



FOURIER TRANSFORM OF BICOMPLEX-VALUED HYPERBOLIC NORM INTEGRABLE FUNCTIONS

ERDEM TOKSOY  *

Abstract. For the purpose of this investigation, the Fourier transforms of hyperbolic norm integrable functions with bicomplex values that are constructed on locally compact abelian groups are utilized. Performing an analysis of these transformations is the purpose of this study. The bicomplex-valued Fourier transform is referred to as the $\mathbb{BC}$ -Fourier transform as a result of the nomenclature it utilizes. It is well defined and unique. There is a relationship between this transform and the convolution operator. The translation and character operators are sometimes included in this transform, which results in outcomes that are comparable to those of the well-known Fourier transform. Many fields employ the Fourier transform to convert a time-dependent function to a frequency-dependent function for time-frequency analysis, including signal processing. Using complex-valued signals is more convenient here. Thus, the $\mathbb{BC}$ -Fourier transform for time-frequency analysis of signals with bicomplex values would be more efficient since bicomplex numbers constitute an extended form of ordinary complex numbers.

Keywords: $\mathbb{BC}$ -Fourier transform, convolution, bicomplex Lebesgue space

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1. INTRODUCTION

Bicomplex numbers, a novel number system, have recently been introduced as an extension of the established complex number system. Price's book [18] is the primary recommended introductory text for bicomplex numbers. We recommend the references [1] and [17], which provide comprehensive explanations of the fundamental principles of bicomplex numbers, with an emphasis on functional analysis. These publications encompass a broad spectrum of applications and offer a thorough introduction to the subject.

The following sources are used to obtain the fundamental information regarding bicomplex numbers: [18], [1], and [17].

In this research, the symbols $\mathbb{R}$ and $\mathbb{C}$ represent real numbers and complex numbers, respectively. Let us now examine two independent imaginary numbers, i and j , where $i^2 = j^2 = -1$. This results in two sets, $\mathbb{C}(i)$ and $\mathbb{C}(j)$, which are identical to the set $\mathbb{C}$. Now, let

us examine an imaginary number k , defined as the product of these two imaginary numbers, possessing the following characteristics:

$$k^2 = 1, ij = ji = k, jk = kj = -i \text{ and } ik = ki = -j.$$

The collection of bicomplex numbers, designated as $\mathbb{BC}$, is characterized as

$$\mathbb{BC} = \{w_1 + jw_2 \mid w_1, w_2 \in \mathbb{C}(i)\}.$$

It is possible to interpret the set $\mathbb{BC}$ as

$$\mathbb{BC} = \{a_1 + a_2i + a_3j + a_4k \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

in accordance with the considerations mentioned previously. In the following ways, the operations of addition and multiplication are, respectively, defined:

$$(w_1 + jw_2) + (w_3 + jw_4) = (w_1 + w_3) + j(w_2 + w_4)$$

and

$$(w_1 + jw_2)(w_3 + jw_4) = (w_1w_3 - w_2w_4) + j(w_1w_4 + w_2w_3).$$

Through the execution of these operations, the set $\mathbb{BC}$ is transformed into a commutative ring and, consequently, a module over itself. The collection of hyperbolic numbers, denoted $\mathbb{D}$, a notable subset of $\mathbb{BC}$, is characterized as

$$\mathbb{D} = \{a_1 + ka_2 \mid a_1, a_2 \in \mathbb{R}\}$$

through the substitution of w_1 and w_2 with a_1 and ia_2 , respectively, within the definition of $\mathbb{BC}$, such that $a_1, a_2 \in \mathbb{R}$. The conditions stated below are satisfied by the crucial elements; these are zero divisors in $\mathbb{BC}$: $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. Here,

$$e_1 + e_2 = 1, e_1e_2 = 0, e_2e_2 = e_2, e_1e_1 = e_1.$$

It is also possible to see e_1 and e_2 as hyperbolic numbers. The above components make up the idempotent basis for bicomplex numbers. The bicomplex number $w_1 + jw_2$ is characterized in a specific form by the expression

$$w_1 + jw_2 = z_1e_1 + z_2e_2,$$

such that $z_1 = w_1 - iw_2, z_2 = w_1 + iw_2 \in \mathbb{C}(i)$. This particular statement is referred to as the "idempotent representation of bicomplex numbers." In that case, each hyperbolic number is assigned the idempotent representation

$$a_1 + ka_2 = e_1b_1 + e_2b_2,$$

where $b_1 = a_1 + a_2, b_2 = a_1 - a_2 \in \mathbb{R}$. The collection of non-negative hyperbolic numbers is denoted by

$$\mathbb{D}^+ = \{a_1e_1 + a_2e_2 \mid a_1, a_2 \geq 0\}.$$

Let a_1 and a_2 belong to $\mathbb{D}$. In the context of hyperbolic numbers, the partial order relation $\preceq$ is stated as

$$a_1 \preceq a_2 \text{ if and only if } a_2 - a_1 \in \mathbb{D}^+.$$

Additionally,

$$a_1 \prec a_2 \text{ if and only if } a_1 \lesssim a_2 \text{ but } a_1 \neq a_2.$$

For further information on the characteristics of the partial order relation $\lesssim$, refer to Section 2.6.2 of [17]. As stated in Section 2.6.3 of [17], let us now give a brief summary of the $\mathbb{D}$ -boundedness. Consider $V \subset \mathbb{D}$. Given that V has a $\mathbb{D}$ -lower bound (or a $\mathbb{D}$ -upper bound) b , it follows that for any $v \in V$, v is comparable to b and $b \lesssim v$ (or $v \lesssim b$). It is said that a set is $\mathbb{D}$ -bounded iff it is $\mathbb{D}$ -bounded from both below and above. We define the concept of $\mathbb{D}$ -infimum, represented by $\inf_{\mathbb{D}} V$, as the greatest $\mathbb{D}$ -lower bound for V if V is a set $\mathbb{D}$ -bounded from below, and the $\mathbb{D}$ -supremum, represented by $\sup_{\mathbb{D}} V$, as the least $\mathbb{D}$ -upper bound for V , if V is a set $\mathbb{D}$ -bounded from above. Here, the greatest $\mathbb{D}$ -lower bound is described as $b \lesssim \inf_{\mathbb{D}} V$ for any $\mathbb{D}$ -lower bound b , despite the fact that not every $\mathbb{D}$ -lower bound is comparable. In this context, the least $\mathbb{D}$ -upper bound is determined as $\sup_{\mathbb{D}} V \lesssim b$ for any $\mathbb{D}$ -upper bound b , regardless of whether all $\mathbb{D}$ -upper bounds are comparable. Let us take V_1 and V_2 that are defined by

$$V_1 = \{v_1 : e_1 v_1 + e_2 v_2 \in V\} \text{ and } V_2 = \{v_2 : e_1 v_1 + e_2 v_2 \in V\},$$

respectively. Assuming that V is $\mathbb{D}$ -bounded from below, it follows that

$$\inf_{\mathbb{D}} V = e_1 \inf V_1 + e_2 \inf V_2.$$

Assuming that V is $\mathbb{D}$ -bounded from above, it follows that

$$\sup_{\mathbb{D}} V = e_1 \sup V_1 + e_2 \sup V_2.$$

Although different modules are defined for bicomplex numbers, the concept of a hyperbolic modulus to be addressed in this study is as follows. For a bicomplex number $w_1 + jw_2$, the hyperbolic modulus $|\cdot|_k$ is represented by the equation

$$|w_1 + jw_2|_k^2 = (w_1 + jw_2)(\overline{w_1} - j\overline{w_2}).$$

For the idempotent representation of a bicomplex number $w_1 + jw_2$, the hyperbolic modulus is given in the form

$$|w_1 + jw_2|_k = |e_1 z_1 + e_2 z_2|_k = e_1 |z_1| + e_2 |z_2|. \quad (1.1)$$

Modules are established over rings, analogous to how vector spaces are constructed over fields. A module U is called a $\mathbb{BC}$ -module if it is defined over a ring $\mathbb{BC}$ (see [8, 20]). Assume that U is a $\mathbb{BC}$ -module. If the following requirements are satisfied, a map $\|u\|_{\mathbb{D}} : U \rightarrow \mathbb{D}^+$ is considered a hyperbolic norm ($\mathbb{D}$ -valued) on U .

- (a) $\|u\|_{\mathbb{D}} = 0$ iff $u = 0$,
- (b) $\|wu\|_{\mathbb{D}} = |w|_k \|u\|_{\mathbb{D}}$ for all $u \in U$, $w \in \mathbb{BC}$,
- (c) $\|u + v\|_{\mathbb{D}} \lesssim \|u\|_{\mathbb{D}} + \|v\|_{\mathbb{D}}$ for all $u, v \in U$.

To describe the hyperbolic ($\mathbb{D}$ -valued) norm of $w_1 + jw_2 \in \mathbb{BC}$, we have the hyperbolic modulus $|\cdot|_k$ with expression (1.1). Assume that $w, \omega \in \mathbb{BC}$ and $\zeta \in \mathbb{D}^+$. The following properties are provided by the hyperbolic ($\mathbb{D}$ -valued) norm:

(i)

$$|w|_k = 0 \text{ iff } w = 0,$$

(ii)

$$|w\omega|_k = |w|_k |\omega|_k,$$

(iii)

$$|w + \omega|_k \lesssim |w|_k + |\omega|_k,$$

(iv)

$$|\zeta|_k = \zeta,$$

(v)

$$|\zeta w|_k = \zeta |w|_k.$$

Suppose that U is a $\mathbb{BC}$ -module. As a result, we get

$$U = e_1 U_1 + e_2 U_2,$$

where $U_1 = e_1 U$ and $U_2 = e_2 U$ are complex vector spaces and $\mathbb{BC}$ -modules. The term given to this equation is the idempotent decomposition of U . Consequently, any u in U can be uniquely expressed as $u = e_1 u_1 + e_2 u_2$ such that u_1 is a member of U_1 and u_2 is a member of U_2 (see [7]). It is assumed that U_1 and U_2 are normed spaces with norms of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Therefore,

$$\|u\|_{\mathbb{D}} = e_1 \|u_1\|_1 + e_2 \|u_2\|_2$$

is a way to endow the $\mathbb{BC}$ -module U with the hyperbolic ($\mathbb{D}$ -valued) norm. In the context of the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$, a sequence (u_n) in U converges to $u_0 \in X$ iff for every $0 \prec \epsilon$, there exists $n_0 \in \mathbb{N}$, where $\|u_n - u_0\|_{\mathbb{D}} \prec \epsilon$ for all $n \geq n_0$. This convergence is denoted by the term $\mathbb{D}$ -convergence (or $\mathbb{D}$ -converges). Additionally, sequences that are $\mathbb{D}$ -non-convergent will be referred to as $\mathbb{D}$ -divergent. According to the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$, a sequence (u_n) in U is a Cauchy sequence iff for every $0 \prec \epsilon$ there exists $n_0 \in \mathbb{N}$, where $\|u_n - u_m\|_{\mathbb{D}} \prec \epsilon$ for all $n, m \geq n_0$. In this context, completeness is defined in a manner that is analogous to the conventional definition, that every Cauchy sequence in U $\mathbb{D}$ -converges to an element in U with regard to the hyperbolic ($\mathbb{D}$ -valued) norm. A bicomplex Banach module is described as a complete real-valued or hyperbolic normed $\mathbb{BC}$ -module, ([13, 7, 8, 16]).

Assume B to be any measurable space with a positive measure μ . Hence, for all values of p from 0 to infinity, the usual Lebesgue space is denoted as $(L^p(B, \mu), \|\cdot\|_p)$, [25, 4, 19]. Consider $\widehat{G}$ as the dual group of G , where G is a locally compact abelian group. Let h be a member of $L^1(G, \nu)$ such that ν is a Haar measure on G . Hence, the Fourier transform $\mathcal{F}h$ is indicated on $\widehat{G}$ by

$$\mathcal{F}h(\gamma) = \int_G h(x) \langle -x, \gamma \rangle d\nu(x)$$

for all $\gamma \in \widehat{G}$, [21].

Consider a σ -algebra Σ constituted by a set B . A bicomplex-valued function $\lambda = e_1 \lambda_1 + e_2 \lambda_2$ on B is referred to as a $\mathbb{BC}$ -measure on Σ if both λ_1 and λ_2 are complex measures on Σ . Specifically, with respect to positive measures λ_1 and λ_2 on Σ , λ is referred to as a $\mathbb{D}$ -measure on Σ , [10, Definition 3.3]. If both h_1 and h_2 are complex measurable functions on B , then the bicomplex-valued function h on B , defined as $h = e_1 h_1 + e_2 h_2$, is said to

be $\mathbb{BC}$ -measurable on B . Notably, h is a $\mathbb{D}$ -measurable function on B if h_1 and h_2 are real measurable functions on B , [10].

For any measurable space B , let $\lambda = e_1\lambda_1 + e_2\lambda_2$ be a $\mathbb{D}$ -measure. Assume that $0 < p < \infty$. Let $L^p(B, \lambda_1)$ and $L^p(B, \lambda_2)$ represent the linear spaces that consist of all equivalence classes of complex-valued measurable functions h_1 and h_2 defined on the set B , where

$$\int_B |h_1(x)|^p d\lambda_1 < \infty$$

and

$$\int_B |h_2(x)|^p d\lambda_2 < \infty,$$

respectively. Hence, $L^p_{\mathbb{BC}}(B, \lambda)$ represents all equivalence classes of bicomplex-valued $\mathbb{BC}$ -measurable functions on B in which the form is $h = e_1h_1 + e_2h_2$, with h_1 belonging to $L^p(B, \lambda_1)$ and h_2 to $L^p(B, \lambda_2)$, [24].

An essential integral operator, the Fourier transform is implemented in numerous disciplines, including signal analysis, time-frequency analysis, quantum mechanics, and the solution of partial and ordinary differential equations. In addition, the Fourier transform, which is defined on locally compact abelian groups G , serves as the foundation for both functional analysis and harmonic analysis, [21, 19]. Recently, the Fourier transform of functions of bicomplex variables was defined in [2]. Subsequently, the inverse of this transform as well as some examples of it were studied in [3]. Afterwards, bicomplex revisions of significant integral transforms, notably the Fourier and Laplace transforms, were examined in [11]. In particular, an examination of the Fourier transforms employed in the study of harmonic analysis reveals that they are defined on locally compact abelian groups. A $\mathbb{BC}$ -Fourier transform of bicomplex-valued hyperbolic norm integrable functions that are defined on the locally compact abelian group will be defined in this paper, and some examples will be provided.

2. BC-FOURIER TRANSFORM

Assume that G is an abelian group that is locally compact. Consequently, a unique $\mathbb{D}$ -Haar measure λ can be expressed as $\lambda = e_1\lambda + e_2\lambda$, where λ represents a unique Haar measure on G . The space $L^p_{\mathbb{BC}}(G, \lambda)$ is established as a $\mathbb{BC}$ -module, for $0 < p < \infty$, [23]. Given that G is a locally compact abelian group, its dual group $\widehat{G}$ is also a locally compact abelian group (see [21]). Therefore, there exists a unique $\mathbb{D}$ -Haar measure $\nu = e_1\nu + e_2\nu$, where ν is a unique Haar measure on $\widehat{G}$.

Definition 2.1. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $h = e_1h_1 + e_2h_2$. Let $\widehat{G}$ be the dual group of G . Then $\mathbb{BC}$ -Fourier transform of h is defined by

$$\mathcal{F}_{\mathbb{BC}}h(\gamma) = e_1\mathcal{F}h_1(\gamma) + e_2\mathcal{F}h_2(\gamma).$$

for all $\gamma \in \widehat{G}$. Since the Fourier transforms of the functions $h_1, h_2 \in L^1(G, \mu)$ are well defined, then the $\mathbb{BC}$ -Fourier transform of any function $h = e_1h_1 + e_2h_2 \in L^1_{\mathbb{BC}}(G, \lambda)$ is well defined.

Remark 2.1. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $h = e_1 h_1 + e_2 h_2$. Take $\widehat{G}$ as the dual group corresponding to G . It is established that

$$|\mathcal{F}h_1(\gamma)| \leq \|h_1\|_1 \quad (2.2)$$

and

$$|\mathcal{F}h_2(\gamma)| \leq \|h_2\|_1 \quad (2.3)$$

for all $\gamma \in \widehat{G}$ (see [19, inequality (4.4.2)]). Combining (2.2), (2.3) and using the definition of $\mathbb{BC}$ -Fourier transform of h , we may write

$$\begin{aligned} |\mathcal{F}_{\mathbb{BC}}h(\gamma)|_k &= e_1 |\mathcal{F}h_1(\gamma)| + e_2 |\mathcal{F}h_2(\gamma)| \\ &\lesssim e_1 \|h_1\|_1 + e_2 \|h_2\|_1 = \|h\|_{1, \mathbb{D}} \end{aligned}$$

for all $\gamma \in \widehat{G}$.

Definition 2.2. Let X be any locally compact Hausdorff space. Then we define $C_0(X, \mathbb{BC})$ as the collection of bicomplex valued functions $h = e_1 h_1 + e_2 h_2$ on X such that $h_1 \in C_0(X)$ and $h_2 \in C_0(X)$.

Proposition 2.1. The set of functions $C_0(X, \mathbb{BC})$ is taken as a $\mathbb{BC}$ -module, utilizing the standard addition operation for functions along with bicomplex scalar multiplication.

Proof. Let $g, h \in C_0(X, \mathbb{BC})$ such that $g = e_1 g_1 + e_2 g_2$ and $h = e_1 h_1 + e_2 h_2$. Consider $\alpha = e_1 \alpha_1 + e_2 \alpha_2 \in \mathbb{BC}$. Obviously, $\alpha_1, \alpha_2 \in \mathbb{C}$. By the definition of $C_0(X, \mathbb{BC})$, we may write $g_i, h_i \in C_0(X)$ for $i = 1, 2$. Since the space $C_0(X)$ is a linear space, then we have

$$g_1 + h_1, g_2 + h_2, \alpha_1 g_1, \alpha_2 g_2 \in C_0(X). \quad (2.4)$$

Besides, we get

$$(g + h) = e_1 (g_1 + h_1) + e_2 (g_2 + h_2) \quad (2.5)$$

and

$$\alpha g = e_1 \alpha_1 g_1 + e_2 \alpha_2 g_2. \quad (2.6)$$

Combining (2.4), (2.5) and (2.6), we obtain $g + h \in C_0(X, \mathbb{BC})$ and $\alpha g \in C_0(X, \mathbb{BC})$. Similarly, it is easy to exhibit various characteristics of the module. $\square$

Remark 2.2. By the definition of the $\mathbb{BC}$ -module $C_0(X, \mathbb{BC})$, we may write (see [1, 16, 7])

$$C_0(X, \mathbb{BC}) = e_1 C_0(X) + e_2 C_0(X).$$

Obviously, the space $C_0(X)$ is a normed space with norm

$$\|h_i\|_{\infty} = \sup_{x \in X} |h_i(x)|$$

for $h_i \in C_0(X)$ with $i = 1, 2$. Let $h \in C_0(X, \mathbb{BC})$, where $h = e_1 h_1 + e_2 h_2$. Hence we can endow the $\mathbb{BC}$ -module $C_0(X, \mathbb{BC})$ with hyperbolic ($\mathbb{D}$ -valued) norm that given by

$$\|h\|_{\infty, \mathbb{D}} = e_1 \|h_1\|_{\infty} + e_2 \|h_2\|_{\infty}.$$

Theorem 2.1. The $\mathbb{BC}$ -Fourier transform of any function $h = e_1 h_1 + e_2 h_2 \in L^1_{\mathbb{BC}}(G, \lambda)$ belongs to $C_0(\widehat{G}, \mathbb{BC})$ and the inequality $\|\mathcal{F}_{\mathbb{BC}}h\|_{\infty, \mathbb{D}} \lesssim \|h\|_{1, \mathbb{D}}$ holds.

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $h = e_1 h_1 + e_2 h_2$. Then $h_1, h_2 \in L^1(G, \mu)$. By the definition of the $\mathbb{BC}$ -Fourier transform, we write

$$\mathcal{F}_{\mathbb{BC}} h = e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2. \quad (2.7)$$

It is known that Fourier transforms of the functions h_1, h_2 belong to $C_0(\widehat{G})$, [21, 1.2.4. Theorem]. Therefore the $\mathbb{BC}$ -Fourier transform of the function h belongs to $C_0(\widehat{G}, \mathbb{BC})$ by Definition 2.2. Since $h_1, h_2 \in L^1(G, \mu)$, then the inequalities

$$\|\mathcal{F} h_1\|_{\infty} \leq \|h_1\|_1 \quad (2.8)$$

and

$$\|\mathcal{F} h_2\|_{\infty} \leq \|h_2\|_1 \quad (2.9)$$

hold (see [21, Theorem 1.2.4]). Combining (2.7), (2.8) and (2.9), we get

$$\begin{aligned} \|\mathcal{F}_{\mathbb{BC}} h\|_{\infty, \mathbb{D}} &= \|e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2\|_{\infty, \mathbb{D}} \\ &= e_1 \|\mathcal{F} h_1\|_{\infty} + e_2 \|\mathcal{F} h_2\|_{\infty} \\ &\lesssim e_1 \|h_1\|_1 + e_2 \|h_2\|_1 = \|h\|_{1, \mathbb{D}}. \end{aligned}$$

□

Proposition 2.2. *The $\mathbb{BC}$ -Fourier transform from $L^1_{\mathbb{BC}}(G, \lambda)$ into $C_0(\widehat{G}, \mathbb{BC})$ is linear.*

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $v, \eta \in \mathbb{BC}$. Then we have $g = e_1 g_1 + e_2 g_2$ and $h = e_1 h_1 + e_2 h_2$, where $g_i, h_i \in L^1(G, \mu)$ with $i = 1, 2$. Hence, we can write

$$\mathcal{F}_{\mathbb{BC}}(g + h) = e_1 \mathcal{F}(g_1 + h_1) + e_2 \mathcal{F}(g_2 + h_2)$$

By using the linearity of the Fourier transform $\mathcal{F}$, we obtain

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(g + h) &= e_1 \mathcal{F}(g_1 + h_1) + e_2 \mathcal{F}(g_2 + h_2) \\ &= e_1 (\mathcal{F} g_1 + \mathcal{F} h_1) + e_2 (\mathcal{F} g_2 + \mathcal{F} h_2) \\ &= (e_1 \mathcal{F} g_1 + e_2 \mathcal{F} g_2) + (e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2) \\ &= \mathcal{F}_{\mathbb{BC}} g + \mathcal{F}_{\mathbb{BC}} h. \end{aligned}$$

Let us take $\mathfrak{c} = e_1 c_1 + e_2 c_2 \in \mathbb{BC}$. Therefore, we have

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\mathfrak{c}h) &= \mathcal{F}_{\mathbb{BC}}((e_1 c_1 + e_2 c_2)(e_1 h_1 + e_2 h_2)) \\ &= \mathcal{F}_{\mathbb{BC}}(e_1(c_1 h_1) + e_2(c_2 h_2)) \\ &= e_1 \mathcal{F}(c_1 h_1) + e_2 \mathcal{F}(c_2 h_2). \end{aligned}$$

By the linearity property of the Fourier transform $\mathcal{F}$, we conclude

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\mathfrak{c}h) &= e_1 \mathcal{F}(c_1 h_1) + e_2 \mathcal{F}(c_2 h_2) \\ &= e_1 c_1 \mathcal{F} h_1 + e_2 c_2 \mathcal{F} h_2 \\ &= (e_1 c_1 + e_2 c_2)(e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2) \\ &= \mathfrak{c} \mathcal{F}_{\mathbb{BC}} h. \end{aligned}$$

□

Definition 2.3. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $g = e_1g_1 + e_2g_2$ and $h = e_1h_1 + e_2h_2$. Then the $\ast'$ convolution of the functions g, h is

$$g \ast' h = e_1 (g_1 \ast h_1) + e_2 (g_2 \ast h_2), \quad (2.10)$$

where $\ast$ is usual convolution on $L^1(G, \mu)$.

Proposition 2.3. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we have

$$\mathcal{F}_{\mathbb{BC}}(g \ast' h) = (\mathcal{F}_{\mathbb{BC}}g)(\mathcal{F}_{\mathbb{BC}}h).$$

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we can write $g = e_1g_1 + e_2g_2$ and $h = e_1h_1 + e_2h_2$, where $g_i, h_i \in L^1(G, \mu)$ with $i = 1, 2$. It is known that the Fourier transform of usual convolution between two functions in $L^1(G, \mu)$ equals the product of their Fourier transforms (see [21]). Therefore, we have

$$\mathcal{F}(g_1 \ast h_1) = \mathcal{F}g_1\mathcal{F}h_1 \quad (2.11)$$

and

$$\mathcal{F}(g_2 \ast h_2) = \mathcal{F}g_2\mathcal{F}h_2. \quad (2.12)$$

By using the definition of $\mathbb{BC}$ -Fourier transform, (2.10), (2.11) and (2.12), we get

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(g \ast' h) &= e_1\mathcal{F}(g_1 \ast h_1) + e_2\mathcal{F}(g_2 \ast h_2) \\ &= e_1\mathcal{F}g_1\mathcal{F}h_1 + e_2\mathcal{F}g_2\mathcal{F}h_2 \\ &= (e_1\mathcal{F}g_1 + e_2\mathcal{F}g_2)(e_1\mathcal{F}h_1 + e_2\mathcal{F}h_2) \\ &= (\mathcal{F}_{\mathbb{BC}}g)(\mathcal{F}_{\mathbb{BC}}h). \end{aligned}$$

□

Theorem 2.2 (Uniqueness Theorem). A function $h \in L^1_{\mathbb{BC}}(G, \lambda)$ is uniquely determined almost everywhere by its $\mathbb{BC}$ -Fourier transform.

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we have idempotent representations $h = e_1h_1 + e_2h_2$ and $\mathcal{F}_{\mathbb{BC}}h = e_1\mathcal{F}h_1 + e_2\mathcal{F}h_2$, where $h_1, h_2 \in L^1(G, \mu)$ and $\mathcal{F}h_1, \mathcal{F}h_2 \in C_0(\widehat{G})$. It is enough to show that if $\mathcal{F}_{\mathbb{BC}}h(\gamma) = 0$ for all $\gamma \in \widehat{G}$, then $h(x) = 0$ almost everywhere on G . Suppose that $\mathcal{F}_{\mathbb{BC}}h(\gamma) = 0$ for all $\gamma \in \widehat{G}$. By the definition of the $\mathbb{BC}$ -Fourier transform, we have

$$\mathcal{F}_{\mathbb{BC}}h(\gamma) = e_1\mathcal{F}h_1(\gamma) + e_2\mathcal{F}h_2(\gamma) = 0,$$

for all $\gamma \in \widehat{G}$. This means that

$$\mathcal{F}h_1(\gamma) = 0$$

and

$$\mathcal{F}h_2(\gamma) = 0,$$

for all $\gamma \in \widehat{G}$. By the uniqueness theorem of Fourier transform (see, [21]), we may write

$$h_1(x) = 0 \quad (2.13)$$

and

$$h_2(x) = 0, \quad (2.14)$$

almost everywhere on G . Using idempotent representation of the function h and combining (2.13) and (2.14), we obtain

$$h(x) = e_1 h_1(x) + e_2 h_2(x) = 0,$$

almost everywhere on G . □

Remark 2.3. Let us take $\mathbb{D}$ -Haar measure $\nu = e_1 \nu + e_2 \nu$ on $\widehat{G}$ such that ν is a unique Haar measure on $\widehat{G}$. Therefore, normalization of $\mathbb{D}$ -Haar measure ν means normalization of Haar measure ν due to the idempotent representation of $\mathbb{D}$ -Haar measure ν .

Theorem 2.3 (Inversion Theorem). Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $\mathcal{F}_{\mathbb{BC}} h \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. If a suitable normalized $\mathbb{D}$ -Haar measure ν is obtained for a fixed $\mathbb{D}$ -Haar measure λ , then

$$\begin{aligned} h(x) &= e_1 \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) + e_2 \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \\ &= e_1 \mathcal{F}^{-1}h_1(x) + e_2 \mathcal{F}^{-1}h_2(x), \end{aligned}$$

holds almost everywhere on G .

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $\mathcal{F}_{\mathbb{BC}} h \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. Thus we have idempotent representations $h = e_1 h_1 + e_2 h_2$ and $\mathcal{F}_{\mathbb{BC}} h = e_1 \mathcal{F}h_1 + e_2 \mathcal{F}h_2$, where $h_1, h_2 \in L^1(G, \mu)$ and $\mathcal{F}h_1, \mathcal{F}h_2 \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. Assume that there exists a suitable normalized $\mathbb{D}$ -Haar measure ν is obtained for a fixed $\mathbb{D}$ -Haar measure λ . By the inversion theorem in [19, Remark 4.4.8], the equalities

$$h_1(x) = \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \quad (2.15)$$

and

$$h_2(x) = \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \quad (2.16)$$

hold almost everywhere on G . We utilize the proofs of Fourier Inversion Theorem I and Fourier Inversion Theorem II in [6] to understand the proof of the inversion theorem for the ordinary Fourier transform on the locally compact abelian groups. Using idempotent representation of the function h and combining (2.15) and (2.16), we get

$$\begin{aligned} h(x) &= e_1 \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) + e_2 \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \\ &= e_1 \mathcal{F}^{-1}h_1(x) + e_2 \mathcal{F}^{-1}h_2(x), \end{aligned}$$

almost everywhere on G . □

Definition 2.4. The character operator is defined in the following manner using Duality theorem (for Duality theorem see, [19, Theorem 4.2.11]):

- (1) Let us take an element γ from $\widehat{G}$ and fix it. Then the function χ_γ from G into $\mathbb{C}$ is defined as

$$\chi_\gamma(x) = \langle x, \gamma \rangle \quad (2.17)$$

for all $x \in G$.

- (2) Let us take an element x from G and fix it. Then the function χ_x from $\widehat{G}$ into $\mathbb{C}$ is defined as

$$\chi_x(\gamma) = \langle x, \gamma \rangle \quad (2.18)$$

for all $\gamma \in \widehat{G}$.

Let us now provide the following proposition expressing some algebraic properties of the $\mathbb{BC}$ -Fourier transform:

Proposition 2.4. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$.

- (1) Let $x \in G$. Then

$$\mathcal{F}_{\mathbb{BC}}(T_x g) = \chi_{-x} \mathcal{F}_{\mathbb{BC}} g. \quad (2.19)$$

- (2) Let $\gamma \in \widehat{G}$. Then

$$\mathcal{F}_{\mathbb{BC}}(\chi_\gamma g) = T_\gamma \mathcal{F}_{\mathbb{BC}} g. \quad (2.20)$$

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$.

- (1) Let $x \in G$. The equality

$$\mathcal{F}(T_x g)(\gamma) = \chi_{-x} \mathcal{F}g(\gamma)$$

holds for all $\gamma \in \widehat{G}$, which is well known by [19, Proposition 4.4.10]. By applying the equality (2.18), we may write

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(T_x g)(\gamma) &= e_1 \mathcal{F}(T_x g_1)(\gamma) + e_2 \mathcal{F}(T_x g_2)(\gamma) \\ &= e_1 \chi_{-x} \mathcal{F}g_1(\gamma) + e_2 \chi_{-x} \mathcal{F}g_2(\gamma) \\ &= \langle -x, \gamma \rangle (e_1 \mathcal{F}g_1(\gamma) + e_2 \mathcal{F}g_2(\gamma)) \\ &= \langle -x, \gamma \rangle \mathcal{F}_{\mathbb{BC}} g(\gamma) = \chi_{-x} \mathcal{F}_{\mathbb{BC}} g(\gamma) \end{aligned}$$

for all $\gamma \in \widehat{G}$. Hence, the equation (2.19) is satisfied.

- (2) Let $\gamma \in \widehat{G}$. As stated in [19, Proposition 4.4.10], the equality

$$\mathcal{F}(\chi_\gamma g) = T_\gamma \mathcal{F}g$$

is valid. Then we have

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\chi_\gamma g) &= e_1 \mathcal{F}(\chi_\gamma g_1) + e_2 \mathcal{F}(\chi_\gamma g_2) \\ &= e_1 T_\gamma \mathcal{F}g_1 + e_2 T_\gamma \mathcal{F}g_2. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\chi_\gamma g)(\eta) &= e_1 (T_\gamma \mathcal{F}g_1)(\eta) + e_2 (T_\gamma \mathcal{F}g_2)(\eta) \\ &= e_1 (T_\gamma \mathcal{F}g_1)(\eta) + e_2 (T_\gamma \mathcal{F}g_2)(\eta) \\ &= (e_1 \mathcal{F}g_1 + e_2 \mathcal{F}g_2)(\eta - \gamma) \\ &= \mathcal{F}_{\mathbb{BC}} g(\eta - \gamma) = T_\gamma \mathcal{F}_{\mathbb{BC}} g(\eta) \end{aligned}$$

for all $\eta \in \widehat{G}$. Therefore, the equation (2.20) is fulfilled.

□

Consider the additive group of real numbers $(\mathbb{R}, +)$ as a locally compact abelian group G . Let's use the Lebesgue measure on $\mathbb{R}$ instead of the Haar measures in the previously

mentioned $\mathbb{D}$ -Haar measure. Let's determine the $\mathbb{BC}$ -Fourier transforms of certain functions in $L^1_{\mathbb{BC}}(G, \lambda)$ space. Regarding the kernel of the Fourier transform, the literature is varied. In particular, we will consider the $K(w, t) = \exp(-iwt)$ function as the kernel for $t \in \mathbb{R}$ and $w \in \widehat{\mathbb{R}} = \mathbb{R}$.

Example 2.1. Let $a, b \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{C}$, $c_1 \neq c_2$. Consider a function $f(t) = c_1 e_1 + c_2 e_1$ for all $t \in [a, b]$. Let us take functions $f_1(t) = c_1$ and $f_2(t) = c_2$ for all $t \in [a, b]$. Therefore, we have $f(t) = c_1 e_1 + c_2 e_2$ for all $t \in [a, b]$. Let dt be a Lebesgue measure on $\mathbb{R}$. Since constant functions are measurable functions in the sense of the Lebesgue measure on $\mathbb{R}$ and

$$\int_a^b |f_j(t)| dt = \int_a^b |c_j| dt = |c_j| (b - a) < \infty,$$

for $j = 1, 2$, then these functions are elements of the space $L^1(\mathbb{R}, dt)$. Hence, the function f is a member of the space $L^1_{\mathbb{BC}}(\mathbb{R}, dx)$, and its $\mathbb{BC}$ -Fourier transform can be described. Then we obtain

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}} f(w) &= e_1 \mathcal{F} f_1(w) + e_2 \mathcal{F} f_2(w) \\ &= e_1 \int_a^b f_1(t) \exp(-iwt) dt + e_2 \int_a^b f_2(t) \exp(-iwt) dt \\ &= e_1 c_1 \int_a^b \exp(-iwt) dt + e_2 c_2 \int_a^b \exp(-iwt) dt \\ &= e_1 c_1 \frac{i}{w} (\exp(-ibt) - \exp(-iat)) + e_2 c_2 \frac{i}{w} (\exp(-ibt) - \exp(-iat)) \\ &= \frac{i}{w} (\exp(-ibt) - \exp(-iat)) (e_1 c_1 + e_2 c_2) \end{aligned}$$

for all $w \in \mathbb{R} - \{0\}$.

Example 2.2. Let a and b are positive real numbers such that $a \neq b$. Let us take the Gaussian functions $h_1(t) = \exp(-at^2)$ and $h_2(t) = \exp(-bt^2)$ for all $t \in \mathbb{R}$. Gaussian functions are also measurable functions in the sense of the Lebesgue measure on $\mathbb{R}$, as they are continuous on the set of all real numbers. The integrals of the Gaussian functions over $\mathbb{R}$ are calculated as

$$\int_{-\infty}^{+\infty} \exp(-at^2) dt = \sqrt{\frac{\pi}{a}} \text{ and } \int_{-\infty}^{+\infty} \exp(-bt^2) dt = \sqrt{\frac{\pi}{b}}.$$

Therefore, $h_1, h_2 \in L^1(\mathbb{R}, dt)$. By utilizing these functions, let us construct a function h as

$$h(t) = e_1 h_1(t) + e_2 h_2(t) = e_1 \exp(-at^2) + e_2 \exp(-bt^2)$$

for all $t \in \mathbb{R}$. We will now compute the ordinary Fourier transform of the function h_1 . We write

$$\begin{aligned}\mathcal{F}h_1(w) &= \int_{-\infty}^{+\infty} \exp(-at^2) \exp(-iwt) dt \\ &= \int_{-\infty}^{+\infty} \exp(-(at^2 + iwt)) dt \\ &= \int_{-\infty}^{+\infty} \exp\left(-\left(\sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}\right)^2 - \frac{w^2}{4a}\right) dt \\ &= \exp\left(-\frac{w^2}{4a}\right) \int_{-\infty}^{+\infty} \exp\left(-\left(\sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}\right)^2\right) dt.\end{aligned}$$

If we make the substitution $u = \sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}$, then we have

$$\mathcal{F}h_1(w) = \frac{1}{\sqrt{a}} \exp\left(-\frac{w^2}{4a}\right) \int_{-\infty}^{+\infty} \exp(-u^2) du.$$

Thus, the integral $\int_{-\infty}^{+\infty} \exp(-u^2) du$ provides the area of the region under the bell-curve $\exp(-u^2)$, and its value is known as $\sqrt{\pi}$. Hence, we obtain the Fourier transforms of the Gaussian function h_1 is given as

$$\mathcal{F}h_1(w) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{-w^2}{4a}\right),$$

and so

$$\mathcal{F}h_2(w) = \sqrt{\frac{\pi}{b}} \exp\left(\frac{-w^2}{4b}\right)$$

is meaningful. Therefore, we get

$$\begin{aligned}\mathcal{F}_{\mathbb{BC}}h(w) &= e_1\mathcal{F}h_1(w) + e_2\mathcal{F}h_2(w) \\ &= e_1\sqrt{\frac{\pi}{a}} \exp\left(\frac{-w^2}{4a}\right) + e_2\sqrt{\frac{\pi}{b}} \exp\left(\frac{-w^2}{4b}\right)\end{aligned}$$

for all $w \in \mathbb{R}$.

We designate $\mathbb{R}$ as the additive group of real numbers and $\mathbb{Z}$ as the subgroup that consists of the integers. The definition of the group $\mathbb{T}$ is $\mathbb{Z}$ the quotient $\mathbb{R}/2\pi\mathbb{Z}$, where $2\pi\mathbb{Z}$ is the group of integer multiples of 2π , as shown in the notation. The functions on $\mathbb{T}$ and 2π -periodic functions on $\mathbb{R}$ are readily identifiable. For comprehensive details regarding the properties of the group $\mathbb{T}$, please consult [12]. Additionally, the n -dimensional form of the group $\mathbb{T}$ is denoted as $\mathbb{T}^n$ and is described as $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Besides, the restriction of the Lebesgue measure of $\mathbb{R}^n$ to $[0, 2\pi)^n$ is the Lebesgue measure dt on $\mathbb{T}^n$. Now, consider $\mathbb{T}^2$ as a locally compact abelian group G . Additionally, the dual group of group $G = \mathbb{T}^2$ turns into group

$\widehat{G} = \mathbb{Z}^2$. The Fourier transform of a function f in the space $L^1(\mathbb{T}^2, dt)$ is of the form

$$\mathcal{F}g(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} g(t_1, t_2) \exp(-i(t_1 k_1 + t_2 k_2)) dt_1 dt_2$$

for all $k = (k_1, k_2) \in \mathbb{Z}^2$ (see, [22, Chapter 7]).

Example 2.3. Let $l, m \in \mathbb{Z}$ and g be a bicomplex trigonometric wave of type

$$g(t_1, t_2) = e_1 \exp(it_1 l) + e_2 \exp(it_2 m)$$

for all $t_1, t_2 \in \mathbb{T}$ that is given in [5, Subsection 3.1]. Since the functions $g_1(t_1, t_2) = \exp(it_1 l)$ and $g_2(t_1, t_2) = \exp(it_2 m)$ are continuous functions, then these functions are measurable functions on $\mathbb{T}^2$. Also, we have

$$\int_0^{2\pi} \int_0^{2\pi} |\exp(it_1 l)| dt_1 dt_2 = 4\pi^2 \text{ and } \int_0^{2\pi} \int_0^{2\pi} |\exp(it_2 m)| dt_1 dt_2 = 4\pi^2.$$

Therefore, the functions g_1 and g_2 belong to $L^1(\mathbb{T}^2, dt)$. Now, we will calculate the $\mathbb{BC}$ -Fourier transform of the bicomplex trigonometric wave g . Firstly, we compute the ordinary Fourier transform of the function g_1 . Let us take $k = (k_1, k_2) \in \mathbb{Z}^2$. We write

$$\mathcal{F}g_1(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \exp(it_1 l) \exp(-i(t_1 k_1 + t_2 k_2)) dt_1 dt_2.$$

By the Fubini's theorem, we get

$$\mathcal{F}g_1(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \exp(-it_2 k_2) \left(\int_0^{2\pi} \exp(it_1 (l - k_1)) dt_1 \right) dt_2$$

Let $k_2 \neq 0$ and $k_1 \neq l$. Then we have

$$\mathcal{F}g_1(k) = \frac{1}{4\pi^2} \left(\frac{i}{k_1 - l} (\exp(2\pi i (l - k_1)) - 1) \right) \int_0^{2\pi} \exp(-it_2 k_2) dt_2.$$

Since $l - k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$, then we conclude

$$\mathcal{F}g_1(k_1, k_2) = 0.$$

If $k_2 = 0$ and $k_1 = l$. Then we have

$$\mathcal{F}g_1(k_1, k_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \exp(-it_2 k_2) \left(\int_0^{2\pi} \exp(it_1 (l - k_1)) dt_1 \right) dt_2 = 1.$$

Hence, we obtain

$$\mathcal{F}g_1(k_1, k_2) = \begin{cases} 1, & k_1 = l \text{ and } k_2 = 0 \\ 0, & \text{otherwise} \end{cases}. \quad (2.21)$$

Similarly, the ordinary Fourier transform of the function g_2 is calculated as

$$\mathcal{F}g_2(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \exp(-it_1 k_1) \left(\int_0^{2\pi} \exp(it_2(m - k_2)) dt_2 \right) dt_1.$$

Let $k_1 \neq 0$ and $k_2 \neq m$. Then we have

$$\mathcal{F}g_2(k) = \frac{1}{4\pi^2} \left(\frac{i}{k_2 - m} (\exp(2\pi i(m - k_2)) - 1) \right) \int_0^{2\pi} \exp(-it_1 k_1) dt_1.$$

Since $m - k_2 \in \mathbb{Z}$ and $k_1 \in \mathbb{Z}$, then we conclude

$$\mathcal{F}g_2(k_1, k_2) = 0.$$

If $k_1 = 0$ and $k_2 = m$. Then we have

$$\mathcal{F}g_1(k_1, k_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \exp(-it_1 k_1) \left(\int_0^{2\pi} \exp(it_2(m - k_2)) dt_2 \right) dt_1 = 1.$$

Hence, we obtain

$$\mathcal{F}g_2(k_1, k_2) = \begin{cases} 1, & k_1 = 0 \text{ and } k_2 = m \\ 0, & \text{otherwise} \end{cases}. \quad (2.22)$$

By combining (2.21) and (2.22), we calculate the $\mathbb{BC}$ -Fourier transform of the bicomplex trigonometric wave g as

$$\mathcal{F}_{\mathbb{BC}}g(k_1, k_2) = e_1 \mathcal{F}g_1(k_1, k_2) + e_2 \mathcal{F}g_2(k_1, k_2).$$

Remark 2.4. Let us take the complex-valued function $g \in L^1(G, \mu)$. We can write this function as the following idempotent representation:

$$g(x) = e_1 g(x) + e_2 g(x)$$

almost everywhere on G . By the definition of the $\mathbb{BC}$ -Fourier transform, we write

$$\mathcal{F}_{\mathbb{BC}}g(\gamma) = e_1 \mathcal{F}g(\gamma) + e_2 \mathcal{F}g(\gamma) = \mathcal{F}g(\gamma)(e_1 + e_2) = \mathcal{F}g(\gamma)$$

for all $\gamma \in \widehat{G}$. Hence, the $\mathbb{BC}$ -Fourier transform of the complex-valued function $g \in L^1(G, \mu)$ corresponds to the ordinary Fourier transform of the function g .

3. DISCUSSION

The kernel function of the known Fourier transform is generalized in the bicomplex sense when looking at [2], which is the fundamental work that defines the bicomplex Fourier transform. Furthermore, the bicomplex Fourier transform is exclusively defined for real-valued functions in [2, Definition 3.2]. Nevertheless, the literature does not contain a valid Fourier transform for bicomplex-valued functions. In particular, the Fourier transform of the bicomplex trigonometric wave that is a bicomplex-valued function, presented in Example 2.3, which means that it cannot be calculated using the bicomplex Fourier transform defined in [2]. As a result, this paper's definition of the $\mathbb{BC}$ -Fourier transform is entirely distinct from that of [2]. Additionally, as the examples show, both functions defined on $\mathbb{T}$ and $\mathbb{R}$ are applicable to the $\mathbb{BC}$ -Fourier transform defined in this paper. The $\mathbb{BC}$ -Fourier transform

provided here yields the answer for bicomplex-valued functions defined on $\mathbb{T}^n$ and $\mathbb{R}^n$ in its most general form. Moreover, Remark 2.4 asserts that the BC-Fourier transform of a complex-valued function defined on a locally compact abelian group G corresponds to the known Fourier transform, which implies that the BC-Fourier transform is a generalization of the known Fourier transform.

4. CONCLUSION

Specifically, an analysis of the Fourier transforms used in the study of harmonic analysis shows that they are defined on locally compact abelian groups. This paper introduces the definition of a BC-Fourier transform for hyperbolic norm integrable functions that have bicomplex values and are specified on a locally compact abelian group. Additionally, the paper includes several examples to illustrate this concept. Furthermore, the paper explores the properties of the BC-Fourier transform in harmonic analysis. Within the field of harmonic analysis, this research will serve as a foundation for forecasting that the results achieved with the well-known Fourier transform can also be achieved with the BC-Fourier transform, providing insights for future investigations.

Conflict of interest. The authors declare no potential conflict of interests.

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(E. Toksoy) ONDOKUZ MAYIS UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, SAMSUN, TURKEY