



A COMPARATIVE STUDY OF SOFT 2-NORMS: LIMITATIONS, REFINED CONCEPTS, AND STRUCTURAL INTEGRATION

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Abstract. The present work aims to further contribute to the study of soft linear structures by analyzing key definitions and providing illustrative examples that highlight both the strengths and limitations of existing approaches. In particular, similarly to classical linear functional analysis, we establish a connection between the refined soft 2-norm and the soft norm by means of a suitable soft basis. This relationship clarifies the structural coherence between these two notions and supports the development of a consistent mathematical framework within soft set theory. Refined soft 2-norm studies contribute to a globally consistent way to measure linear dependence across all parameters, establish connections with soft 2-inner products, and provide concrete constructions that extend classical 2-norm theory to soft vector spaces with parametric structure. In this work, a careful comparison between the soft 2-norm and the refined soft 2-norm is presented to highlight their conceptual and structural differences. Furthermore, while both norms were attempted to be related to Gram-type 2-inner products, the refined soft 2-norm was found to naturally provide a consistent derivation from the soft 2-inner product thanks to its parameter-coordinated structure. Therefore, the refined version not only strengthens the link between geometry and algebra in soft vector spaces, but also fills a gap in the literature by providing an example where global parametric dependence is clearly characterized. Another aim of this paper is to investigate an extension theorem for soft linear functionals in refined soft 2-normed spaces. By introducing an induced soft norm obtained from a fixed soft vector, we establish a Hahn–Banach type extension theorem in the framework of refined soft 2-normed spaces. Furthermore, we discuss consequences related to soft bounded linear functionals and the dual structure of these spaces. Our results reinforce the importance of refined concepts in soft set theory and contribute to the ongoing effort to establish a solid and coherent mathematical framework for handling uncertainty.

Keywords: Soft set, Soft normed space, Refined soft 2-normed space, Refined soft 2-inner product space.

MSC (2020): 03E72, 08A72, 46B99.

1. INTRODUCTION

The theory of soft sets, introduced by Molodtsov in 1999 [16], has emerged as an effective mathematical framework for dealing with uncertainty, vagueness, and incomplete information. Unlike classical approaches such as fuzzy sets, rough sets, and intuitionistic fuzzy sets, soft set theory avoids the need for additional structures such as membership functions or probability measures. Instead, it is based on a parameterized family of subsets, which provides greater flexibility in modeling real-world problems involving uncertain or approximate data.

Since its introduction, soft set theory has attracted considerable attention, and numerous extensions and applications have been developed. Researchers have investigated algebraic structures of soft sets, including soft groups, soft rings, and soft vector spaces, as well as topological and analytical aspects such as soft topology, soft metric spaces, and soft normed spaces. These developments have significantly enriched the theoretical foundation of the field and expanded its applicability to areas such as decision-making, optimization, data analysis, and engineering sciences [5], [4], [7], [6], [2], [19], [18], [3], [20].

In particular, the study of soft linear structures has played a central role in the advancement of soft set theory. Concepts such as soft linear dependence, soft bases, and soft inner products have been proposed to extend classical linear algebra into the soft setting. However, transferring classical notions directly into the soft framework often leads to subtle difficulties. Several examples in the literature demonstrate that naive generalizations may fail to preserve fundamental properties, thereby necessitating refined definitions and careful theoretical analysis.

The development of soft set theory continued with Kadhim's introduction of the concept of a soft 2-normed space [11]. However, after some time, Sanabria addressed the deficiency in Kadhim's definition by redefining the soft 2-normed space within the refined framework [11], [17]. Recent works have focused on addressing these challenges by introducing more robust notions, such as refined soft linear dependence and improved formulations of soft inner product and soft 2-inner product spaces. These refinements have proven essential in preventing inconsistencies and ensuring that the soft analogues faithfully reflect the behavior of their classical counterparts. In particular, examples provided by Sanabria and other authors have clarified critical issues and guided the development of a consistent soft theory [17].

The concept of 2-normed spaces was first investigated by Gähler in 1965 [10]. Later, Lewandowska studied the properties of 2-normed sets and constructed two 2-normed sets of linear operators in 2003 [14]. He also proved several important theorems on generalized 2-normed spaces [13], [12]. Furthermore, in 2007, he introduced a new structure called generalized 2-normed spaces [1].

Motivated by these developments, the present work aims to further contribute to the study of soft linear structures by analyzing key definitions and providing illustrative examples that highlight both the strengths and limitations of existing approaches. Specifically, it establishes a link between the soft 2-norm and the soft norm through a suitable soft basis, similar to classical linear functional analysis. This relationship clarifies the structural consistency between these two notions and supports the development of a coherent soft framework within

soft set theory. Furthermore, both norms were attempted to be related to Gram-type 2-inner products, and it was observed that the refined soft 2-norm naturally provides a consistent derivation from the soft 2-inner product thanks to its parameter-coordinate structure. Therefore, it was seen that the refined version not only strengthens the link between geometry and algebra in soft vector spaces but also clearly characterizes global parametric dependence by means of an example. Our results reinforce the importance of refined concepts in soft set theory and support the ongoing effort to establish a solid and coherent mathematical framework for handling uncertainty.

In summary, this study presents a careful comparison between the soft 2-norm and the refined soft 2-norm is presented to highlight their conceptual and structural differences. The soft 2-norm is defined parameterwise, allowing each parameter to be treated independently; as a result, linear dependence in a single parameter does not necessarily yield a zero norm, and its construction is often simpler and softer for computational or illustrative purposes. In contrast, the refined soft 2-norm imposes a stronger, global condition: the norm vanishes only if the soft vectors are linearly dependent across all parameters. This global perspective captures the notion of refined linear dependence more rigorously, providing a more theoretically consistent framework. Further, while both norms can be connected to Gram-type 2-inner products, the refined soft 2-norm naturally ensures a coherent derivation from a soft 2-inner product due to its parameter-coordinated structure. Therefore, the refined version not only strengthens the link between geometry and algebra in soft vector spaces but also fills a gap in the literature by offering an example where global parametric dependence is explicitly captured and visualized. Furthermore, we know that one of the most fundamental results in functional analysis is the Hahn–Banach Theorem, which guarantees the extension of bounded linear functionals from a subspace to the whole space without increasing the norm. This theorem has numerous applications in analysis, optimization, and operator theory. Motivated by these developments, in this work we study the extension of soft bounded linear functionals in refined soft 2-normed spaces and establish a refined soft 2-normed Hahn–Banach type theorem for such spaces.

2. PRELIMINARIES

This section is devoted to recalling various concepts from soft set theory together with essential notions such as refined soft 2-normed spaces and refined soft 2-inner product spaces.

Let Q be a universe and P be a set of parameters, $P(Q)$ denotes the power set of Q and B be a non-empty subset of P .

Definition 2.1. [16] *We define a soft set over Q as a pair (G, P) where G maps each element of P to a subset of Q , i.e. $G : P \rightarrow P(Q)$.*

Definition 2.2. [15] *Consider a soft set (G, P) defined over the universe Q . If for every parameter $p \in P$, $G(p) = Q$, then (G, P) is called an absolute soft set, often denoted by $\tilde{Q}$.*

Definition 2.3. [15] *For a soft set (G, P) over Q if each parameter $p \in P$ satisfies $G(p) = \emptyset$, then $G(p)$ is referred to as a null soft set, usually written as Φ .*

Definition 2.4. [15] *Let Q be a non-empty set and P be a nonempty parameter set. A function $q : P \rightarrow Q$ is called a soft element of Q . We say that a soft element q belongs to a*

soft set G over Q , denoted $q \in Q$, if for every parameter $p_1 \in P$, the valued q_{p_1} is an element of $G(p_1)$. In other words, a soft set G can be viewed as assigning to each parameter of $p_1 \in P$ a subset of Q , namely $G(p_1)$, and a soft element belongs to G if its values are contained in these subsets for all parameters. Furthermore, if (G, P) is a soft set in which every $G(p_1)$ contains exactly one element (i.e., is a singleton) for each $p_1 \in P$, then the entire soft set can be uniquely represented by a single soft element. In this case, knowing the soft element is enough to reconstruct the soft set.

Definition 2.5. [5] Let Q be a linear space over a field $\mathbb{R}$, and the parameter set P be the real number set $\mathbb{R}$. The soft set (G, P) is defined as a soft vector and is denoted by $\tilde{q}(p_1)$ if there exists one $p_1 \in P$ such that $G(p_1) = \{q\}$ for some $q \in Q$ and $G(p'_1) = \emptyset$ for all $p'_1 \in P \setminus \{p_1\}$. The set of all soft vectors over $\tilde{Q}$ will be defined by $SV(\tilde{Q})$.

Definition 2.6. [4] Let $\tilde{Q}$ be a soft linear space, P be a nonempty set of parameters and (G, P) be a soft linear space on Q . A finite set of soft vectors $\{\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n\}$ of (G, P) is said to be linearly dependent in G if there exist soft scalars $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n$ not all equal to $\tilde{0}$, such that $\tilde{\alpha}_1 \tilde{q}_1 + \tilde{\alpha}_2 \tilde{q}_2 + \dots + \tilde{\alpha}_n \tilde{q}_n = \tilde{\theta}$.

Definition 2.7. [8] Let $F : SV(\tilde{Q}) \rightarrow SV(\tilde{Q})$ be an operator. Then F is said to be soft linear if

- i) $F(\tilde{q} + \tilde{w}) = F(\tilde{q}) + F(\tilde{w})$ for every soft vectors $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- ii) for every soft scalar $\tilde{\alpha}$, $F(\tilde{\alpha}\tilde{q}) = \tilde{\alpha}F(\tilde{q})$ for every soft vector $\tilde{q} \in \tilde{Q}$.

Definition 2.8. [9] A soft linear functional F is a soft linear operator such that $F : SV(\tilde{Q}) \rightarrow K$ where $\tilde{Q}$ is a soft linear space and $K = \mathbb{R}(P)$ if $\tilde{Q}$ is a real soft linear space and $K = \mathbb{C}(P)$ if $\tilde{Q}$ is a complex soft linear space.

Definition 2.9. [11] Let $\tilde{Q}$ be an absolute soft linear space. A soft 2-norm on $\tilde{Q}$ is a mapping $\|\cdot, \cdot\| : SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$ which satisfies the following conditions:

- i) $\|\tilde{q}, \tilde{w}\| \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are linearly dependent in $\tilde{Q}$,
- ii) $\|\tilde{q}, \tilde{w}\| = \|\tilde{w}, \tilde{q}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\|\tilde{q}, \tilde{\alpha}\tilde{w}\| = |\tilde{\alpha}| \|\tilde{q}, \tilde{w}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- iv) $\|\tilde{q}, \tilde{w} + \tilde{z}\| \leq \|\tilde{q}, \tilde{w}\| + \|\tilde{q}, \tilde{z}\|$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$,

The absolute soft linear space $\tilde{Q}$ with a refined soft 2-norm $\|\cdot, \cdot\|$ on $\tilde{Q}$ is said to be a soft 2-normed space and is denoted by $(\tilde{Q}, \|\cdot, \cdot\|, P)$ or $(\tilde{Q}, \|\cdot, \cdot\|)$.

However, it is not possible to show that this definition yields the soft 2-inner product. Inspired by Sanabria's example, we present a new example that explains and illustrates this situation:

Example 2.1. Let $Q = \mathbb{R}^2$ be a vector space over $\mathbb{R}$ and let the set of parameters be $P = \{1, 2\}$. Define the soft vectors $\tilde{q}, \tilde{w} : P \rightarrow Q$

$$\tilde{q}(p) = \begin{cases} (1, 0), & p = 1 \\ (1, 2), & p = 2 \end{cases}; \quad \tilde{w}(p) = \begin{cases} (2, 0), & p = 1 \\ (2, 3), & p = 2 \end{cases}.$$

Observe that $p = 1$ we have $\tilde{w}(1) = 2\tilde{q}(1)$; hence, the soft vectors $\{\tilde{q}(p), \tilde{w}(p)\}$ are linearly dependent in the soft vector space $\tilde{Q}$ over the parameter set P . Consider now the standard 2-inner product induced by the usual inner product on $\mathbb{R}^2$, $\langle \tilde{q}(p), \tilde{q}(p) | \tilde{w}(p) \rangle = \|\tilde{q}(p)\|^2 \|\tilde{w}(p)\|^2 - |\langle \tilde{q}(p), \tilde{w}(p) \rangle|^2$. For $p = 2$, $\tilde{q}(2) = (1, 2)$ and $\tilde{w}(2) = (2, 3)$, and straightforward computations yield $\|\tilde{q}(2)\|^2 = 5$, $\|\tilde{w}(2)\|^2 = 13$, $|\langle \tilde{q}(2), \tilde{w}(2) \rangle| = 8$. Thus,

$$\langle \tilde{q}(2), \tilde{q}(2) | \tilde{w}(2) \rangle = 5.13 - 64 = 1 > 0.$$

Therefore, although the value of the induced soft 2-inner product is strictly positive for $p = 2$, the set of soft vectors $\{\tilde{q}(p), \tilde{w}(p)\}$ is soft linearly dependent, which contradicts the first condition in the definition of a soft 2-inner product given in [11]. This example demonstrates that the standard 2-inner product does not necessarily generate a soft 2-inner product that satisfies the linear independence requirement stated in Definition 5.1 of [11], thereby supporting the observation presented in Example 5.2.

This issue was identified by Sanabria in [17], where he introduced the notion of refined linear dependence, thereby preventing the emergence of a significant problem in the development of soft theory.

Definition 2.10. [17] Suppose Q is a linear space, P is a nonempty parameter set, and (G, P) is a soft linear space on Q . A finite set of soft vectors $\{\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n\}$ in (G, P) is called refined linearly dependent in G if there exist soft scalars $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n$ with $\tilde{\alpha}_i \neq \tilde{0}$ for at least one $i \in \{1, 2, \dots, n\}$, such that $\tilde{\alpha}_1 \tilde{q}_1 + \tilde{\alpha}_2 \tilde{q}_2 + \dots + \tilde{\alpha}_n \tilde{q}_n = \tilde{\theta}$ for every $p \in P$.

Definition 2.11. [17] Suppose $\tilde{Q}$ is an absolute soft linear space. We define a refined soft 2-norm on $\tilde{Q}$ to be a mapping $\|\cdot, \cdot\| : SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$, provided that it satisfies the conditions listed below:

- i) $\|\tilde{q}, \tilde{w}\| \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent in $\tilde{Q}$,
- ii) $\|\tilde{q}, \tilde{w}\| = \|\tilde{w}, \tilde{q}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\|\tilde{q}, \tilde{\alpha}\tilde{w}\| = |\tilde{\alpha}| \|\tilde{q}, \tilde{w}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- iv) $\|\tilde{q}, \tilde{w} + \tilde{z}\| \leq \|\tilde{q}, \tilde{w}\| + \|\tilde{q}, \tilde{z}\|$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$.

A refined soft 2-normed space is an absolute soft linear space $\tilde{Q}$ together with a refined soft 2-norm $\|\cdot, \cdot\|$ and it is represented by $(\tilde{Q}, \|\cdot, \cdot\|, P)$ or $(\tilde{Q}, \|\cdot, \cdot\|)$.

Definition 2.12. [17] For an absolute soft linear space $\tilde{Q}$, a refined soft 2-inner product is a function $\langle \cdot, \cdot | \cdot \rangle : SV(\tilde{Q}) \times SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$ that fulfills the conditions listed below:

- i) $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent in $\tilde{Q}$,
- ii) $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle = \langle \tilde{w}, \tilde{w} | \tilde{q} \rangle$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\langle \tilde{q}, \tilde{w} | \tilde{z} \rangle = \langle \tilde{w}, \tilde{q} | \tilde{z} \rangle$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$,
- iv) $\langle \tilde{\alpha}\tilde{q}, \tilde{w} | \tilde{z} \rangle = |\tilde{\alpha}| \langle \tilde{q}, \tilde{w} | \tilde{z} \rangle$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- v) $\langle \tilde{q}_1 + \tilde{q}_2, \tilde{y} | \tilde{z} \rangle = \langle \tilde{q}_1, \tilde{y} | \tilde{z} \rangle + \langle \tilde{q}_2, \tilde{y} | \tilde{z} \rangle$ for all $\tilde{q}_1, \tilde{q}_2, \tilde{w}, \tilde{z} \in \tilde{Q}$.

Let $\tilde{Q}$ be an absolute soft linear space. If $\tilde{Q}$ is endowed with a refined soft 2-inner product $\langle \cdot, \cdot | \cdot \rangle$, then $\tilde{Q}$ is said to be a refined soft 2-inner product space and it is represented by $(\tilde{Q}, \langle \cdot, \cdot | \cdot \rangle, P)$ or $(\tilde{Q}, \langle \cdot, \cdot | \cdot \rangle)$.

3. MAIN RESULTS

Before introducing new definitions and theorems related to the concept of refined soft 2-norm, we present two new examples to better understand the relationship between the concepts of soft 2-norm and refined soft 2-norm. Then, based on the definition of refined soft 2-normed spaces given by Sanabria, we present new definitions and results concerning refined soft sequences [17]. We begin by extending certain classical convergence results from functional analysis to the framework of refined soft sequences.

Example 3.1. Let $Q = \mathbb{R}^2$ be a real vector space, and let the parameter set be $P = \{\text{optimism}, \text{pessimism}, \text{stability}\}$. A soft vector is a mapping

$$\tilde{q} : P \rightarrow \mathbb{R}^2.$$

Let the base vector be $x = (2, 3)$. Define a soft scalar function $\tilde{\alpha} : P \rightarrow \mathbb{R}$ by $\tilde{\alpha}(\text{optimism}) = 1.5$, $\tilde{\alpha}(\text{pessimism}) = 0.5$, $\tilde{\alpha}(\text{stability}) = 1.0$. Define the refined soft vector

$$\tilde{q}(p) = \tilde{\alpha}(p)x$$

for all $p \in P$. Explicitly, $\tilde{q}(\text{optimism}) = 1.5(2, 3) = (3, 4.5)$, $\tilde{q}(\text{pessimism}) = 0.5(2, 3) = (1, 1.5)$, $\tilde{q}(\text{stability}) = 1.0(2, 3) = (2, 3)$.

Now define another refined soft vector by $\tilde{w}(p) = 2\tilde{q}(p)$ for all $p \in P$. Clearly, the soft vectors $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent because they are linearly dependent for every parameter. Now, define the refined soft 2-norm by

$$\|\tilde{q}, \tilde{w}\|(p) = \left| \det \begin{pmatrix} \tilde{x}_1(p) & \tilde{y}_1(p) \\ \tilde{x}_2(p) & \tilde{y}_2(p) \end{pmatrix} \right|.$$

Equivalently, $\|\tilde{q}, \tilde{w}\|(p) = |\tilde{x}_1(p)\tilde{y}_2(p) - \tilde{x}_2(p)\tilde{y}_1(p)|$. Explicitly:

Parameter	$\tilde{q}(p)$	$\tilde{w}(p)$
optimism	(3, 4.5)	(6, 9)
pessimism	(1, 1.5)	(2, 3)
stability	(2, 3)	(4, 6)

In all cases optimism, pessimism or stability, we find $\|\tilde{q}, \tilde{w}\|(p) = 0$. For every parameter $p \in P$, $\|\tilde{q}, \tilde{w}\|(p) = 0$. Therefore, $\|\tilde{q}, \tilde{w}\| = \tilde{0}$. Since the vectors are linearly dependent for all parameters, $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent. Thus, in this example: $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent. Furthermore, this determinant-based definition also satisfies all other axioms of the refined soft 2-norm.

Now define another soft vector: $\tilde{z}(\text{optimism}) = (6, 9)$, $\tilde{z}(\text{pessimism}) = (1, 0)$, $\tilde{z}(\text{stability}) = (4, 6)$. Then we obtain: the soft vectors $\tilde{w}$ and $\tilde{z}$ are dependent at optimism, dependent at stability, and independent at pessimism. Therefore, $\tilde{w}$ and $\tilde{z}$ are not refined linearly dependent, but according to Kadhim's approach, they are linearly dependent. Recall, $\|\tilde{w}, \tilde{z}\|(p) = |\tilde{w}_1(p)\tilde{z}_2(p) - \tilde{w}_2(p)\tilde{z}_1(p)|$, we get

$$\|\tilde{w}, \tilde{z}\| = \begin{cases} 0, & \text{optimism} \\ 3, & \text{pessimism} \\ 0, & \text{stability} \end{cases}.$$

This also contradicts Kadhim's definition of soft 2-normed space. This example shows clearly: Some parameters may produce linear dependence, but unless dependence holds for every parameter, the vectors are not refined linearly dependent, and therefore the refined soft 2-norm does not vanish.

Example 3.2. Let $\tilde{Q} = \widetilde{\mathbb{R}^2}$, $P = \{1, 2\}$. We define a refined soft 2-norm parameterwise as:

$$\|\tilde{q}, \tilde{w}\|(p) = \|\tilde{q}(p)\| \|\tilde{w}(p)\| \sin \theta(\tilde{q}(p)\tilde{w}(p)),$$

where $\cos \theta(\tilde{q}(p)\tilde{w}(p)) = \frac{\langle \tilde{q}(p)\tilde{w}(p) \rangle}{\|\tilde{q}(p)\| \|\tilde{w}(p)\|}$. Define soft vectors $\tilde{q}, \tilde{w} : P \rightarrow \tilde{Q}$ as

$$\tilde{q}(p) = \begin{cases} (1, 2), & p = 1 \\ (1, 0), & p = 2 \end{cases}$$

and

$$\tilde{w}(p) = \begin{cases} (2, 4), & p = 1 \\ (0, 1), & p = 2 \end{cases}.$$

They are not refined linearly dependent, because refined linear dependence requires linear dependence for all parameters. But they are linearly dependent. On the other hand, we obtain $\|\tilde{q}, \tilde{w}\|(1) = 0$ for $p = 1$ and $\|\tilde{q}, \tilde{w}\|(2) = 1$ for $p = 2$. This would also contradict Kadhim's approach.

Therefore, these two examples show us that Sanabria's definition of refined linear dependence is a crucial requirement for soft 2-norm spaces.

Definition 3.1. A soft sequence $\tilde{q}_n(p)$ in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ is said to be soft Cauchy if for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Definition 3.2. A refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ is bounded if one can find a sequence of soft elements $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ for which, for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, there exists a non-negative constant L such that

$$\|\tilde{q}_n(p), \tilde{w}(p)\| \lesssim L$$

for all natural numbers n .

Theorem 3.1. In a refined soft 2-normed space, any sequence that is soft convergent must be a soft Cauchy sequence.

Proof. Let $\{\tilde{q}_n(p)\}$ be a soft convergent sequence in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ that is

$$\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| \rightarrow 0 \text{ for every } \tilde{w} \in SV(\tilde{Q}) \text{ and } p \in P,$$

which implies $\tilde{q}_n(p) \rightarrow \tilde{q}(p)$. Since the sequence $\{\tilde{q}_n(p)\}$ is soft convergent, it follows that

$$\begin{aligned} \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| &= \|\tilde{q}_n(p) + \tilde{q}(p) - \tilde{q}(p) - \tilde{q}_m(p), \tilde{w}(p)\| \\ &\lesssim \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| + \|\tilde{q}_m(p) - \tilde{q}(p), \tilde{w}(p)\| \\ &\rightarrow 0. \end{aligned}$$

for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. $\square$

Theorem 3.2. *If $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ are soft Cauchy sequences in the refined soft 2-normed space $\tilde{Q}$, and $\{\tilde{\beta}_n(p)\}$ is a soft real Cauchy sequence in $\tilde{\mathbb{R}}$, then both sequences $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$ and $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$ are soft Cauchy sequences in $\tilde{Q}$.*

Proof. Let $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ be soft Cauchy sequences in $\tilde{Q}$. By Definition 3.1, for every $\tilde{z}(p) \in SV(\tilde{Q})$ and $p \in P$

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{z}(p)\| \rightarrow 0 \text{ and } \|\tilde{w}_n(p) - \tilde{w}_m(p), \tilde{z}(p)\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Consider the sequence $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$. For every $\tilde{z}(p) \in SV(\tilde{Q})$ and $p \in P$, we have

$$\begin{aligned} & \|(\tilde{q}_n(p) + \tilde{w}_n(p)) - (\tilde{q}_m(p) + \tilde{w}_m(p)), \tilde{z}(p)\| \\ & \leq \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{z}(p)\| + \|\tilde{w}_n(p) - \tilde{w}_m(p), \tilde{z}(p)\| \\ & \rightarrow 0 \text{ as } n, m \rightarrow \infty. \end{aligned}$$

Hence, $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$ is a soft Cauchy sequence. Let $\{\tilde{q}_n(p)\}$ be soft Cauchy sequences in $\tilde{Q}$ and $\{\tilde{\beta}_n(p)\}$ is a soft real Cauchy sequence in $\tilde{\mathbb{R}}$. From Theorem 3.3, there exists a constant $L > 0$ such that $\|\tilde{q}_n(p), \tilde{w}(p)\| \leq L$ for all $n \in \mathbb{N}$ and $\tilde{w}(p) \in SV(\tilde{Q})$. Moreover, since $\{\tilde{\beta}_n(p)\}$ is soft real Cauchy in $\mathbb{R}$, it converges, and hence $\{\tilde{\beta}_n(p)\}$ is bounded. We now consider the sequence $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$. For any $\tilde{w}(p) \in SV(\tilde{Q})$,

$$\begin{aligned} \|\tilde{\beta}_n \tilde{q}_n(p) - \tilde{\beta}_m \tilde{q}_m(p), \tilde{w}(p)\| &= \|\tilde{\beta}_n \tilde{q}_n(p) + \tilde{\beta}_n \tilde{q}_m(p) - \tilde{\beta}_n \tilde{q}_m(p) - \tilde{\beta}_m \tilde{q}_m(p), \tilde{w}(p)\| \\ &\leq |\tilde{\beta}_n - \tilde{\beta}_m| \|\tilde{q}_m(p), \tilde{w}(p)\| + |\tilde{\beta}_n| \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \\ &\rightarrow 0 \end{aligned}$$

This shows that $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$ is a soft Cauchy sequences in $\tilde{Q}$. Hence, the set of soft Cauchy sequences in a soft normed space is closed under addition and scalar multiplication. $\square$

Theorem 3.3. *Let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$. Then, for each $\tilde{w}(p) \in SV(\tilde{Q})$, the sequence $\{\tilde{q}_n(p)\}$ is bounded.*

Proof. Since $\{\tilde{q}_n(p)\}$ is a soft Cauchy sequence in $\tilde{Q}$, for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, we have

$$\lim_{m, n \rightarrow \infty} \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| = 0.$$

which means: $\forall \tilde{\varepsilon} > 0 \exists n_0 \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{\varepsilon}$ for all $n, m > n_0$. Take $\tilde{\varepsilon} = \tilde{1}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{1}$$

for all $n, m > n_0$. Now, consider the finitely many terms before n_0 . Define

$$L = \max \left\{ \|\tilde{q}_1(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\|, \|\tilde{q}_2(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\|, \dots, \|\tilde{q}_{n_0-1}(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\| \right\}.$$

Note that L is finite because it is the maximum of finitely many real numbers. For any $n \in \mathbb{N}$, we estimate $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\|$. If $n < n_0$, then $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \lesssim L$ from definition of L . If $n \geq n_0$, then $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \lesssim 1$. Thus, for all n , $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| < L + 1$. Therefore, we obtain

$$\begin{aligned} \|\widetilde{q}_n(p) - \widetilde{q}_m(p), \widetilde{w}(p)\| &= \|\widetilde{q}_n(p) + \widetilde{q}_{n_0}(p) - \widetilde{q}_{n_0}(p) - \widetilde{q}_m(p), \widetilde{w}(p)\| \\ &\leq \|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| + \|\widetilde{q}_m(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \\ &< 2(L + 1) \end{aligned}$$

for any $n, m \in \mathbb{N}$. Thus, the sequence $\{\widetilde{q}_n(p)\}$ is bounded by the constant $2(L + 1)$. $\square$

Theorem 3.4. *Let $\widetilde{Q}$ be a refined soft 2-normed space and let $\{\widetilde{q}_n(p)\}$ be a soft Cauchy sequence in $\widetilde{Q}$. If a soft subsequence $\{\widetilde{q}_{n_k}(p)\}$ of $\{\widetilde{q}_n(p)\}$ converges to $\{\widetilde{q}(p)\}$ then the sequence $\{\widetilde{q}_n(p)\}$ also converges to $\{\widetilde{q}(p)\}$ in $\widetilde{Q}$.*

Proof. For any $\widetilde{w}(p) \in SV(\widetilde{Q})$ and $p \in P$, we can write

$$\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}(p)\| = \|\widetilde{q}_n(p) + \widetilde{q}_{n_k}(p) - \widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\|.$$

Now let n be large and choose k such that $n_k > n$. Because $\{\widetilde{q}_n(p)\}$ is Cauchy, $\|\widetilde{q}_n(p) - \widetilde{q}_{n_k}(p), \widetilde{w}(p)\| \rightarrow 0$ as $n, n_k \rightarrow \infty$. And since the subsequence $\{\widetilde{q}_{n_k}(p)\}$ converges to $\{\widetilde{q}(p)\}$, $\|\widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \rightarrow 0$ for every $\widetilde{w}(p) \in SV(\widetilde{Q})$. Hence, we have

$$\begin{aligned} \|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}(p)\| &= \|\widetilde{q}_n(p) + \widetilde{q}_{n_k}(p) - \widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{q}_n(p) - \widetilde{q}_{n_k}(p), \widetilde{w}(p)\| + \|\widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\rightarrow 0. \end{aligned}$$

Hence, the sequence $\{\widetilde{q}_n(p)\}$ converges to $\{\widetilde{q}(p)\}$ in $\widetilde{Q}$. $\square$

Lemma 3.1. *Let $\widetilde{Q}$ be a refined soft 2-normed space and let $\{\widetilde{q}(p)\}, \{\widetilde{w}(p)\}, \{\widetilde{z}(p)\} \in SV(\widetilde{Q})$. Then, the following inequality holds for every $p \in P$:*

$$|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|.$$

Proof. Let $\{\widetilde{q}(p)\}, \{\widetilde{w}(p)\}, \{\widetilde{z}(p)\} \in SV(\widetilde{Q})$ and $p \in P$. By the properties of a refined soft 2-norm, we have

$$\begin{aligned} \|\widetilde{q}(p), \widetilde{w}(p)\| &= \|\widetilde{q}(p) + \widetilde{z}(p) - \widetilde{z}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\| + \|\widetilde{z}(p), \widetilde{w}(p)\|. \end{aligned}$$

This implies $|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|$. Similarly, we obtain

$$\begin{aligned} \|\widetilde{z}(p), \widetilde{w}(p)\| &= \|\widetilde{z}(p) + \widetilde{q}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{z}(p) - \widetilde{q}(p), \widetilde{w}(p)\| + \|\widetilde{q}(p), \widetilde{w}(p)\|. \end{aligned}$$

Hence, $|||\widetilde{z}(p), \widetilde{w}(p)\| - \|\widetilde{q}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{z}(p) - \widetilde{q}(p), \widetilde{w}(p)\|$. Combining the above two inequalities, we obtain

$$|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|$$

for every $p \in P$. $\square$

Theorem 3.5. Let $\tilde{Q}$ be a refined soft 2-normed space and let $\{\tilde{q}_n(p)\}$ be soft sequence with $\{\tilde{q}_n(p)\} \rightarrow \{\tilde{q}(p)\}$ in $\tilde{Q}$ for $n \rightarrow \infty$. Then, the following statements hold:

- a) The soft sequence $\{\tilde{q}_n(p)\}$ is bounded,
- b) Every subsequence of $\{\tilde{q}_n(p)\}$ converges to $\{\tilde{q}(p)\}$,
- c) If the soft scalar sequence $\{\tilde{\alpha}_n(p)\}$ converges to $\tilde{\alpha}(p)$, then the sequence $\{\tilde{\alpha}_n(p) \tilde{q}_n(p)\}$ converges to $\tilde{\alpha}(p) \tilde{q}(p)$,
- d) If the sequence $\{\tilde{w}_n(p)\}$ converges to $\{\tilde{w}(p)\}$, then the sequence $\{\tilde{q}_n(p) + \tilde{w}_n(p)\}$ converges to $\tilde{q}(p) + \tilde{w}(p)$,
- e) $\|\tilde{q}_n(p), \tilde{w}(p)\| \rightarrow \|\tilde{q}(p), \tilde{w}(p)\|$ for every $\{\tilde{w}(p)\} \in SV(\tilde{Q})$ and $p \in P$.

Proof. The proof can be easily obtained by a method similar to the above proofs. □

Example 3.3. Let $\tilde{Q} = \mathbb{R}^2$ be equipped with the 2-norm:

$$\|\tilde{q}(p), \tilde{w}(p)\| = \left| \sum_{i=1}^2 \tilde{q}_i(p), \tilde{w}_i(p) \right|$$

where $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p)) \in SV(\mathbb{R}^2)$, $\tilde{w}(p) = (\tilde{w}_1(p), \tilde{w}_2(p)) \in SV(\mathbb{R}^2)$ and $p \in P$ is refined soft 2-norm.

Theorem 3.6. Let $\tilde{Q}$ be a refined soft 2-normed space. Assume that $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ are soft sequences such that $\{\tilde{q}_n(p)\} \rightarrow \{\tilde{q}(p)\}$ and $\{\tilde{w}_n(p)\} \rightarrow \{\tilde{w}(p)\}$ in $\tilde{Q}$ as $n \rightarrow \infty$. Further, assume that the soft scalar sequences $\{\tilde{\alpha}_n(p)\} \rightarrow \{\tilde{\alpha}(p)\}$ and $\{\tilde{\beta}_n(p)\} \rightarrow \{\tilde{\beta}(p)\}$ as $n \rightarrow \infty$. Then, it follows that $\tilde{\alpha}_n(p) \tilde{q}_n(p) + \tilde{\beta}_n(p) \tilde{w}_n(p) \rightarrow \tilde{\alpha}(p) \tilde{q}(p) + \tilde{\beta}(p) \tilde{w}(p)$ in $\tilde{Q}$ as $n \rightarrow \infty$.

Proof. For every $\epsilon > 0$, we can state $\|\tilde{q}_n(p_n) - \tilde{q}(p), \tilde{z}(p)\| \rightarrow 0$ for every $\tilde{z} \in SV(\tilde{Q})$ and $p \in P$ and $\|\tilde{w}_n(p) - \tilde{w}(p), \tilde{z}(p)\| \rightarrow 0$ for every $\tilde{z} \in SV(\tilde{Q})$ and $p \in P$. Further, for every $\epsilon > 0$, we can write $|\tilde{\alpha}_n(p) - \tilde{\alpha}(p)| < \epsilon$ and $|\tilde{\beta}_n(p) - \tilde{\beta}(p)| < \epsilon$. Then, it follows that

$$\begin{aligned} & \left\| \left(\tilde{\alpha}_n(p) \tilde{q}_n(p) + \tilde{\beta}_n(p) \tilde{w}_n(p) \right) - \left(\tilde{\alpha}(p) \tilde{q}(p) + \tilde{\beta}(p) \tilde{w}(p) \right), \tilde{z}(p) \right\| \\ & \leq \|\tilde{\alpha}_n(p) \tilde{q}_n(p) - \tilde{\alpha}(p) \tilde{q}(p), \tilde{z}(p)\| + \|\tilde{\beta}_n(p) \tilde{w}_n(p) - \tilde{\beta}(p) \tilde{w}(p), \tilde{z}(p)\| \\ & \rightarrow 0 \end{aligned}$$

from Theorem 3.5. Thus, the proof is complete. □

Theorem 3.7. Let $\tilde{Q}$ be a refined soft 2-normed space. If $\{\tilde{q}_n(p)\}$ is a soft Cauchy sequence in $\tilde{Q}$, then for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, the soft real sequence $(\|\tilde{q}_n(p), \tilde{w}(p)\|_{n=1}^\infty)$ is a real soft Cauchy sequence.

Proof. Let $\tilde{Q}$ be a refined soft 2-normed space and let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in $\tilde{Q}$. By definition, for every soft real number $\tilde{\epsilon} \geq \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{\epsilon}$$

for all $n, m > k$ and for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. Fix an arbitrary element $\tilde{w}(p) \in SV(\tilde{Q})$. Then, for all $n, m > k$, we have

$$||\|\tilde{q}_n(p), \tilde{w}(p)\| - \|\tilde{q}_m(p), \tilde{w}(p)\||| \leq \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\|.$$

Using the soft Cauchy property of $\{\tilde{q}_n(p)\}$, it follows that $||\|\tilde{q}_n(p), \tilde{w}(p)\| - \|\tilde{q}_m(p), \tilde{w}(p)\||| < \tilde{\epsilon}$ for all $n, m > k$. Hence, the real soft sequence $\|\tilde{q}_n(p), \tilde{w}(p)\|$ satisfies the Cauchy condition. Therefore, it is a real soft Cauchy sequence. $\square$

Definition 3.3. Let $\tilde{Q}$ be a refined soft 2-normed space. If every soft Cauchy sequence in $\tilde{Q}$ converges to a soft element of $\tilde{Q}$, then $\tilde{Q}$ is called a complete refined soft 2-normed space, or a refined soft 2-Banach space.

Definition 3.4. Let $\tilde{Q}$ be a refined soft 2-normed space. Recall that we assume that $\tilde{Q}$ has finite dimension d , with $2 \leq d < \infty$, unless stated otherwise. Let $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ be a soft basis of $\tilde{Q}$. Then every soft vector $\tilde{q}(p) \in SV(\tilde{Q})$ can be expressed uniquely in the form

$$\tilde{q}(p) = \sum_{i=1}^d \tilde{\alpha}_i(p) \cdot \tilde{a}_i(p) \text{ where } \tilde{\alpha}_i(p) \text{ are soft real scalars for } i = 1, 2, \dots, d.$$

Further, since $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, it is soft refined linearly independent and $\text{span}\tilde{Q}$. Consequently, the above representation of $\tilde{q}(p)$ is unique. Let $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$ be given by $\tilde{q}(p) = \sum_{i=1}^d \tilde{\alpha}_i(p) \cdot \tilde{a}_i(p)$ and $\tilde{w}(p) = \sum_{i=1}^d \tilde{\beta}_i(p) \cdot \tilde{a}_i(p)$ for $p \in P$. Then the soft 2-norm $\|\tilde{q}(p), \tilde{w}(p)\|$ can be expressed in terms of the soft basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ and the corresponding soft coordinates $\tilde{\alpha}_i, \tilde{\beta}_i$.

Furthermore, it follows from the results of [4] that if $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, then the set $\{\tilde{a}_1(p), \tilde{a}_2(p), \dots, \tilde{a}_d(p)\}$ constitutes a basis of $\tilde{Q}(p) = Q$ for all $p \in P$.

Lemma 3.2. A soft sequence $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ is said to soft converge to $\tilde{q}(p)$ if and only if $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$.

Proof. If $\{\tilde{q}_n(p)\}$ is soft convergent to $\tilde{q}(p)$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. Conversely, suppose that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. We now show that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| \rightarrow 0$ for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. Since $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, there exist soft scalars $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_d\}$ such that $\tilde{w}(p) = \tilde{\alpha}_1(p) \cdot \tilde{a}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{a}_2(p) + \dots + \tilde{\alpha}_d(p) \cdot \tilde{a}_d(p)$. Using the properties of the refined soft 2-norm, we obtain

$$\begin{aligned} \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| &= \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{\alpha}_1(p) \cdot \tilde{a}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{a}_2(p) + \dots + \tilde{\alpha}_d(p) \cdot \tilde{a}_d(p)\| \\ &\leq |\tilde{\alpha}_1(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_1(p)\| + |\tilde{\alpha}_2(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_2(p)\| + \dots + \\ &\quad |\tilde{\alpha}_d(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_d(p)\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof. $\square$

From the above lemma, we can also derive the following result.

Lemma 3.3. A soft sequence $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ is said to soft converge to $\tilde{q}(p)$ if and only if $\max \{\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. The proof is immediate. $\square$

Lemma 3.4. For a $\tilde{q}(p) = \sum_{i=1}^2 \tilde{\alpha}_i(p) \cdot \tilde{e}_i(p)$ and $\tilde{w}(p) = \sum_{i=1}^2 \tilde{\beta}_i(p) \cdot \tilde{e}_i(p)$, we obtain $\|\tilde{q}(p), \tilde{w}(p)\| = \left| \tilde{\alpha}_1(p) \tilde{\beta}_1(p) - \tilde{\beta}_2(p) \tilde{\alpha}_2(p) \right| \|\tilde{e}_1(p), \tilde{e}_2(p)\|$, $\tilde{e}_1(p) = \left(\tilde{1}(p), \tilde{0}(p) \right) = (1, 0)$ and $\tilde{e}_2(p) = \left(\tilde{0}(p), \tilde{1}(p) \right) = (0, 1)$.

Proof. For every $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$, we find

$$\begin{aligned} \|\tilde{q}(p), \tilde{w}(p)\| &= \left\| \sum_{i=1}^2 \tilde{\alpha}_i(p) \cdot \tilde{e}_i(p), \sum_{i=1}^2 \tilde{\beta}_i(p) \cdot \tilde{e}_i(p) \right\| \\ &= \left\| \tilde{\alpha}_1(p) \cdot \tilde{e}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{e}_2(p), \tilde{\beta}_1(p) \cdot \tilde{e}_1(p) + \tilde{\beta}_2(p) \cdot \tilde{e}_2(p) \right\| \\ &= \left| \tilde{\alpha}_1(p) \tilde{\beta}_1(p) - \tilde{\beta}_2(p) \tilde{\alpha}_2(p) \right| \|\tilde{e}_1(p), \tilde{e}_2(p)\|. \end{aligned}$$

$\square$

Theorem 3.8. A two-dimensional refined soft linear 2-normed space is said to be a refined soft 2-Banach space if it is complete over the underlying field.

Proof. Let $\tilde{Q}$ be a refined soft 2-normed space with standard soft basis $\{\tilde{e}_1, \tilde{e}_2\}$. Let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in $\tilde{Q}$. Thus there exists a natural number $k \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \lesssim \tilde{\epsilon} \text{ and } \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}'(p)\| \lesssim \tilde{\epsilon}$$

for all $n, m > k$ and for every $\tilde{w}(p), \tilde{w}'(p) \in SV(\tilde{Q})$ and $p \in P$. Here, $\tilde{w}$ and $\tilde{w}'$ are assumed to be refined linearly independent. If they were refined linearly dependent, this representation would not be necessary. Let $\tilde{q}_n(p) = \tilde{q}_{n1}(p) \cdot \tilde{e}_1(p) + \tilde{q}_{n2}(p) \cdot \tilde{e}_2(p)$, $\tilde{w}(p) = \tilde{w}_1(p) \cdot \tilde{e}_1(p) + \tilde{w}_2(p) \cdot \tilde{e}_2(p)$ and $\tilde{w}'(p) = \tilde{w}'_1(p) \cdot \tilde{e}_1(p) + \tilde{w}'_2(p) \cdot \tilde{e}_2(p)$. From Lemma 3.4, we obtain

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| = |\tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \|\tilde{e}_1(p), \tilde{e}_2(p)\|$$

and similarly

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}'(p)\| = |\tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \|\tilde{e}_1(p), \tilde{e}_2(p)\|.$$

$\{\tilde{e}_1, \tilde{e}_2\}$ is refined linearly independent; therefore

$$|\tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

and

$$|\tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

as $n, m \rightarrow \infty$. From here, we obtain

$$|\tilde{w}'_2(p) \tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) \tilde{w}'_2(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

and

$$|-\tilde{w}_2(p) \tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) + \tilde{w}_2(p) \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

as $n, m \rightarrow \infty$. By adding the two equations, we get

$$\left| \left(\widetilde{w}_2(p) \widetilde{w}'_1(p) - \widetilde{w}_1(p) \widetilde{w}'_2(p) \right) (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_{m2}(p)) \right| \rightarrow 0$$

as $n, m \rightarrow \infty$. Since $\widetilde{w}$ and $\widetilde{w}'$ are linearly independent, this condition holds only when $|\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_{m2}(p)| \rightarrow 0$ as $n, m \rightarrow \infty$. This implies that $\{\widetilde{\alpha}_{n2}\}$ is a Cauchy sequence of soft real numbers. By a similar argument, $\{\widetilde{\alpha}_{n1}\}$ can also be shown to be real soft Cauchy. Since $\{\widetilde{\alpha}_{n1}\}$ and $\{\widetilde{\alpha}_{n2}\}$ are real soft Cauchy sequences, there are real soft numbers $\widetilde{b}_1$ and $\widetilde{b}_2$ such that $\widetilde{\alpha}_{n1}(p) \rightarrow \widetilde{b}_1(p)$ and $\widetilde{\alpha}_{n2}(p) \rightarrow \widetilde{b}_2(p)$ as $n \rightarrow \infty$. If we take $\widetilde{q}(p) = \widetilde{\alpha}_1(p) \cdot \widetilde{e}_1(p) + \widetilde{\alpha}_2(p) \cdot \widetilde{e}_2(p) \in SV(\widetilde{Q})$. Then we get

$$\begin{aligned} \left\| \widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}''(p) \right\| &= \left\| \begin{aligned} &(\widetilde{\alpha}_{n1}(p) - \widetilde{\alpha}_1(p)) \widetilde{e}_1(p) + (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_2(p)) \widetilde{e}_2(p), \widetilde{w}_1''(p) \widetilde{e}_1(p) \\ &+ \widetilde{w}_2''(p) \widetilde{e}_2(p) \end{aligned} \right\| \\ &= \left| \widetilde{w}_2''(p) (\widetilde{\alpha}_{n1}(p) - \widetilde{\alpha}_1(p)) - \widetilde{w}_1''(p) (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_2(p)) \right| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus, we obtain $\widetilde{q}_n(p) - \widetilde{q}(p)$ as $n \rightarrow \infty$. Therefore, $\widetilde{Q}$ is a soft 2-Banach space. $\square$

Similarly to classical 2-normed spaces, one can relate a refined soft 2-normed space to a soft normed linear space through a basis, using the norm. If $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$ is a soft basis of the refined soft 2-normed space $\widetilde{Q}$, we can define the following norm on $\widetilde{Q}$:

$$\|\widetilde{q}(p)\|_0 = \max \{ \|\widetilde{q}(p), \widetilde{a}_i(p)\| : i = 1, 2, \dots, d \}. \quad (3.1)$$

One can observe that the norm $\|\cdot\|_0$ defined above is a soft norm on $\widetilde{Q}$.

Example 3.4. Let $\widetilde{Q}$ be a refined soft 2-normed space with refined 2-norm $\|\cdot, \cdot\|$. We can define another soft norm $\|\cdot\|_1$ on $\widetilde{Q}$ by $\|\widetilde{q}(p)\|_1 = \sum_{i=1}^d \|\widetilde{q}_i(p), \widetilde{a}_i(p)\|$. This norm is well-defined and satisfies all the properties of a soft norm.

Theorem 3.9. Let $\widetilde{Q}$ be a refined soft 2-normed space with refined 2-norm $\|\cdot, \cdot\|$. The soft sequence $\{\widetilde{q}_n\}$ is soft convergent to $\widetilde{q}$ in $\widetilde{Q}$ if and only if $\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$.

Proof. If $\{\widetilde{q}_n\}$ is soft convergent to $\widetilde{q}$ in $\widetilde{Q}$, then it is clear that $\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. From 3.1, we can write

$$\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 = \max \{ \|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| : i = 1, 2, \dots, d \} \rightarrow 0$$

as $n \rightarrow \infty$. Conversely, suppose that $\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. From 3.1, we get $\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Let $\widetilde{Q}$ be a refined soft 2-normed space with a soft basis $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$. Using the $\|\cdot\|_0$ norm generated by the two-norm, the open ball centered at $\widetilde{q}$ in $\widetilde{Q}$ with soft radius $\widetilde{r} \succ \widetilde{0}$ is defined as

$$S_{\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}}(\widetilde{q}(p), \widetilde{r}(p)) = \{ \widetilde{w}(p) : \|\widetilde{q}(p) - \widetilde{w}(p)\|_0 \prec \widetilde{r}(p) \}.$$

This is called the soft open ball generated by the soft basis $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$. We can also express the definition of convergence in a refined soft 2-normed space as follows:

Classical: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall \tilde{\epsilon} > \tilde{0}, \exists n \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow \tilde{q}_n(p) \in S_{\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}}(\tilde{q}(p), \tilde{r}(p))$.

Using the 0–norm: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall \tilde{\epsilon} > \tilde{0}, \exists n \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow \|\tilde{q}_n(p) - \tilde{q}(p)\|_0 < \tilde{\epsilon}$.

Component-wise via the basis vectors: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall i = 1, 2, \dots, d, \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$.

Thus, convergence in a refined soft 2-normed space can be described either in terms of the 0–norm generated by the basis or equivalently via the convergence of the 2-norms with respect to each basis vector.

Example 3.5. Let $\tilde{Q} = \widetilde{\mathbb{R}^2}$ with the refined soft 2-norm defined by

$$\|\tilde{q}(p), \tilde{w}(p)\| = |\tilde{q}_1(p), \tilde{w}_2(p) - \tilde{q}_2(p), \tilde{w}_1(p)|,$$

for $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p)) \in SV(\widetilde{\mathbb{R}^2})$, $\tilde{w} = (\tilde{w}_1(p), \tilde{w}_2(p)) \in SV(\widetilde{\mathbb{R}^2})$. Let $\tilde{i} = (\tilde{1}, \tilde{0}), \tilde{j} = (\tilde{0}, \tilde{1})$ be the soft standard basis of $\widetilde{\mathbb{R}^2}$. Then for $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p))$ we have $\|\tilde{q}(p), \tilde{i}(p)\| = |\tilde{q}_2(p)|$ and $\|\tilde{q}(p), \tilde{j}(p)\| = |\tilde{q}_1(p)|$. This generates a soft norm. Using these, we can define $\|\tilde{q}(p)\|_1 = \sum_{i=1}^2 \|\tilde{q}_i(p), \tilde{e}_i(p)\| = |\tilde{q}_1(p)| + |\tilde{q}_2(p)|$.

Lemma 3.5. $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space if and only if $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space, where $\|\cdot\|$ is the soft norm generated by the refined soft 2-norm $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$.

Proof. Assume that $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space and that $\{\tilde{q}_n\}$ is a soft Cauchy sequence in the soft normed space $(\tilde{Q}, P, \|\cdot\|)$. Then, for every $\tilde{\epsilon} > \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p)\| < \tilde{\epsilon}$ for all $n, m > k$, and for every $p \in P$. From the equality $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$, we obtain $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{a}_i(p)\| < \tilde{\epsilon}$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$, for every $p \in P$, and $i = 1, 2, \dots, d$. Hence, since $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space, it follows that $\tilde{q}_n(p) \rightarrow \tilde{q}(p) \in SV(\tilde{Q})$ as $n \rightarrow \infty$. If $\{\tilde{q}_n\}$ is soft convergent to $\tilde{q}$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. Consequently, $\|\tilde{q}_n(p) - \tilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space.

Conversely, assume that $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space and that $\{\tilde{q}_n\}$ is a soft Cauchy sequence in the refined soft 2-normed space $(\tilde{Q}, P, \|\cdot, \cdot\|)$. Then, for every $\tilde{\epsilon} > \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{a}_i(p)\| < \tilde{\epsilon}$ for all $n, m > k$, and for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$, and for every $p \in P$. From the equality $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$, we obtain $\|\tilde{q}_n(p) - \tilde{q}_m(p)\|_0 < \tilde{\epsilon}$ for every $p \in P$. Hence, since $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space, it follows that $\tilde{q}_n(p) \rightarrow \tilde{q}(p) \in SV(\tilde{Q})$ as $n \rightarrow \infty$. If $\{\tilde{q}_n\}$ is soft convergent to $\tilde{q}$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p)\|_0 \rightarrow 0$. Consequently, $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ for every $i = 1, 2, \dots, d$. as $n \rightarrow \infty$. Therefore, $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space. $\square$

Any refined soft 2-normed space $(\tilde{Q}, P, \|\cdot, \cdot\|)$ whose soft 2-norm satisfies the condition

$$\|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 + \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 = 2 \left(\|\tilde{q}(p), \tilde{z}(p)\|^2 + \|\tilde{w}(p), \tilde{z}(p)\|^2 \right)$$

will be regarded as a refined soft 2-inner product space, with the refined soft 2-inner product defined by

$$\langle \tilde{q}(p), \tilde{w}(p) | \tilde{z}(p) \rangle = \frac{1}{4} \left(\|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 - \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 \right)$$

[17].

Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space. For any nonzero soft vector $\tilde{z}(p) \in SV(\tilde{Q})$, let $M(\tilde{z}(p))$ denote the soft subspace of $\tilde{Q}$ generated by $\tilde{z}(p)$. Let $\tilde{Q}_{\tilde{z}(p)} = \tilde{Q}/M(\tilde{z}(p))$ be the corresponding soft quotient space. For each $\tilde{q}(p) \in SV(\tilde{Q})$, we denote the equivalence class of $\tilde{q}(p)$ with respect to $M(\tilde{z}(p))$ by $(\tilde{q}(p))_{\tilde{z}(p)}$. The space $\tilde{Q}_{\tilde{z}(p)}$ becomes a soft linear space under the operations defined by soft scalar multiplication

$$\tilde{\alpha}(p) (\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{\alpha}(p) \tilde{q}(p))_{\tilde{z}(p)}$$

and addition

$$(\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} = (\tilde{q}(p) + \tilde{w}(p))_{\tilde{z}(p)}.$$

If $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$ satisfy $(\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{w}(p))_{\tilde{z}(p)}$, then $\|\tilde{q}(p), \tilde{z}(p)\| = \|\tilde{w}(p), \tilde{z}(p)\|$ since $\|\tilde{q}(p), \tilde{z}(p)\| - \|\tilde{w}(p), \tilde{z}(p)\| \leq \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\| = 0$. Thus, the function $\|\cdot\|_{\tilde{z}(p)}$ on $\tilde{Q}_{\tilde{z}(p)}$ defined by $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p), \tilde{z}(p)\|$ is well defined.

Theorem 3.10. $\|\cdot\|_{\tilde{z}(p)}$ is a soft norm on $\tilde{Q}_{\tilde{z}(p)}$.

Proof. $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = 0$ if and only if $\|\tilde{q}(p), \tilde{z}(p)\| = 0$ if and only if $\tilde{q}(p) \in M(\tilde{z}(p))$ i.e., if and only if $(\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{0}(p))_{\tilde{z}(p)}$.

$\left\| \tilde{\alpha}(p) (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \left\| (\tilde{\alpha}(p) \tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{\alpha}(p) \tilde{q}(p), \tilde{z}(p)\| = |\tilde{\alpha}(p)| \|\tilde{q}(p), \tilde{z}(p)\| = |\tilde{\alpha}(p)| \left\| \tilde{q}(p)_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}$ for $\tilde{\alpha}(p) \in \mathbb{R}$.

$\left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \left\| (\tilde{q}(p) + \tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\| \leq \|\tilde{q}(p), \tilde{z}(p)\| + \|\tilde{w}(p), \tilde{z}(p)\| = \left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}.$ $\square$

Theorem 3.11. Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space. This refined soft 2-normed space is a refined soft 2-inner product space if and only if for every non-zero soft vector $\tilde{z}(p) \in SV(\tilde{Q})$, $(\tilde{Q}_{\tilde{z}(p)}, \|\cdot\|_{\tilde{z}(p)})$ is a soft inner product space.

Proof. Assume that $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space where the refined soft 2-norm obeys the parallelogram identity. Let $\tilde{z}(p)$ be an arbitrary non-zero vector in $SV(\tilde{Q})$. For any elements $(\tilde{q}(p))_{\tilde{z}(p)}, (\tilde{w}(p))_{\tilde{z}(p)}$ in the quotient space $\tilde{Q}_{\tilde{z}(p)}$, we utilize the functional definition of the induced norm $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p), \tilde{z}(p)\|$. By applying the refined soft 2-norm properties, the following identity holds:

$$\begin{aligned} & \left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \\ &= \|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 + \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 \\ &= 2 \left(\|\tilde{q}(p), \tilde{z}(p)\|^2 + \|\tilde{w}(p), \tilde{z}(p)\|^2 \right) \end{aligned}$$

$$= 2 \left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2.$$

This result demonstrates that the induced soft norm $\|\cdot\|_{\tilde{z}(p)}$ on the space $\tilde{Q}_{\tilde{z}(p)}$ satisfies the standard parallelogram law, effectively reducing the refined soft 2-normed structure to a classical soft normed structure. Conversely, assume that for every non-zero $\tilde{z}(p) \in SV(\tilde{Q})$, the induced soft norm $\|\cdot\|_{\tilde{z}(p)}$ on $\tilde{Q}_{\tilde{z}(p)}$ satisfies the parallelogram law. For any $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$, we can write:

$$\begin{aligned} & \|(\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)}, \tilde{z}(p)\|^2 + \|(\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)}, \tilde{z}(p)\|^2 \\ &= \left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \\ &= 2 \left(\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \right) \\ &= 2 \left(\| \tilde{q}(p), \tilde{z}(p) \|^2 + \| \tilde{w}(p), \tilde{z}(p) \|^2 \right). \end{aligned}$$

□

Now we will give the Hahn-Banach extension theorem for refined soft 2-norm spaces. This theorem is related to the generalization of the Hahn-Banach extension theorem-one of the fundamental results of linear functional analysis-to the refined 2-norm, as well as to the properties of the dual space. This theorems is presented through the connection established between the refined soft 2-norm and the soft norm in a way that remains consistent with the context of the study.

Theorem 3.12. (Refined Soft 2-Normed Hahn-Banach Extension) Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space and let $\tilde{w}(p) \in SV(\tilde{Q})$. Let $\tilde{M} \subset \tilde{Q}$ be a soft linear subspace and let $\tilde{f} : \tilde{M} \rightarrow \tilde{\mathbb{R}}$ be a soft linear functional satisfying $|\tilde{f}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p) \in SV(\tilde{M})$ and some soft constant $\tilde{K} \gtrsim \tilde{0}$. Then there exists a soft linear functional $\tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}}$ such that $\tilde{f}(\tilde{q}) = \tilde{F}(\tilde{q})$ and $|\tilde{F}(\tilde{q})(p)| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p) \in SV(\tilde{M})$.

Proof. Define $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$. From Theorem 3.11, $(\tilde{Q}, \|\cdot, \cdot\|_{\tilde{w}(p)})$ is a soft normed linear space. From the assumption, we have $|\tilde{f}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \| = \tilde{K} \| \tilde{q}(p) \|_{\tilde{w}(p)}$ for all $\tilde{q}(p) \in SV(\tilde{M})$. Thus, $\tilde{f}$ is a bounded soft linear functional on the soft normed space $(\tilde{M}, \|\cdot, \cdot\|_{\tilde{w}(p)})$. By the Hahn-Banach Theorem, the soft functional $\tilde{f}$ can be extended to a soft linear functional $\tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}}$ such that $|\tilde{F}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p) \|_{\tilde{w}(p)}$ for all $\tilde{q}(p) \in SV(\tilde{Q})$. Since $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$, we obtain $|\tilde{F}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$. Therefore, the soft linear functional $\tilde{f}$ admits an extension $\tilde{F}$ on $\tilde{Q}$ preserving the bound. □

Theorem 3.13. Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space and let $\tilde{w}(p) \in SV(\tilde{Q})$ be a fixed nonzero soft vector. Define

$$\tilde{Q}_{\tilde{w}(p)}^* = \left\{ \tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}} : \tilde{F} \text{ is a soft linear and } |\tilde{F}(\tilde{q})(p)| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \| \text{ for some soft constant } \tilde{K} \gtrsim \tilde{0} \right\}.$$

Then $\tilde{Q}_{\tilde{w}(p)}^*$ forms the soft dual space of the soft normed space $(\tilde{Q}, \|\cdot, \cdot\|_{\tilde{w}(p)})$, where $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$.

Proof. From the definition, $\|\tilde{q}(p)\|_{\tilde{w}(p)} = \|\tilde{q}(p), \tilde{w}(p)\|$ defines a soft norm on $(\tilde{Q}, \|\cdot\|_{\tilde{w}(p)})$, which is a soft normed linear space. By the Hahn–Banach Theorem, every bounded soft linear functional defined on a subspace of $\tilde{Q}$ can be extended to the whole space while preserving the bound. Therefore, the soft set $\tilde{Q}_{\tilde{w}(p)}^*$ coincides with the dual space of the soft normed space $(\tilde{Q}, \|\cdot\|_{\tilde{w}(p)})$. $\square$

4. CONCLUSION

In conclusion, this work contributes to the development of a consistent soft linear framework by clarifying the relationship between soft norms and soft 2-norms through an appropriate soft basis. The analysis demonstrates that refined definitions are essential for preserving fundamental structural properties in soft linear spaces. These findings provide a solid foundation for future research, particularly in the study of completeness, convergence, continuity, and linear operators in soft normed and soft 2-normed spaces. Further investigations may also explore applications of these refined structures in optimization, decision-making, and other uncertainty-based mathematical models.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

Conflict of interest. The authors declare no potential conflict of interests.

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