



EULER AND CHEN-INVARIANT TYPE INEQUALITIES ON SUBMANIFOLDS IN GOLDEN STATISTICAL MANIFOLDS

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Abstract. The purpose of this research paper is to analyze the statistical manifolds coupled with Golden structure. On golden-like statistical manifolds, we derive several basic attributes for the intrinsic and extrinsic invariants of statistical submanifolds in golden-like statistical manifolds. Also, we derive an Euler inequality for submanifolds in golden-like statistical manifolds. Moreover, we used particular types of second fundamental forms of Chen-invariant type inequalities for these classes of submanifolds in golden-like statistical manifolds to characterize the equality case. We strengthen our idea with an example as well.

Keywords: Statistical manifolds, Submanifolds, Euler inequality, Chen-invariant.

MSC (2020): 53C15, 53C20, 53C25, 53C42.

1. INTRODUCTION

Statistical manifolds are fundamental in the field of information geometry, a branch of mathematics that studies the geometric forms of probability distribution. These manifolds provide a framework for analyzing statistical models and estimating parameters in a system that considers the core geometry of the statistical space. This is the situation of normal linear regression that is statistically significant. The statistical framework of classical information geometry can be seen as a means of expanding the geometric basis of regression. Refer to Vos and Marriott (2010) for wider settings [46].

Statisticians, have mostly concentrated on applications pertaining to contemporary statistical practice, however, as the introduction illustrates, information geometry has expanded to a wide range of fields and has significantly influenced numerous other fields. Naturally, there are many applications for the general concepts of geometric dualistic structures and ideas of divergence that we have explored here [46].

Amari [3, 4] examines how divergences can be effectively utilized to fit neural network models. It estimates hidden layers in a neural network using a dual geometric version in

Received: 2026.01.29

Revised: 2026.06.24

Accepted: 2026.08.24

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information geometry . It specifically makes use of the normal mixture model, expert mixtures, and stochastic multilayer perceptron models.

Statistical manifolds act for the compilation of probability distributions, constantly equipped with a Riemannian metric that quantifies the directions and distances in this space. The Riemannian metric determines the curvature and structure of the manifold, activating the development of logical statistical methods adapted to the underlying geometry. One of the central persons in the evolution of information geometry and statistical manifolds is Shun-ichi Amari [3, 4]. Amari's study has been instrumental in establishing the theoretical basics of this pasture and has paved the method for applications in numerous fields, including machine learning, signal processing and physics. Golden-like statistical manifolds are a specialized category of statistical manifolds that have possession similar to the golden geometry, known for its exceptional mathematical attributes, and the golden structure itself has its algebraic roots in the broader theory of polynomial structures on manifolds [18]. The phrase "golden-like" mentions to structures with certain geometric possessions or connections to the golden ratio, a mathematical constant that become visible in numerous natural occurrence and creative endeavors, with further applications of the golden ratio on Riemannian manifolds discussed in [21]. Statistical submanifolds in golden-like statistical structure has practical inferences. Firstly, in model selection, by scouting submanifolds, practitioners can better choose statistical models that describe data without overfitting, especially in high-dimensional spaces. Secondly, in Bayesian inference, Bayesian techniques can benefit from the geometric viewpoints provided by submanifolds, particularly in specifying prior distributions that regard the geometric form of the underlying statistical manifold. Thirdly, in machine learning, algorithms operating manifold learning methods can utilize submanifolds to catch complex data patterns while staying computationally feasible. [3, 4, 8, 34].

Intrinsic invariants are possessions that are inherent to a geometric item and are independent of how the item appears from the outside. For instance, the curvature properties of a surface or the length of a curve. Such invariants are directly related to the geometric structure of the surface or curve. Extrinsic invariants are related to the position of a geometric item in space. That is, these possessions are related to the item's position and association to other items in its domain. For instance, how a surface is laid out in space or the position of a curve in a plane. Likewise invariants can be identified by outer observations and measurements. Presently, in the comparison relationship problems between the intrinsic and extrinsic invariants, statistical manifolds are being studied very actively. Golden semi-Riemannian manifolds are an interesting section of study in modern mathematics, combining classical geometrical structures with the unique possessions of the golden ratio. This structure generates a major field for further research, promising insights that transcend conventional frontiers in mathematics and physics. In [9], B.-Y. Chen discovered new intrinsic curvature invariants, very popularly known as Chen's invariants and derived optimal relationships between these intrinsic invariants and the extrinsic invariants, a line of study later extended to the Lagrangian setting [32] and to pseudo-Riemannian ambient spaces via δ -invariants [12]. Currently, this is a very active area of research [9, 42, 43].

In [2], Amari introduced the concept of statistical manifolds via information geometry. An analogue to conjugate connections in affine geometry (see [29]), statistical manifolds

are endowed with a pair of dual torsion-free connections. Generally sectional curvature of a statistical manifold is not very easy to define. Thus, in [31], B. Opozda showed how to define sectional curvature of a statistical manifold, building on Bochner-type techniques for statistical structures developed earlier [30]. The relationships between the intrinsic and extrinsic curvature invariants are very important in submanifold theory. In this connection, B.-Y. Chen obtained the relationship between the Gaussian and the mean curvatures [10], [11]. In [40], [41], K. Takano studied the statistical submersions with certain structures. In [44], A.D. Vîlcu and G.E. Vîlcu examined the statistical submanifolds of quaternionic statistical manifolds. In the same paper, they proposed several open problems. For similar studies on curvature inequalities for submanifolds and Casorati curvatures in various ambient spaces, we refer [5], [26], [27], [7], [14], [38], [39], [25], [29], [36], [1], [37], [6], [13], [16], [17], [19], [22], [23], [24], [33], [45], [47]. Now coming towards the actual theme of the current paper. In one of their earlier submitted papers, the first and second author of this paper defined the Golden-like statistical manifolds and obtained the optimal inequalities using Casorati curvatures.

In this paper, we obtain some sharp relationships between extrinsic and intrinsic invariants for statistical submanifolds of Golden-like statistical manifolds. At the end, we talk about their equality cases.

2. SOME BASIC CONCEPTS

Definition 2.1. [15, 20, 35] *Let (F, μ) be a semi-Riemannian manifold and let ϱ be $(1, 1)$ tensor field on F . If ϱ satisfies the following equality*

$$\varrho^2 = \varrho + I. \quad (2.1)$$

At that point the tensor field ϱ is named a Golden structure on F . On condition that the metric μ is ϱ compatible, the (F, μ, ϱ) is named a Golden semi-Riemannian manifold.

The following equalities are true for the ϱ compatible metric g

$$\mu(\varrho X_1, Y_1) = \mu(X_1, \varrho Y_1), \quad (2.2)$$

$$\begin{aligned} \mu(\varrho X_1, \varrho Y_1) &= \mu(\varrho^2 X_1, Y_1) \\ &= \mu(\varrho X_1, Y_1) + \mu(X_1, Y_1), \quad X_1, Y_1 \in TF. \end{aligned} \quad (2.3)$$

Let F be a semi-Riemannian manifold and $\hat{\nabla}$ be the torsion-free affine connection. The triple $(F, \hat{\nabla}, \mu)$ is named a statistical manifold if $\hat{\nabla}\mu$ is symmetric. Let's define a new ∇^* affine connection as follows:

$$X_1\mu(Y_1, Z_1) = \mu(\hat{\nabla}_{X_1} Y_1, Z_1) + \mu(\hat{\nabla}_{X_1}^* Z_1, Y_1), \quad (2.4)$$

for any vector fields X_1, Y_1 and Z_1 on F . The affine connection $\hat{\nabla}^*$ is called conjugate to $\hat{\nabla}$ in accordance with μ . The affine connection $\hat{\nabla}^*$ is torsion-free, $\hat{\nabla}^*\mu$ is symmetric and provides $\hat{\nabla}^0 = \frac{\hat{\nabla} + \hat{\nabla}^*}{2}$. We indicate by Ξ and Ξ^* the curvature tensors on F in accordance with the affine connection $\hat{\nabla}$ and its conjugate $\hat{\nabla}^*$, respectively. Moreover, the curvature tensor field $\hat{\Xi}^0$ associated with $\hat{\nabla}^0$ is called semi-Riemannian curvature tensor. In this case the following

relationship is given

$$\mu(\hat{\Xi}(X_1, Y_1)Z_1, W_1) = -\mu(Z_1, \hat{\Xi}^*(X_1, Y_1)W_1),$$

, where $\hat{\Xi}(X_1, Y_1)Z_1 = [\hat{\nabla}_{X_1}, \hat{\nabla}_{Y_1}]Z_1 - \hat{\nabla}_{[X_1, Y_1]}Z_1$.

The following definition is given for the statistical structure $(\hat{\nabla}, g)$ on F :

$$\hat{S}(X_1, Y_1)Z_1 = \frac{1}{2}[\hat{\Xi}(X_1, Y_1)Z_1 + \hat{\Xi}^*(X_1, Y_1)Z_1], \quad (2.5)$$

for any $X_1, Y_1, Z_1 \in TF$.

Usually, dual connections are not metric. as in the instance of semi-Riemannian geometry, section curvature cannot be defined in the statistical setting. Therefore, Oposda proposed two presumptions of section curvature in this type manifolds (see [30, 31]).

Let F be a statistical manifold and π a plane section in TF with orthonormal basis $\{X_1, Y_1\}$, in that case the sectional K -curvature is defined in [31] as

$$\hat{K}(\pi) = \frac{1}{2} \left[\mu(\hat{\Xi}(X_1, Y_1)Y_1, X_1) + \mu(\hat{\Xi}^*(X_1, Y_1)Y_1, X_1) - 2\mu(\hat{\Xi}^0(X_1, Y_1)Y_1, X_1) \right].$$

Definition 2.2. Suppose that (F, μ, ϱ) is a Golden semi-Riemannian manifold equipped with a tensor field ϱ^* of type $(1, 1)$ providing

$$\mu(\varrho X_1, Y_1) = \mu(X_1, \varrho^* Y_1), \quad (2.6)$$

for vector fields X_1 and Y_1 on F . In view of (2.6), we can easily obtain

$$(\varrho^*)^2 X_1 = \varrho^* X_1 + X_1, \quad (2.7)$$

$$\mu(\varrho X_1, \varrho^* Y_1) = \mu(\varrho X_1, Y_1) + \mu(X_1, Y_1). \quad (2.8)$$

In that case (F, μ, ϱ) is called Golden-like statistical manifold.

In accordance with equations (2.6), (2.7) and (2.8), we easily derive the following proposition:

Proposition 2.1. (F, μ, ϱ) is a Golden-like statistical manifold if and only if so is (F, μ, ϱ^*) .

A remarkable fact is that, choosing $\varrho = \varrho^*$ in the Golden-like statistical manifold yields a Golden semi-Riemannian manifold.

Definition 2.3. Let $(F = F_a(\zeta_a) \times F_b(\zeta_b), g, \varrho)$ be a Golden-like statistical manifold. In that case the Riemannian curvature tensor $\hat{\Xi}$ of F is given as follows([35]):

$$\begin{aligned} \hat{\Xi}(X_1, Y_1)Z_1 &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) \{ \mu(Y_1, Z_1)X_1 - \mu(X_1, Z_1)Y_1 + \mu(\varrho Y_1, Z_1)\varrho X_1 \\ &\quad - \mu(\varrho X_1, Z_1)\varrho Y_1 \} + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) \{ \mu(\varrho Y_1, Z_1)X_1 - \mu(\varrho X_1, Z_1)Y_1 \\ &\quad + \mu(Y_1, Z_1)\varrho X_1 - \mu(X_1, Z_1)\varrho Y_1 \}. \end{aligned} \quad (2.9)$$

We can get the curvature tensor $\hat{\Xi}^*$ in accordance with dual connection precisely by modifying ϱ by ϱ^* .

By using the sectional curvature, the scalar curvature of N for orthonormal tangent frame $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and normal frame $\{\varsigma_{d+1}, \dots, \varsigma_q\}$, respectively, is defined as

$$\tau = 2 \sum_{1 \leq i < j \leq d} K(\varsigma_i \wedge \varsigma_j). \quad (2.10)$$

The average or mean curvature vector fields of N are also provided by

$$\Upsilon = \frac{1}{d} \sum_{i=1}^d \sigma(\varsigma_i, \varsigma_i) = \frac{1}{d} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^\gamma \right) \xi_\gamma, \quad \sigma_{ij}^\gamma = \mu(\sigma(\varsigma_i, \varsigma_j), \varsigma_\gamma)$$

and

$$\Upsilon^* = \frac{1}{d} \sum_{i=1}^d \sigma^*(\varsigma_i, \varsigma_i) = \frac{1}{d} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^{*\gamma} \right) \xi_\gamma, \quad \sigma_{ij}^{*\gamma} = \mu(\sigma^*(\varsigma_i, \varsigma_j), \varsigma_\gamma),$$

for $d+1 \leq \gamma \leq q$.

Moreover, we have $2\Upsilon^0 = \Upsilon + \Upsilon^*$, and $2\sigma^0 = \sigma + \sigma^*$, wherein the the mean curvature Υ^0 and second fundamental form σ^0 are obtained in accordance with Levi-Civita connection ∇^0 on N .

For N , the squared mean curvatures can be expressed as

$$\begin{aligned} \|\Upsilon\|^2 &= \frac{1}{d^2} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^\gamma \right)^2, \\ \|\Upsilon^*\|^2 &= \frac{1}{d^2} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^{*\gamma} \right)^2. \end{aligned}$$

Let $\hat{\Xi}$ and $\hat{\Xi}^*$ be the curvature tensors according to $\hat{\nabla}$ and $\hat{\nabla}^*$, respectively. In that case, we have the following relations on statistical manifolds.

$$\begin{aligned} \mu(\hat{\Xi}(X_1, Y_1)Z_1, W_1) &= \mu(\Xi(X_1, Y_1)Z_1, W_1) + \mu(\sigma(X_1, Z_1), \sigma^*(Y_1, W_1)) \\ &\quad - \mu(\sigma^*(X_1, W_1), \sigma(Y_1, Z_1)), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \mu(\hat{\Xi}^*(X_1, Y_1)Z_1, W_1) &= \mu(\Xi^*(X_1, Y_1)Z_1, W_1) + \mu(\sigma^*(X_1, Z_1), \sigma(Y_1, W_1)) \\ &\quad - \mu(\sigma(X_1, W_1), \sigma^*(Y_1, Z_1)), \end{aligned} \quad (2.12)$$

for all vector fields X_1, Y_1, Z_1, W_1 .[44]

3. MAIN INEQUALITIES

With a double of conjugate connections, here we derive some sharp relationships for statistical submanifolds of Golden-like statistical manifolds.

Lemma 3.1. [27] *Let $x_1, x_2, x_3, \dots, x_d$ be $d \geq 4$ real numbers. In that case, we have*

$$\sum_{1 \leq \alpha < \beta \leq d} x_\alpha x_\beta - x_1 x_2 - x_3 x_4 \leq \frac{d-3}{2(d-2)} \left(\sum_{\alpha=1}^d x_\alpha \right)^2.$$

Equality holds if and only if $x_1 + x_2 = x_3 + x_4 = x_5 = \dots = x_d$.

Theorem 3.1. Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, we have the following inequality

$$\begin{aligned} & (\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\ \geq & \left(-\frac{(1-\beta)\zeta_p - \beta\zeta_q}{2\sqrt{5}} \right) ([d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] - [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\ & - [1 + \Psi(\pi_2) + \Theta(\pi_2)]) + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\ & - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0). \end{aligned} \quad (3.13)$$

Proof. Let $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and $\{\varsigma_{d+1}, \dots, \varsigma_q\}$ be the orthonormal frames of $T_x N$ and $T_x^\perp N$, respectively. The scalar curvature of N corresponding to the sectional K -curvature is

$$\begin{aligned} \tau = & \frac{1}{2} \sum_{1 \leq i < j \leq d} \left[g(\Xi(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) + g(\Xi^*(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) \right. \\ & \left. - 2g(\Xi^0(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) \right]. \end{aligned}$$

Make use of (2.9) and Gauss equation for Ξ and Ξ^* . After discharging basic simple computations, we obtain

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-1) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} + \sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - 2\sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma \right]. \end{aligned}$$

Using (2.7), we get

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} + \sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - 2\sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma \right], \end{aligned}$$

which can be put down as follows

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + 2 \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\sigma_{ii}^{0\gamma} \sigma_{jj}^{0\gamma} - (\sigma_{ij}^{0\gamma})^2] - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma \\ & + (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}]. \end{aligned}$$

By utilizing Gauss equation for the Levi-Civita connection, we obtain

$$\begin{aligned} \tau &= \tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ &\quad + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - 2\hat{\tau}^0 \\ &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}]. \end{aligned} \quad (3.14)$$

Let π_1 and π_2 be mutually orthogonal two-planes spanned by $\{\varsigma_1, \varsigma_2\}$ and $\{\varsigma_3, \varsigma_4\}$, respectively. We denote

$$\mu(\phi_{\varsigma_1, \varsigma_1})\mu(\phi_{\varsigma_2, \varsigma_2}) = \Psi(\pi_1)$$

and

$$\mu(\phi_{\varsigma_3, \varsigma_3})\mu(\phi_{\varsigma_4, \varsigma_4}) = \Psi(\pi_2).$$

Also, as in [28], for an orthonormal bases $\{\varsigma_1, \varsigma_2\}$ and $\{\varsigma_3, \varsigma_4\}$ of two-plane sections, we denote

$$\Theta(\pi_1) = \mu(\phi_{\varsigma_1, \varsigma_2})\mu(\phi_{\varsigma_1, \varsigma_2}^*)$$

and

$$\Theta(\pi_2) = \mu(\phi_{\varsigma_3, \varsigma_4})\mu(\phi_{\varsigma_3, \varsigma_4}^*),$$

where $\Theta(\pi_1)$ and $\Theta(\pi_2)$ are real numbers in $[0, 1]$. In that case, the sectional curvature $K(\pi_1)$ of the plane π_1 is given by

$$\begin{aligned} K(\pi_1) &= \frac{1}{2} \left[\mu(\Xi(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) + \mu(\Xi^*(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) \right. \\ &\quad \left. - 2g(\Xi^0(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) \right]. \end{aligned} \quad (3.15)$$

By utilizing (2.9) and Gauss equation for Ξ and Ξ^* and set down systems in (3.15), we derive

$$\begin{aligned} K(\pi_1) &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \mu(\phi_{e_1, e_1})\mu(\phi_{e_2, e_2}) - \mu(\phi_{e_1, e_2})\mu(\phi_{e_2, e_1})] \\ &\quad - K^0(\pi_1) + \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^{*\gamma} + \sigma_{11}^{*\gamma} \sigma_{22}^\gamma - 2\sigma_{12}^{*\gamma} \sigma_{12}^\gamma \right]. \end{aligned}$$

By utilizing $\sigma + \sigma^* = 2\sigma^0$, we obtain

$$\begin{aligned} K(\pi_1) &= \left(-\frac{(1-\beta)\zeta_p - \beta\zeta_q}{2\sqrt{5}} \right) [1 + g(\phi_{e_1, e_1})g(\phi_{e_2, e_2}) \\ &\quad - g(\phi_{e_1, e_2})g(\phi_{e_2, e_1})] - K^0(\pi_1) + 2 \sum_{\gamma=d+1}^q \left[\sigma_{11}^{0\gamma} \sigma_{22}^{0\gamma} - (\sigma_{12}^{0\gamma})^2 \right] \\ &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left\{ \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] + \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right] \right\}. \end{aligned}$$

By utilizing Gauss equation for Levi-Civita connection, we obtain

$$\begin{aligned}
 K(\pi_1) &= K^0(\pi_1) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \mu(\phi_{\varsigma_1, \varsigma_1})\mu(\phi_{\varsigma_2, \varsigma_2}) \\
 &\quad + \mu(\phi_{\varsigma_1, \varsigma_2})\mu(\phi^*_{\varsigma_1, \varsigma_2})] - 2\hat{K}^0(\pi_1) - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.16}$$

The on the top equation can be noted in the following form:

$$\begin{aligned}
 K(\pi_1) &= K^0(\pi_1) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] - 2\hat{K}^0(\pi_1) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.17}$$

Similarly, the sectional curvature $K(\pi_2)$ of the plane π_2 can be written as

$$\begin{aligned}
 K(\pi_2) &= K^0(\pi_2) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] - 2\hat{K}^0(\pi_2) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^\gamma \sigma_{44}^\gamma - (\sigma_{34}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} - (\sigma_{34}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.18}$$

The relations (3.14), (3.17) and (3.18) imply that

$$\begin{aligned}
 &(\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\
 &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
 &\quad + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}] \\
 &\quad - \left\{ \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \right. \\
 &\quad \left. - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right] \right\} - \left\{ \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \right. \\
 &\quad \left. - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^\gamma \sigma_{44}^\gamma - (\sigma_{34}^\gamma)^2 \right] - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} - (\sigma_{34}^{*\gamma})^2 \right] \right\} \\
 &\quad + 2\hat{K}^0(\pi_1) + 2\hat{K}^0(\pi_2) - 2\hat{\tau}^0.
 \end{aligned}$$

$$\begin{aligned}
&\geq \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_q}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \\
&\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left\{ \left[\sigma_{ii}^\gamma \sigma_{jj}^\gamma - \sigma_{11}^\gamma \sigma_{22}^\gamma - \sigma_{33}^\gamma \sigma_{44}^\gamma \right] \right. \\
&\quad \left. + \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - \sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - \sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} \right] \right\} \\
&\quad + 2\hat{K}^0(\pi_1) + 2\hat{K}^0(\pi_2) - 2\hat{\tau}^0,
\end{aligned}$$

where we have used the known relation $\sigma + \sigma^* = 2\sigma^0$.

Now, we apply Lemma 3.1 for σ and σ^* to obtain the following inequality:

$$\begin{aligned}
&(\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\
&\geq \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \\
&\quad - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0).
\end{aligned}$$

From this relation, we can easily derive our desired inequality (3.13). □

The equality occasion of inequality (3.13) is shaped by the following:

Theorem 3.2. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, the equality holds for (3.13) identically if and only if we have*

- (i) $\sigma_{11}^\gamma + \sigma_{22}^\gamma = \sigma_{33}^\gamma + \sigma_{44}^\gamma = \sigma_{55}^\gamma = \cdots = \sigma_{dd}^\gamma$.
- (ii) $\sigma_{11}^{*\gamma} + \sigma_{22}^{*\gamma} = \sigma_{33}^{*\gamma} + \sigma_{44}^{*\gamma} = \sigma_{55}^{*\gamma} = \cdots = \sigma_{dd}^{*\gamma}$.
- (iii) $\sigma_{ij}^\gamma = \sigma_{ij}^{*\gamma} = 0$, $1 \leq i < j \leq d$, $i \neq j$, $(i, j) \neq (1, 2), (2, 1), (3, 4), (4, 3)$,

for $\gamma \in \{d+1, d+2, \dots, q\}$.

Corollary 3.1. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional totally real statistical submanifold. In that case, we have*

$$\begin{aligned} & (\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\ & \geq (d(d-2) - 2) \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] \\ & \quad + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0). \end{aligned} \quad (3.19)$$

The equality case of (3.19) can be discussed as Theorem 3.2.

Next, we derive Euler inequality. Before that, we provide the following

Proposition 3.1. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional totally real statistical submanifold.. In that case, we have*

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 & \geq \frac{8}{d^2} \left\{ 2Ric^0(X, X) - Ric(X, X) \right. \\ & \quad + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & \quad \left. + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}, \end{aligned} \quad (3.20)$$

where $\{X = \varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ is an orthonormal basis of $T_x N$.

Proof. Let $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and $\{\varsigma_{d+1}, \dots, \varsigma_q\}$ be the orthonormal frames of $T_x N$ and $T_x^\perp N$, respectively. In that instance, equations (2.5), (2.11), and (2.12) are used to determine the scalar curvature:

$$\begin{aligned} \tau &= 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & \quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\ & \quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}], \end{aligned}$$

where $\hat{K}^0$ is the sectional curvature of N in accordance with Levi-Civita connection.

Using $\sigma + \sigma^* = 2\sigma^0$, we get

$$\begin{aligned} \tau &= 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) + \text{tr}^2(\phi) - \text{tr}(\phi^*)] \\ & \quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\ & \quad - \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + \frac{1}{2} [\|\sigma\|^2 + \|\sigma^*\|^2]. \end{aligned} \quad (3.21)$$

To introduce an inequality in (3.21), we compute $\|\sigma\|^2$ and $\|\sigma^*\|^2$ as follows:

$$\begin{aligned}
\|\sigma\|^2 &= \sum_{\gamma=d+1}^q \sum_{i,j=1}^d (\sigma_{ij}^\gamma)^2 \\
&= \sum_{\gamma=d+1}^q \{(\sigma_{11}^\gamma)^2 + (\sigma_{12}^\gamma)^2 + \cdots + (\sigma_{1d}^\gamma)^2 + (\sigma_{21}^\gamma)^2 + (\sigma_{22}^\gamma)^2 \\
&\quad + \cdots + (\sigma_{2d}^\gamma)^2 + \cdots + (\sigma_{d1}^\gamma)^2 + (\sigma_{d2}^\gamma)^2 + \cdots + (\sigma_{dd}^\gamma)^2\} \\
&= \frac{1}{2} \sum_{\gamma=d+1}^q \{(\sigma_{11}^\gamma + \sigma_{22}^\gamma + \cdots + \sigma_{dd}^\gamma)^2 + (\sigma_{11}^\gamma - \sigma_{22}^\gamma - \cdots - \sigma_{dd}^\gamma)^2\} \\
&\quad + 2 \sum_{\gamma=d+1}^q \left\{ \sum_{j=2}^d (\sigma_{1j}^\gamma)^2 + \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \right\} \\
&\geq \frac{1}{2} \sum_{\gamma=d+1}^q (\sigma_{11}^\gamma + \sigma_{22}^\gamma + \cdots + \sigma_{dd}^\gamma)^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \\
&= \frac{d^2}{2} \|\Upsilon\|^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma).
\end{aligned}$$

Similarly,

$$\|\sigma^*\|^2 \geq \frac{d^2}{2} \|\Upsilon\|^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^{*\gamma})^2 - \sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma}).$$

Combining the two inequalities yields

$$\begin{aligned}
\|\sigma\|^2 + \|\sigma^*\|^2 &\geq \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon\|^2] + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \\
&\quad + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^{*\gamma})^2 - \sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma}) \\
&= \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon\|^2] + \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} \{(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 \\
&\quad - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) \\
&\quad + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) \\
&\quad + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma)\}.
\end{aligned}$$

We put this inequality into (3.21), it follows that

$$\begin{aligned}
\tau &\geq 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\
&\quad - \frac{d^2}{4} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} \{(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 \\
&\quad - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma)\}. \quad (3.22)
\end{aligned}$$

After rearranging (3.22), we arrive at

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{4}{d^2} \left\{ 2\tau^0 - \tau \right. \\ & + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(e_i \wedge e_j) \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} [(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) \\ & \left. + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma) \right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{8}{d^2} \left\{ 2\text{Ric}^0(X, X) - \text{Ric}(X, X) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) \right. \\ & - \text{tr}(\phi^*) + \text{tr}^2(\phi)] + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) \\ & \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}, \end{aligned}$$

where $\{X = \varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ is an orthonormal basis of $T_x N$. □

Proposition 3.2. Suppose that (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, the equality of (3.20) holds at $x \in N$ if and only if $2\sigma_{11} = d\Upsilon$, $2\sigma_{11}^* = d\Upsilon^*$ and $\sigma_{1j} = 0$, $\sigma_{1j}^* = 0$ for $j = 2, \dots, d$.

By using Proposition 3.1, we have

Theorem 3.3. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold.

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{8}{d^3} \left\{ 2\tau^0 - \tau + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}^2(\phi^*) \right. \\ & + \text{tr}(\phi)] + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\ & \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}. \end{aligned} \tag{3.23}$$

Theorem 3.4. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case the equality of 3.23 holds at $x \in N$ if and only if either (i) $\sigma = \sigma^* = 0$, if $d \geq 3$ or (ii) $\sigma = \mu \otimes \Upsilon$ and $\sigma^* = \mu \otimes \Upsilon^*$, if $d = 2$.

Example 3.1. Consider the Golden-like statistical manifold $(\mathbb{R}_d^{2d+q}, \hat{\nabla}, g, \phi)$ with standard coordinates and the semi-Riemannian metric g with the signature $(\underbrace{-, \dots, -}_{d\text{-times}}, \underbrace{+, \dots, +}_{(d+q)\text{-times}})$. The tensor field ϕ type $(1, 1)$ on $\mathbb{R}_d^{2d+q}$ is the following:

$$\phi = \frac{1}{2} \begin{bmatrix} \lambda_{ij} & \sqrt{5}\lambda_{ij} & 0 \\ \sqrt{5}\lambda_{ij} & \lambda_{ij} & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

where β is the Golden mean. In that case, ϕ is golden structure on $\mathbb{R}_d^{2d+q}$. Furthermore, the conjugate tensor field ϕ^* is defined as

$$\phi^* = \frac{1}{2} \begin{bmatrix} \lambda_{ij} & -\sqrt{5}\lambda_{ij} & 0 \\ -\sqrt{5}\lambda_{ij} & \lambda_{ij} & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

Next, let N be any d -dimensional statistical submanifold of the Golden-like statistical manifold $(\mathbb{R}_d^{2d+q}, \hat{\nabla}, \mu, \phi)$. In that case, the inequality (3.23) is satisfied. In particular, the statistical submanifold $N = \mathbb{R}_1^3$ of $\mathbb{R}_2^5$ attains equality in (3.23) because N is a totally geodesic submanifold of $\mathbb{R}_2^5$ in accordance with the Levi-Civita connection, that is, $\sigma = \sigma^* = 0$.

To justify this claim explicitly, take $N = \mathbb{R}_1^3$ to be embedded in $F = \mathbb{R}_2^5$ as the standard linear (affine) subspace determined by fixing the last two ambient coordinates to be constant. On $F = \mathbb{R}_2^5$ with the standard flat statistical structure, the dual connections $\hat{\nabla}$ and $\hat{\nabla}^*$ both coincide with the ordinary directional derivative D of $\mathbb{R}_2^5$ (in flat coordinates the connection coefficients vanish identically, and $\hat{\nabla} = \hat{\nabla}^* = \hat{\nabla}^0 = D$). For tangent vector fields X_1, Y_1 on N , extended to F so that their components along the two fixed (normal) directions vanish identically, the directional derivative $D_{X_1}Y_1$ again has vanishing components in those directions, i.e. $D_{X_1}Y_1$ is tangent to N at every point. By the Gauss formula

$$\hat{\nabla}_{X_1}Y_1 = \nabla_{X_1}Y_1 + \sigma(X_1, Y_1), \quad \hat{\nabla}_{X_1}^*Y_1 = \nabla_{X_1}^*Y_1 + \sigma^*(X_1, Y_1),$$

the normal-valued terms $\sigma(X_1, Y_1)$ and $\sigma^*(X_1, Y_1)$ are exactly the components of $\hat{\nabla}_{X_1}Y_1 = D_{X_1}Y_1$ and $\hat{\nabla}_{X_1}^*Y_1 = D_{X_1}^*Y_1$ transverse to N . Since $D_{X_1}Y_1$ has been shown to be tangent to N , these transverse components vanish for all X_1, Y_1 tangent to N , and hence $\sigma \equiv 0$ and $\sigma^* \equiv 0$ on N . This confirms that $N = \mathbb{R}_1^3$ is totally geodesic in $F = \mathbb{R}_2^5$ with respect to both dual connections, as claimed.

Corollary 3.2. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is a d -dimensional totally real statistical submanifold.. In that case, we have

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 &\geq \frac{8}{d^3} \left\{ 2\tau^0 - \tau + d(d-2) \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) \right. \\ &\quad \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge e_j) \right\}. \end{aligned} \quad (3.24)$$

The equality case of (3.24) can be discussed as Theorem 3.4

4. CONCLUSION

This topic focuses on Euler and Chen type inequalities in submanifold geometry, as well as intrinsic and extrinsic curvature in statistical manifolds with Golden structures, which fills a niche that is unique. This study fills a gap in the literature by extending statistical

manifold results to geometric settings and analyzing the situation of statistical manifolds with golden structures. This study provides formal characterizations of various Pinching theorems with trivial normal connections, thereby strengthening the theoretical value of the optimal bounds. We turn up the results relative to Chen-invariant, expanding the range of ambient spaces where sharp bounds are obtained. The goal is to understand the relationship between intrinsic scalar curvature and extrinsic type invariants, as well as identify examples of equality or rigidity.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

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