\lambda = 0$, (or) $\lambda > 0$ for $n > 2$. This indicates that λ is either steady or expanding. As a result, we can say this.

Theorem 5.1. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds admitting the condition $S(\xi, X).P = 0$ are either steady or expanding, and ξ is a vector field that is space-like.*

Furthermore, the vector field ξ is assumed to be a time-like on the (ϵ) -para Kenmotsu manifold. We therefore have $\lambda = 0$ (or) $\lambda < 0$ in light of (5.46). This suggests that λ is either steady or shrinking the same. Therefore, we can say that:

Theorem 5.2. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds that satisfy $S(\xi, X).P = 0$ and possess ξ as a time-like vector field are characterised as either steady or shrinking.*

6. AN EXAMPLE OF A THREE-DIMENSIONAL INDEFINITE PARA-KENMOTSU MANIFOLD

We look at a three-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3 . For each point in M_3 , let e_1, e_2 , and e_3 be a set of linearly independent vectors provided by

$$e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}. \tag{6.47}$$

The definition of the Riemannian metric g is

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j; i, j = 1, 2, 3. \end{cases}$$

The 1-form, denoted by η , is defined as:

$$g(X, e_3) = \epsilon \eta(X), \text{ for each } X \in \chi(M_3).$$

Let ϕ be a $(1, 1)$ -tensor field on M_3 with the following definition:

$$\phi^2(e_1) = e_1, \phi^2(e_2) = e_2, \phi^2(e_3) = 0; \text{ and}$$

$$\phi(e_1) = -\epsilon e_1, \phi(e_2) = -\epsilon e_2, \phi(e_3) = 0.$$

We can get this by using the linearity of ϕ and g :

$$\eta(e_3) = 1, g(\xi, \xi) = \epsilon \text{ where } \epsilon = \pm 1,$$

$$\phi^2(X) = X - \eta(X) e_3,$$

$$\text{and } g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y),$$

for all $X, Y \in \chi(M_3)$. Therefore, for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost paracontact structure on M_3 .

Now, from (6.47) we have

$$[e_1, e_3] = e_1, [e_2, e_3] = e_2, \text{ and } [e_1, e_2] = 0.$$

Koszul's formula gives us the Levi-Civita connection ∇ of the metric tensor g as follows:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X g(Y, Z) + Y g(Z, X) - Z g(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \quad (6.48)$$

Next, we obtain the following using the Koszul formula mentioned above:

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = \epsilon e_1;$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = \epsilon e_2;$$

$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

Based on the parameters listed above, the manifold under evaluation satisfies the conditions (2.12) and (2.13). This demonstrates that M_3 is an indefinite para-Kenmotsu manifold with the structure $(\phi, \xi, \eta, g, \epsilon)$.

Additionally, using (1.1) and the values mentioned before, we obtain

$$(L_\xi g)(e_1, e_1) = 2, (L_\xi g)(e_2, e_2) = 2 \text{ and } (L_\xi g)(e_3, e_3) = 0.$$

For a three-dimensional manifold, using (2.20), we get

$$S(X, \xi) = -2 \eta(X). \quad (6.49)$$

Using (2.22), we obtain

$$S(X, Y) = -(\epsilon + \lambda) \epsilon \eta(X) + \eta(X). \quad (6.50)$$

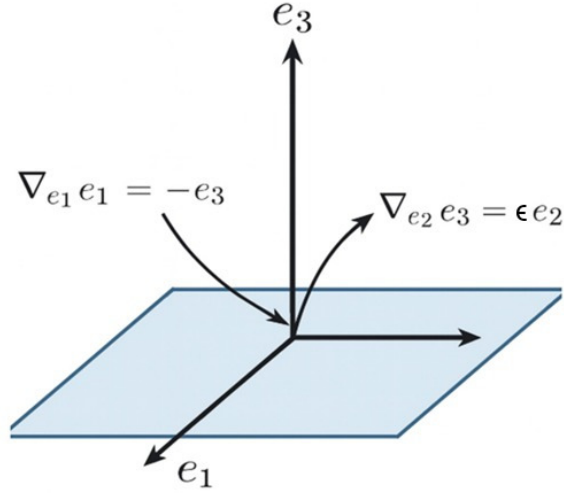


FIGURE 1. Classification of 3-Manifold with Covariant Derivatives.

From the preceding two equations, we obtain

$$\lambda = 2 \epsilon. \quad (6.51)$$

For $\epsilon = 1$, we have $\lambda = 2$. From (2.22), we have

$$S(X, Y) = -3 g(X, Y) + \eta(X) \eta(Y). \quad (6.52)$$

Similarly, for $\epsilon = -1$, we get $\lambda = -2$, and from (2.22), we get

$$S(X, Y) = 3 g(X, Y) + \eta(X) \eta(Y). \quad (6.53)$$

Hence from (2.22), we have

$$S(e_1, e_1) = -(\epsilon + \lambda) \epsilon, \quad S(e_2, e_2) = -(\epsilon + \lambda) \epsilon, \quad S(e_3, e_3) = -(\epsilon + \lambda) \epsilon + 1,$$

where $r = \sum_{i=1}^3 S(e_i, e_i) g(e_i, e_i)$. This gives

$$r = -3 (\epsilon + \lambda) + \epsilon. \quad (6.54)$$

From the above equation, we get $r = -8$, for $\epsilon = 1$ and $\lambda = 2$; and for $\epsilon = -1$ and $\lambda = -2$, we get $r = 8$.

Clearly for $\epsilon = 1$, the manifold has negative scalar curvature shows that the Ricci soliton is expanding, and the manifold has positive scalar curvature for $\epsilon = -1$. This proves that the Ricci soliton is steady.

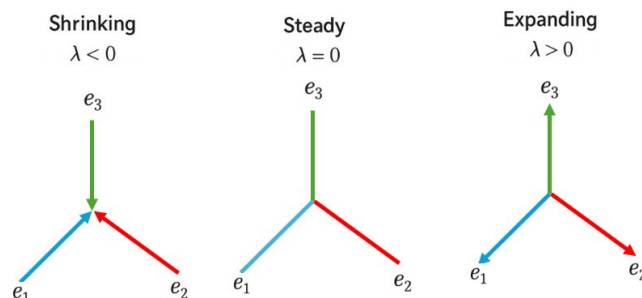


FIGURE 2. Classification of Ricci solitons based on the sign of λ

7. COMPARISON

TABLE 7.1. Comparison of Ricci solitons on indefinite para-Sasakian and para-Kenmotsu manifolds

Property	indefinite para-Sasakian	indefinite para-Kenmotsu
Structure equations	$\nabla_X \phi Y = -g(X, Y)\xi + \epsilon \eta(Y) X$	$\nabla_X \phi Y = -g(X, \phi Y)\xi - \epsilon \eta(Y) \phi X$
Nature of ξ	$\nabla_X \xi = \epsilon \phi X$	$\nabla_X \xi = \epsilon(X - \eta(X)) \xi$
Ricci Soliton form	$L_V g + 2S + 2\lambda g = 0$	$L_V g + 2S + 2\lambda g = 0$
Ricci Soliton equation	$S(X, \xi) = -(n-1)\eta(X)$	$S(X, Y) = -\lambda \epsilon \eta(X)$
Type of Soliton	$\lambda = (1-n)\epsilon$	$\lambda = (n-2)\epsilon$
Geometric behaviour	Shrinking or steady depending on ϵ	Space-like steady or expanding depending on ϵ

8. APPLICATION

Geometric Flow Modeling: The potential vector field $V = \beta \xi$ indicates that the metric evolves along the vector field ξ .

1. For $\epsilon = 1$, the soliton expands. So the manifold stretches along ξ , whereas orthogonal directions evolve differently or remain constant.
2. This simulates geometric flows that are anisotropic, meaning that one preferred direction changes noticeably.

Practical use: Anisotropic cosmologies and manifolds with directed deformation are examples of spaces or spacetimes with directional expansion.

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(S. Sunitha Devi) DEPARTMENT OF MATHEMATICS, KONERU LAKSHMAIAH EDUCATION FOUNDATION, VI-JAYAWADA, INDIA

(B. V. S. T. Sai) SCHOOL OF APPLIED SCIENCES, V V I T UNIVERSITY, NAMBUR, GUNTUR DISTRICT, INDIA

(V. Kavitha) DEPARTMENT OF MATHEMATICS, VARDHAMAN COLLEGE OF ENGINEERING, HYDERABAD, INDIA

(K. L. Sai Prasad) DEPARTMENT OF MATHEMATICS, GAYATRI VIDYA PARISHAD COLLEGE OF ENGINEERING FOR WOMEN, VISAKHAPATNAM, INDIA