



NEW RESULTS ON RIEMANNIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. In this paper, we study Riemannian concircular structure manifolds. Some basic concepts and some basic properties such manifolds are established. Also in the last section, some flat and some symmetric properties of a Riemannian concircular structure manifold are studied.

Keywords: Riemannian manifolds, Riemannian concircular structure manifolds, curvature tensors, symmetric spaces, torse-forming vector field, concircular vector field

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1. INTRODUCTION

In the study of the geometry of Riemannian manifolds, intrinsic vector fields such as torse-forming, torqued, concircular, recurrent, and parallel vector fields play a fundamental role. In the year 1939, in a Riemannian manifold M the notion of concircular vector fields introduced by Fialkow [13]. Indeed, in a Riemannian manifold a concircular vector field seems the theory of a transformation. For example, a concircular vector field plays an important role in the study of a concircular mapping [25]. Subsequently, in 1965, Tashiro provided a classification of Riemannian manifolds admitting concircular vector fields, which has since become a cornerstone in this field [24]. Following these foundational works, the study of manifolds equipped with specific vector fields was expanded by various authors. For instance, Yano developed the theory of torse-forming vector fields [26]. Then Mihai and Mihai [18] investigated some applications of the existence of torse formations on Sasakian and Kenmotsu manifolds. Furthermore, Chen and his co-authors extensively studied the properties of torqued and parallel vector fields, particularly in the context of Ricci solitons ([9], [10], [12]). The Riemannian manifolds with different vector fields have been studied by several authors ([4], [13], [17], [16], [19], [24], [25], [26]). Recently, Chaubey and Suh introduced a new class of Riemannian manifolds known as Riemannian concircular structures manifolds (briefly, $(R_iCS_t)_n$ -manifolds [6]. They also studied of semisymmetric and the

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conformally flat $(R_iCS_t)_n$ -manifolds. Also Chen et al. studied almost Ricci solitons on $(R_iCS_t)_3$ -manifolds [11].

In differential geometry, the Riemann curvature tensor $\mathcal{R}_{cur}$ is the most common way used to express the curvature of Riemannian manifolds. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field). The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. In the similar way with the help of projective and conharmonical transformations, the notion of projective curvature and conharmonical curvature have been defined [21]. Apart from conformal curvature tensor, the projective curvature tensor and conharmonical curvature tensor are another important tensors from the differential geometric point of view.

A central theme in the analysis of Riemannian manifolds is the classification of spaces based on curvature symmetry conditions. The most basic of these is "flatness" ($\mathcal{R}_{cur} = 0$) and "local symmetry" ($\nabla \mathcal{R}_{cur} = 0$) [23]. Here $\mathcal{R}_{cur}$ denotes the curvature tensor of the manifold M , ∇ is the Levi-Civita connection of the Riemannian metric $\tilde{g}$ and $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)$ acts as a derivation on $\mathcal{R}_{cur}$ for all vector fields ϑ_1 and ϑ_2 on M . If the manifold M satisfies the condition $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur} = 0$, then M is called a semisymmetric manifold and M satisfies the condition $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric} = 0$, then M is called a Ricci semisymmetric manifold. Hence, every Ricci semisymmetric manifold automatically satisfies the Ricci symmetric condition $\nabla \mathcal{S}_{Ric} = 0$, although the converse does not hold in general. Also every semisymmetric manifold is Ricci semisymmetric, but in general the converse part is not true. The properties of symmetric spaces have been studied by several authors in Riemannian and semi-Riemannian manifolds ([2], [3], [7], [15], [19], [22]). Apart from, using these definitions, if a (semi-)Riemannian manifold satisfying $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = 0$ and $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$ is called projective symmetric and projective Ricci symmetric. In an n -dimensional Riemannian manifold $(M, \tilde{g})$ the projective tensor $\mathcal{P}_{cur}$ is defined by

$$\begin{aligned} \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 &= \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 \\ &\quad - \frac{1}{n-1} \{ \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\vartheta_2 \}. \end{aligned}$$

We also define the endomorphism $\vartheta_1 \wedge_{\tilde{A}} \vartheta_2$ of $\chi(M)$ by

$$(\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\vartheta_3 = \tilde{A}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{A}(\vartheta_1, \vartheta_3)\vartheta_2,$$

where $\tilde{A}$ is a $(0, 2)$ -tensor field and symmetric. For a $(0, k)$ -tensor field $\tilde{T}$, $k < 1$ and a $(0, 2)$ -tensor field $\tilde{A}$ on $(M, \tilde{g})$, we define the tensor $\tilde{Q}(\tilde{A}, \tilde{T})$ by

$$\begin{aligned} \tilde{Q}(\tilde{A}, \tilde{T})(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_k; \vartheta_1, \vartheta_2) &= -\tilde{T}((\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_k) - \dots - \\ &\quad -\tilde{T}(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_{k-1}, (\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\tilde{\vartheta}_k). \end{aligned}$$

In particular, a (semi-)Riemannian manifold satisfying $\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = L_{\mathcal{R}_{cur}} \tilde{Q}(\tilde{g}, \mathcal{R}_{cur})$, $\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = L_{\mathcal{S}_{Ric}} \tilde{Q}(\tilde{g}, \mathcal{S}_{Ric})$, $\mathcal{R}_{cur} \cdot \mathcal{P}_{cur} = L_{\mathcal{P}_{cur}} \tilde{Q}(\tilde{g}, \mathcal{P}_{cur})$, $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = L_1 \tilde{Q}(\tilde{g}, \mathcal{R}_{cur})$ and $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = L_2 \tilde{Q}(\tilde{g}, \mathcal{S}_{Ric})$ is respectively called pseudosymmetric, Ricci pseudosymmetric, projective pseudosymmetric, projective Ricci pseudosymmetric and Ricci pseudosymmetric due to the projective curvature tensor, where $L_{\mathcal{R}_{cur}}$, $L_{\mathcal{S}_{Ric}}$, $L_{\mathcal{P}_{cur}}$, L_1 , L_2 are the associated scalars.

On the other hand, a Riemannian manifold is flat if and only if its curvature tensor vanishes identically. As is well known that the necessary and sufficient condition for some curvature tensors defined with the help of the curvature tensor to be flat is that these tensors vanish identically. In fact, a Riemannian manifold is conformally flat if each point has a neighborhood that can be mapped to flat space by a conformal transformation. Thus, a Riemannian manifold is conformally flat if and only if its conformal curvature tensor vanishes identically. The projective curvature tensor is an invariant under transformations preserving geodesics on semi-Riemannian manifolds. Moreover, it exhibits geometric features distinct from other generalized curvature tensors. Also a Riemannian manifold is projectively flat (respectively conharmonically flat) if and only if its projective curvature tensor (respectively conharmonic curvature tensor) vanishes identically.

In the year 1987, the notion of quasi-Einstein metric introduced by Koiso [14]. For an n -dimensional Riemannian manifold M if the Ricci tensor $\mathcal{S}_{Ric}$ satisfies

$$\mathcal{S}_{Ric} = \check{a}\check{g} + \check{b}\hat{\eta} \otimes \hat{\eta}, \quad (1.1)$$

where $\check{a}$ and $\check{b}$ are non-zero smooth functions on M and $\hat{\eta}$ is a 1-form ([8], [5]). If the non-vanishing Ricci tensor $\mathcal{S}_{Ric}$ of M satisfies equation (1.1), then the manifold M is referred to as a quasi-Einstein manifold [5]. In deed, if $\check{b} = 0$, then the manifold M reduces to the Einstein manifold, while the Ricci flat manifold is obtained in the particular case $\check{a} = 0$. In [5], Chaki and Maity studied manifolds under the following assumptions: (i) $\check{a}$ and $\check{b}$ are constants and the generator of $(QE)_n$ -manifold is recurrent, and (ii) $\check{a} + \check{b} = 0$, $\vartheta_4 = \frac{1}{2\check{a}}grad\check{a}$ and $\nabla_{\vartheta_1}\vartheta_4 = -\vartheta_1 + A(\vartheta_1)\vartheta_4$, where A is a dual 1-form of ϑ_4 .

The present paper is organized as follows: After introduction in section 2, we provide the basic concepts and properties of $(R_iCS_t)_n$ -manifolds. In section 3, we investigate the flatness and symmetry properties of these manifolds, specifically examining conditions such as projective flatness, projective symmetry, and various forms of pseudosymmetry within the framework of Riemannian concircular structures. Also, we give a non-trivial example.

2. Basic concepts of a $(R_iCS_t)_n$ -manifold

In this section, we present the fundamental definitions and geometric properties of the $(R_iCS_t)_n$ -manifold, as established in the foundational works of Chaubey and Suh [6].

Let M be an n -dimensional Riemannian manifold, $\check{g}$ is a Riemannian metric and ξ a torse-forming vector field on M satisfying

$$(\nabla_{\vartheta_1}\hat{\eta})(\vartheta_2) = \check{\alpha}\check{g}(\vartheta_1, \vartheta_2) + \pi(\vartheta_1)\hat{\eta}(\vartheta_2), \quad \forall \vartheta_1, \vartheta_2 \in \chi_\nu(M), \quad (2.2)$$

where $\hat{\eta}(\cdot) = \check{g}(\cdot, \xi)$ is a 1-form associated with ξ and π is also a 1-form. If 1-form π is closed on M , then ξ is said to be a concircular vector field ([13, 25, 26]). Also, for the torse-forming vector field ξ on M , we can say the followings:

- * if $\pi(\xi) = 0$, then ξ is a torqued vector field [10],
- * if $\pi = 0$, then ξ is a concircular vector field (in Fialkow's sense) [9, 13],
- * if the 1-form π is closed, then ξ is a concircular vector field (in Yano's sense) [25],
- * if $\alpha = 0$, then ξ is a recurrent vector field [20],
- * if $\pi = 0$, then ξ is a concurrent vector field and $\alpha = 1$ [12],

* if $\pi = 0$ and $\alpha = 0$, then ξ is a parallel vector field [12, 13].

Set $\vartheta_2 = \xi$ in the equation (2.2), we find

$$\check{\alpha}\hat{\eta}(\vartheta_1) + \pi(\vartheta_1) = 0, \quad (2.3)$$

Using the equation (2.3) in the equation (2.2), we obtain

$$(\nabla_{\vartheta_1}\hat{\eta})(\vartheta_2) = \check{\alpha}\{\tilde{g}(\vartheta_1, \vartheta_2) - \hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2)\}, \quad (2.4)$$

which implies that

$$\nabla_{\vartheta_1}\xi = \check{\alpha}\{\vartheta_1 - \hat{\eta}(\vartheta_1)\xi\}, \quad (2.5)$$

where $\check{\alpha}$ is a non-zero scalar and $\nabla_{\vartheta_1}\check{\alpha} = \tilde{g}(\vartheta_1, \tilde{D}\check{\alpha}) = \vartheta_1(\check{\alpha}) = \mu u(\vartheta_1)$ for a certain smooth function $\tilde{\mu}$ on M . Here $\tilde{D}$ denotes the gradient operator of $\tilde{g}$. From the equation (2.4), it follows directly that the 1-form $\hat{\eta}$ is closed. Taking the covariant derivative of the equation (2.3) along ϑ_2 and using the relation $\vartheta_1(\alpha) = \mu u(\vartheta_1)$ together with the equation (2.4), we infer that π is also closed. Hence, the unit torse-forming vector field ξ defined in the equation (2.2) can be regarded as a unit concircular vector field on M in the sense of Yano.

Let the Riemannian manifold M admit a $(1, 1)$ tensor field ϕ such that

$$\check{\alpha}\phi\vartheta_1 = \nabla_{\vartheta_1}\xi, \quad \check{\alpha} \neq 0,$$

which gives

$$\phi\vartheta_1 = \vartheta_1 - \hat{\eta}(\vartheta_1)\xi, \quad (2.6)$$

where the equation (2.5) is used. Thus, we obtain

$$\phi^2 = I - \hat{\eta} \otimes \xi.$$

In view of the equation (2.6), we have

$$\begin{aligned} \tilde{g}(\phi\vartheta_1, \phi\vartheta_2) &= \tilde{g}(\vartheta_1, \vartheta_2) - \hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2), \\ \hat{\eta}(\vartheta_2) &= \tilde{g}(\vartheta_2, \xi). \end{aligned}$$

Also, it follows that if M admits a unit concircular vector field ξ , a $(1, 1)$ tensor field ϕ and a 1-form $\hat{\eta}$, then we get

$$\hat{\eta}(\xi) = 1, \quad \tilde{g}(\phi\cdot, \phi\cdot) = \tilde{g}(\cdot, \cdot) - \hat{\eta} \otimes \hat{\eta}. \quad (2.7)$$

On an n -dimensional Riemannian manifold M , if quadruple $(\phi, \xi, \hat{\eta}, \tilde{g})$ satisfies the equation (2.7), then the manifold M is called $(R_iCS_t)_n$ -manifold. Hence the structure $(\phi, \xi, \hat{\eta}, \tilde{g})$ defines a (R_iCS_t) on M . Also we have the followings:

$$\begin{aligned} \phi\xi &= 0, \quad \hat{\eta}(\phi\vartheta_1) = 0, \quad \text{rank}(\phi) = n - 1, \\ (\nabla_{\vartheta_1}\phi)(\vartheta_2) &= \check{\alpha}[2\hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2)\xi - \tilde{g}(\vartheta_1, \vartheta_2)\xi - \hat{\eta}(\vartheta_2)\vartheta_1], \quad \forall \vartheta_1, \vartheta_2 \in \chi_\nu(M). \end{aligned}$$

Also the curvature identities

$$\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\xi = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\}, \quad (2.8)$$

and

$$\mathcal{R}_{cur}(\xi, \vartheta_1)\vartheta_2 = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_2)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_2)\xi\}, \quad (2.9)$$

hold for any vector fields ϑ_1, ϑ_2 . Using the equations (2.8) and (2.9), we obtain

$$\hat{\eta}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3) = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_3) - \hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_3)\}, \quad (2.10)$$

and

$$\mathcal{S}_{Ric}(\vartheta_1, \xi) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\hat{\eta}(\vartheta_1), \quad (2.11)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi_\nu(M)$ and $(\check{\alpha}^2 + \tilde{\mu}) \neq 0$ [6].

3. Main results

In this section, we investigate the geometric properties of the $(R_iCS_t)_n$ -manifolds under various curvature conditions, including projective flatness, projective symmetry and pseudosymmetry. Firstly, we consider a projectively flat $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{P}_{cur} = 0, \quad (3.12)$$

where $\mathcal{P}_{cur}$ is the projective curvature tensor of dimension n and is defined by

$$\mathcal{P}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 - \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.13)$$

From (3.12) and (3.13), it follows that

$$\mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.14)$$

Putting $\vartheta_5 = \xi$ in (3.14), we have

$$\mathcal{R}_{cur}(\xi, \vartheta_6)\vartheta_4 = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\xi - \mathcal{S}_{Ric}(\xi, \vartheta_4)\vartheta_6\}. \quad (3.15)$$

Taking the inner product of (3.15) with ϑ_7 , we have

$$\tilde{g}(\mathcal{R}_{cur}(\xi, \vartheta_6)\vartheta_4, \vartheta_7) = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7) - \mathcal{S}_{Ric}(\xi, \vartheta_4)\tilde{g}(\vartheta_6, \vartheta_7)\}. \quad (3.16)$$

From (3.13), we find

$$\begin{aligned} (\check{\alpha}^2 + \tilde{\mu})\{\tilde{g}(\vartheta_6, \vartheta_7)\hat{\eta}(\vartheta_4) - \tilde{g}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7)\} &= \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7) \\ &\quad + (n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_6, \vartheta_7)\hat{\eta}(\vartheta_4)\}. \end{aligned} \quad (3.17)$$

Putting $\vartheta_7 = \xi$ in (3.17), we have

$$\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_6, \vartheta_4),$$

which means that the manifold M is an Einstein manifold. Thus we can state the following:

Theorem 3.1. *Let M be a projectively flat $(R_iCS_t)_n$ -manifold. Then the manifold M is an Einstein manifold.*

As is well known that a Riemannian manifold of dimension n , ($n > 3$) is projectively flat if and only if it is of constant curvature $\tilde{c}$. Then we can write

$$\mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \tilde{c}\{\tilde{g}(\vartheta_6, \vartheta_4)\vartheta_5 - \tilde{g}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.18)$$

Comparing (3.14) and (3.18), we get

$$\tilde{c} = -(\check{\alpha}^2 + \tilde{\mu}),$$

which means that the manifold is of constant curvature $-(\check{\alpha}^2 + \tilde{\mu})$. Since $\check{\alpha}$ is constant scalar then the smooth function $\tilde{\mu}$ must be a constant. Thus we have the following:

Corollary 3.1. *A $(R_iCS_t)_n$ -manifold is projectively flat if and only if the smooth function $\tilde{\mu}$ is a constant.*

Now we consider the condition $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = 0$ on a $(R_iCS_t)_n$ -manifold. Thus we have

$$(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = 0,$$

which turns to

$$\begin{aligned} & \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 = 0. \end{aligned} \quad (3.19)$$

Putting $\vartheta_1 = \xi$ in (3.19), we obtain

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\xi + \tilde{g}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4)\xi \right. \\ & - \tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4)\xi + \tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\vartheta_3 \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\vartheta_3 + \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\vartheta_5 \} \\ & + \frac{(\check{\alpha}^2 + \tilde{\mu})}{n-1} \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\xi - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4)\xi \right. \\ & + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\vartheta_3 + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4)\xi \\ & \left. - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\vartheta_3 + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\vartheta_5 \right\} = 0. \end{aligned} \quad (3.20)$$

Taking the inner product of (3.20) with ξ , we have

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4) - \tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4) \right. \\ & + \tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_5) - \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_3) + \tilde{g}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4) \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\hat{\eta}(\vartheta_3) + \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\hat{\eta}(\vartheta_5) \} \\ & + \frac{(\check{\alpha}^2 + \tilde{\mu})}{n-1} \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4) \right. \\ & + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_5) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_3) \\ & \left. + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\hat{\eta}(\vartheta_3) + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\hat{\eta}(\vartheta_5) \right\} = 0. \end{aligned} \quad (3.21)$$

Putting $\vartheta_3 = \xi$ in (3.21), we have

$$(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\xi, \vartheta_5)\vartheta_4) + \frac{1}{n-1}\mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\xi, \vartheta_5)\vartheta_4) = 0.$$

Using the equation (2.9), we get

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \{ \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4) - \tilde{g}(\vartheta_4, \vartheta_5)\hat{\eta}(\vartheta_2) \} \\ & + \frac{1}{n-1}(\check{\alpha}^2 + \tilde{\mu}) \{ \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4) - \mathcal{S}_{Ric}(\vartheta_4, \vartheta_5)\hat{\eta}(\vartheta_2) \} = 0. \end{aligned} \quad (3.22)$$

Putting $\vartheta_4 = \xi$ in (3.22), we have

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5)$$

which means that the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.2. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{P} \cdot \mathcal{R} = 0$. Then the manifold M is an Einstein manifold.*

Now we consider the condition $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$ on a $(R_iCS_t)_n$ -manifold. Thus we have

$$(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5) = 0,$$

which turns to

$$\mathcal{S}_{Ric}(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5) = 0. \quad (3.23)$$

Putting $\vartheta_1 = \xi$ in (3.23), we have

$$\mathcal{S}_{Ric}(\mathcal{P}_{cur}(\xi, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{P}_{cur}(\xi, \vartheta_2)\vartheta_5) = 0. \quad (3.24)$$

Using (3.13) in (3.24), we obtain

$$\begin{aligned} & (\alpha^2 + \tilde{\mu})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_5) + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_3)\} \\ & + (n-1)(\alpha^2 + \tilde{\mu})^2\{\tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_5) + \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_3)\} = 0. \end{aligned} \quad (3.25)$$

Putting $\vartheta_5 = \xi$ in (3.25), we get

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3) = -(n-1)(\alpha^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_3),$$

which means that the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.3. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$. Then the manifold M is an Einstein manifold.*

Now we consider pseudo-symmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_{\mathcal{R}_{cur}}Q(\tilde{g}, \mathcal{R}_{cur})$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = \mathcal{L}_{\mathcal{R}_{cur}}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4,$$

and

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_{\mathcal{R}_{cur}}\{(\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, (\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)(\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_4\}, \end{aligned}$$

or,

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_{\mathcal{R}_{cur}}\{\tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_1 - \tilde{g}(\vartheta_1, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_2 \\ & - \tilde{g}(\vartheta_2, \vartheta_3)\mathcal{R}_{cur}(\vartheta_1, \vartheta_5)\vartheta_4 + \tilde{g}(\vartheta_1, \vartheta_3)\mathcal{R}_{cur}(\vartheta_2, \vartheta_5)\vartheta_4 \\ & - \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_1)\vartheta_4 + \tilde{g}(\vartheta_1, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_2)\vartheta_4 \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_1 + \tilde{g}(\vartheta_1, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\} \end{aligned} \quad (3.26)$$

Putting $\vartheta_1 = \vartheta_4 = \xi$ in (3.26), we get

$$(\check{\alpha}^2 + \tilde{\mu})(1 + \mathcal{L}_{\mathcal{R}_{cur}})\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\vartheta_3\} = 0,$$

which means that

$$\mathcal{L}_{\mathcal{R}_{cur}} = -1.$$

Thus we can state the following:

Theorem 3.4. *Every $(R_iCS_t)_n$ -manifold M is a pseudo-symmetric manifold satisfying*

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = -Q(\tilde{g}, \mathcal{R}_{cur}).$$

Now we consider pseudo-symmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_q Q(\mathcal{S}_{Ric}, \mathcal{R}_{cur}),$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = \mathcal{L}_q((\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4,$$

and

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_q\{(\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}((\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, (\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)(\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_4\}, \end{aligned}$$

or,

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_q\{\mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_2 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\mathcal{R}_{cur}(\vartheta_1, \vartheta_5)\vartheta_4 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\mathcal{R}_{cur}(\vartheta_2, \vartheta_5)\vartheta_4 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_1)\vartheta_4 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_2)\vartheta_4 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_1 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\}. \end{aligned} \quad (3.27)$$

Putting $\vartheta_1 = \vartheta_4 = \xi$ in (3.27), we get

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\vartheta_3 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\} \\ = & \mathcal{L}_q(\check{\alpha}^2 + \tilde{\mu})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\vartheta_3 - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_5 - (n-1)(\check{\alpha}^2 + \tilde{\mu})\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\}. \end{aligned} \quad (3.28)$$

Again putting $\vartheta_3 = \xi$ and taking the inner product with ξ in (3.28), we obtain

$$\begin{aligned} \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) &= \frac{-(n-1)(\alpha^2 + \tilde{\mu})\mathcal{L}_{\check{\alpha}} - (1 - \check{\alpha}^2 - \tilde{\mu})}{\mathcal{L}_q}\tilde{g}(\vartheta_2, \vartheta_5) \\ &+ \frac{(1 - \check{\alpha}^2 - \tilde{\mu})}{\mathcal{L}_q}\hat{\eta}(\vartheta_2)\hat{\eta}(\vartheta_5), \end{aligned}$$

which means that the manifold M is an $\hat{\eta}$ -Einstein manifold. Thus we can state the following:

Theorem 3.5. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_q Q(\mathcal{S}_{Ric}, \mathcal{R}_{cur})$. Then the manifold M is an $\hat{\eta}$ -Einstein manifold.*

Now we consider pseudo-Riccisymmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = \mathcal{L}_{\mathcal{S}_{Ric}} Q(\tilde{g}, \mathcal{S}_{Ric}),$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5) = \mathcal{L}_{\mathcal{S}_{Ric}}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5),$$

and

$$\begin{aligned} & \mathcal{S}_{Ric}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5) \\ &= \mathcal{L}_{\mathcal{S}}\{\mathcal{S}_{Ric}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, (\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_5)\}. \end{aligned} \quad (3.29)$$

Putting $\vartheta_1 = \xi$ in (3.29), we get

$$\begin{aligned} & (\alpha^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_3)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) - \tilde{g}(\vartheta_2, \vartheta_3)\mathcal{S}_{Ric}(\xi, \vartheta_5) \\ & + \hat{\eta}(\vartheta_5)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3) - \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{S}_{Ric}(\xi, \vartheta_3)\} \\ &= \mathcal{L}_{\mathcal{S}_{Ric}}\{\tilde{g}(\vartheta_2, \vartheta_3)\mathcal{S}_{Ric}(\xi, \vartheta_5) - \hat{\eta}(\vartheta_3)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) \\ & + \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{S}_{Ric}(\xi, \vartheta_3) - \hat{\eta}(\vartheta_5)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\}. \end{aligned} \quad (3.30)$$

Again putting $\vartheta_3 = \xi$ in (3.30), we have

$$(\check{\alpha}^2 + \tilde{\mu} + \mathcal{L}_{\mathcal{S}_{Ric}})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) + (n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5)\} = 0.$$

Since $\alpha^2 + \tilde{\mu} + \mathcal{L}_{\mathcal{S}_{Ric}} \neq 0$, we have

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5),$$

which means that the manifold M is an Einstein manifold. Thus we have the following:

Theorem 3.6. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = \mathcal{L}_{\mathcal{S}_{Ric}} Q(\tilde{g}, \mathcal{S}_{Ric})$. Then the manifold M is an Einstein manifold.*

Let M be a ϕ -symmetric $(R_iCS_t)_n$ -manifold which is satisfying the condition

$$\phi^2((\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0. \quad (3.31)$$

Then we have

$$-(\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3) + \hat{\eta}((\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0. \quad (3.32)$$

Putting $\vartheta_3 = \xi$ in (3.32), we have

$$\begin{aligned} (\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \xi) &= \nabla_{\vartheta_4} \mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \xi) - \mathcal{R}_{cur}(\nabla_{\vartheta_4} \vartheta_1, \vartheta_2, \xi) \\ &\quad - \mathcal{R}_{cur}(\vartheta_1, \nabla_{\vartheta_4} \vartheta_2, \xi) - \mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \nabla_{\vartheta_4} \xi), \end{aligned}$$

which turns to

$$\begin{aligned} & \nabla_{\vartheta_4}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\} \\ & + (\check{\alpha}^2 + \tilde{\mu})\{\nabla_{\vartheta_4}(\hat{\eta}(\vartheta_1))\vartheta_2 + \hat{\eta}(\vartheta_1) \nabla_{\vartheta_4} \vartheta_2 \\ & - \nabla_{\vartheta_4}(\hat{\eta}(\vartheta_2))\vartheta_1 - \hat{\eta}(\nabla_{\vartheta_4} \vartheta_1)\vartheta_2\} \\ & - \check{\alpha}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\} - \check{\alpha}\mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \vartheta_4) = 0. \end{aligned} \quad (3.33)$$

Taking the inner product with ϑ_5 in (3.33), we get

$$\begin{aligned} & \nabla_{\vartheta_4}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5) - \hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_5)\} \\ & + (\check{\alpha}^2 + \tilde{\mu})\{\nabla_{\vartheta_4}(\hat{\eta}(\vartheta_1))\tilde{g}(\vartheta_2, \vartheta_5) + \hat{\eta}(\vartheta_1)\tilde{g}(\nabla_{\vartheta_4}\vartheta_2, \vartheta_5) \\ & - \nabla_{\vartheta_4}(\hat{\eta}(\vartheta_2))\tilde{g}(\vartheta_1, \vartheta_5) - \hat{\eta}(\nabla_{\vartheta_4}\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5)\} \\ & - \check{\alpha}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5) - \hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_5)\} - \check{\alpha}\tilde{g}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \vartheta_4), \vartheta_5) = 0. \end{aligned} \quad (3.34)$$

Putting $\vartheta_2 = \vartheta_5 = e_i$ in (3.34), we obtain

$$\mathcal{S}_{Ric}(\vartheta_1, \vartheta_4) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_1, \vartheta_4)$$

which means that the manifold M is an Einstein manifold. So we have the following:

Theorem 3.7. *Let M be a ϕ -symmetric $(R_iCS_t)_n$ -manifold. Then the manifold M is an Einstein manifold.*

Now let M be a locally ϕ -symmetric $(R_iCS_t)_3$ -manifold which is satisfying the condition $\phi^2((\nabla_{\vartheta_4}\mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0$, which means that all vector fields are orthogonal to ξ . Firstly as is well known that if the manifold M is of dimension 3, we always have

$$\begin{aligned} \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 &= \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\vartheta_2 \\ &+ \tilde{g}(\vartheta_2, \vartheta_3)Q\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)Q\vartheta_2 - \frac{r}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\}. \end{aligned} \quad (3.35)$$

Using (3.35) in (3.32), we get

$$\begin{aligned} & \tilde{g}((\nabla_{\vartheta_4}Q)\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}((\nabla_{\vartheta_4}Q)\vartheta_1, \vartheta_3)\vartheta_2 \\ & + \tilde{g}(\vartheta_2, \vartheta_3)(\nabla_{\vartheta_4}Q)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)(\nabla_{\vartheta_4}Q)\vartheta_2 \\ & - \frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} = 0. \end{aligned}$$

From the last equation, we obtain

$$\begin{aligned} & \frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} \\ & - (\frac{r}{2} + 2(\check{\alpha}^2 + \tilde{\mu}))\check{\alpha}\{\tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_1, \vartheta_4)\xi - \tilde{g}(\vartheta_1, \vartheta_3)\tilde{g}(\vartheta_2, \vartheta_4)\xi\}, \end{aligned} \quad (3.36)$$

and

$$\hat{\eta}((\nabla_{\vartheta_4}\mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = -(\frac{r}{2} + 2(\check{\alpha}^2 + \tilde{\mu}))\check{\alpha}\{\tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_1, \vartheta_4) - \tilde{g}(\vartheta_1, \vartheta_3)\tilde{g}(\vartheta_2, \vartheta_4)\}, \quad (3.37)$$

respectively. Thus from (3.36) and (3.37), we have

$$\frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} = 0,$$

which means that $\vartheta_4(r) = 0$. Thus, the scalar curvature r is a constant. So, we have the following:

Theorem 3.8. *A $(R_iCS_t)_3$ -manifold is locally ϕ -symmetric if and only if the scalar curvature r is constant.*

Now, we give a non-trivial example of a $(R_iCS_t)_3$ -manifold.

Example 3.1. We consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are standard coordinates in $\mathbb{R}^3$. The vector fields

$$\theta_1 = e^{-f(z)} \frac{\partial}{\partial x}, \quad \theta_2 = e^{-f(z)} \frac{\partial}{\partial y}, \quad \theta_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M . Here, $f(z)$ is a non-vanishing smooth function of z and $f'(z) \neq 0$. Suppose that the associated metric $\tilde{g}$ of M^3 is defined by

$$\tilde{g} = e^{2f(z)} \{dx \otimes dx + dy \otimes dy\} + dz \otimes dz$$

and

$$\tilde{g}(\theta_1, \theta_1) = \tilde{g}(\theta_2, \theta_2) = \tilde{g}(\theta_3, \theta_3) = 1,$$

$$\tilde{g}(\theta_1, \theta_2) = \tilde{g}(\theta_1, \theta_3) = \tilde{g}(\theta_2, \theta_3) = 0.$$

Since the metric $\tilde{g}$ is a Riemannian metric, thus M equipped with $\tilde{g}$ is a three dimensional Riemannian manifold. Let u be a 1-form associated with the unit vector field $\xi = \theta_3$ such that $\tilde{g}(\theta_i, \theta_3) = \tilde{u}(\theta_i)$. Then we have

$$\nabla_{\theta_i} \theta_3 = f'(z) \{\theta_i - \tilde{g}(\theta_i, \xi) \xi\}, \quad (3.38)$$

for $i = 1, 2, 3$, where ∇ is the Levi-Civita connection of the metric $\tilde{g}$. This infers that $(M, \tilde{g})$ admits a concircular vector field ξ . Let φ is a $(1, 1)$ -tensor field on M such that $\phi(\theta_1) = \theta_1$, $\phi(\theta_2) = \theta_2$, $\phi(\theta_3) = 0$ and $f'(z)\phi\theta_i = \nabla_{\theta_i} \theta_3$, then by straight forward calculations we notice that the relations $\phi^2 = I - \tilde{u} \otimes \theta_3$, $\tilde{g}(\theta_3, \theta_3) = 1$ and $\tilde{g}(\phi \cdot, \phi \cdot) = \tilde{g}(\cdot, \cdot) - \tilde{u} \otimes \tilde{u}$ for all $\theta_i, i = 1, 2, 3$ hold on M . Here I is the identity transformation [6]. Now, by direct calculations, we have

$$[\theta_1, \theta_2] = 0, \quad [\theta_1, \theta_3] = f'(z)\theta_1, \quad [\theta_2, \theta_3] = f'(z)\theta_2.$$

Also, non-zero components of ∇ on M are

$$\nabla_{\theta_1} \theta_1 = -f'(z)\theta_3, \quad \nabla_{\theta_1} \theta_3 = f'(z)\theta_1,$$

$$\nabla_{\theta_2} \theta_2 = -f'(z)\theta_3, \quad \nabla_{\theta_2} \theta_3 = f'(z)\theta_2.$$

From (3.38), we conclude that M endowed with the structure $(\phi, \theta_3, \tilde{u}, \tilde{g})$ is a three-dimensional Riemannian concircular structure manifold.

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