



## LINEAR DIOPHANTINE FUZZY IDEAL THEORY IN $d$ -ALGEBRAS

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**Abstract.** This study presents a comprehensive investigation into the integration of linear Diophantine fuzzy sets within the framework of  $d$ -algebraic structures, offering a refined perspective on the interaction between fuzzy logic and abstract algebra. The concepts of (weak) linear Diophantine fuzzy  $d$ -subalgebras and (weak) linear Diophantine fuzzy  $d$ -ideals are rigorously defined and formalized, supported by illustrative examples to enhance conceptual clarity. A thorough examination is conducted to identify and characterize the algebraic properties of these structures, with a particular focus on establishing the necessary and sufficient conditions under which a structural transformation between the two notions is feasible. Furthermore, the study delves into two distinguished classes of ideals in  $d$ -algebras through the lens of linear Diophantine fuzzy theory, emphasizing their structural alignment and interaction with classical  $d$ -ideals.

**Keywords:**  $d$ -algebra, Linear Diophantine fuzzy set, Weak linear Diophantine fuzzy set, Linear Diophantine fuzzy subalgebra, Linear Diophantine fuzzy ideal

**MSC (2020):** 08A72, 03G25, 16D25, 94D05.

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### 1. INTRODUCTION

In real world problems, the limitations of classical mathematical tools become apparent due to the inherent vagueness and uncertainty present in many complex systems. The introduction of fuzzy set theory by Zadeh [23] marked a significant shift in mathematical modeling, enabling more flexible and nuanced representations of imprecise information. Fuzzy logic, derived from fuzzy sets, provides a framework where truth values range in the interval  $[0, 1]$ , offering an alternative to classical binary logic and probability theory for managing uncertainty and ambiguity. Atanassov [5] extended the concept of fuzzy sets by introducing intuitionistic fuzzy sets, in which the sum of an element's membership degree and non-membership degree does not exceed 1.

Two significant classes of logic algebras, namely BCK-algebras and BCI-algebras, were introduced in the 1960s by Imai and Iseki [11], [12], and have since been the subject of extensive study by numerous researchers. It is well established that BCK-algebras constitute

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Received: 2025.11.24

Revised: 2026.06.02

Accepted: 2026.07.02

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a proper subclass of BCI-algebras. Among the generalizations of BCK-algebras, the concept of  $d$ -algebras was introduced by Neggers and Kim [16]. Furthermore, the notions of  $d$ -subalgebras,  $d$ -ideals, and several related concepts, along with the relationships among them, were defined and examined by Neggers et al. [17]. Several studies integrating different types of ideals and fuzzy structures in the context of BCK-algebras and  $d$ -algebras can be found in [2], [3], [14], [21], [22].

Riaz and Hashmi [19] proposed the concept of a linear Diophantine fuzzy set by incorporating reference parameters inspired by the classical linear Diophantine equation  $ax + by = c$  from number theory. This approach eliminates the constraints typically imposed on membership and non-membership degrees, thereby enabling decision-makers to assign values freely without predefined restrictions. Diophantine equations, particularly linear Diophantine equations, play a fundamental role in number theory and algebra due to their simplicity and wide range of applications in cryptography, coding theory, and optimization. Recently, there has been increasing interest in combining the flexibility of fuzzy logic with the structural rigor of Diophantine equations, leading to the notion of linear Diophantine fuzzy sets. These sets aim to represent collections of fuzzy elements subject to linear Diophantine constraints, enriching fuzzy modeling with arithmetic consistency.

The interplay between algebraic structures and linear Diophantine fuzzy sets was first introduced by Kamacı [13] in 2021. In his pioneering work, he examined finite linear Diophantine fuzzy substructures within classical algebraic systems such as groups, rings, and fields. Subsequently, Al-Tahan et al. [4] extended this line of research by investigating linear Diophantine fuzzy subpolygroups of a polygroup. Inspired by recent developments on linear Diophantine fuzzy substructures and the basic literature on fuzzy algebraic structures, Muhiuddin et al. focused on the study of linear Diophantine fuzzy sets in the context of BCK/BCI algebras [15]. Other researchers applied linear Diophantine fuzzy sets to different algebraic structures [1], [6], [20]. Building upon the application of linear Diophantine fuzzy sets to BE-algebras, Öztürk et al. [18] introduced weak linear Diophantine fuzzy BE-algebras and investigated their fundamental properties. However, extending the theory to  $d$ -algebras is not merely a direct translation. The order relation in  $d$ -algebras, defined by  $\nu * \omega = 0$ , together with the additional  $d$ -condition, yields structural properties that do not arise in the BE-algebra setting. These features enable us to establish new relationships between linear Diophantine fuzzy  $d$ -subalgebras and  $d$ -ideals, particularly in  $d$ -algebras and edge  $d$ -algebras, and to investigate the behavior of weak linear Diophantine level sets within this distinct algebraic framework.

Recently, linear Diophantine fuzzy theory has attracted considerable attention due to its applicability in decision-making and uncertainty modeling. Several aggregation operators and decision-making approaches have been developed in this framework [7, 9, 10]. In particular, roughness and approximation-based extensions of linear Diophantine fuzzy sets have also been investigated, providing new tools for handling uncertainty in more complex environments [8]. These studies demonstrate the growing importance of linear Diophantine fuzzy structures in both theoretical and applied research, further motivating their study in different algebraic systems.

The primary motivation of the present study is to investigate the intersection between linear Diophantine fuzzy sets and  $d$ -algebraic structures. This integration not only enhances the theoretical framework of fuzzy algebra but also introduces novel directions for applying fuzzy arithmetic logic in contexts characterized by uncertainty within a structurally defined setting. This work aims to contribute to the ongoing synthesis of fuzzy set theory and algebraic systems, paving the way for further advancements in the theoretical foundations and applications of fuzzy algebra.

Linear Diophantine fuzzy sets generalize several existing fuzzy models, including intuitionistic and Pythagorean fuzzy sets, by introducing the reference parameters  $\tau$  and  $\varphi$ . These parameters provide additional flexibility in handling membership and non-membership information. Motivated by this feature, we investigate linear Diophantine fuzzy ideals in  $d$ -algebras and establish their basic properties.

The structure of this paper is organized as follows. In Section 2, we provide the necessary preliminaries and recall fundamental definitions related to  $d$ -algebras and linear Diophantine fuzzy sets. Section 3 is devoted to the introduction of linear Diophantine fuzzy  $d$ -subalgebras and linear Diophantine fuzzy  $d$ -ideals in  $d$ -algebras. In this section, illustrative examples are provided and several basic properties are examined. In Section 4, we define the notions of linear Diophantine fuzzy  $d^\#$ -ideal and linear Diophantine fuzzy commutative  $d$ -ideal in  $d$ -algebras. In Section 5, we present the definitions of weak linear Diophantine fuzzy  $d$ -subalgebras and weak linear Diophantine fuzzy  $d$ -ideals with illustrative examples, and further provide the necessary condition for nonempty weak linear Diophantine level sets to constitute  $d$ -subalgebras.

## 2. PRELIMINARIES

In this section, we provide a foundational background by presenting several fundamental results concerning fuzzy sets and intuitionistic fuzzy sets, linear Diophantine fuzzy sets,  $d$ -algebras and some types of  $d$ -ideals. These preliminary concepts will serve as a basis for the subsequent theoretical developments in the paper.

**Definition 2.1.** ([16]) *A non-empty set  $(\Upsilon; *, 0)$  of type  $(2, 0)$  is called a  $d$ -algebra if  $\forall v, \omega \in \Upsilon$ , the following conditions hold:*

- (1)  $v * v = 0$ ,
- (2)  $0 * v = 0$ ,
- (3)  $v * \omega = 0$  and  $\omega * v = 0$  implies  $v = \omega$ .

A BCK-algebra is a  $d$ -algebra  $(\Upsilon; *, 0)$  satisfying the following additional axioms:  $\forall v, \omega, \eta \in \Upsilon$ ,

- (4)  $((v * \omega) * (v * \eta)) * (\eta * \omega) = 0$ ,
- (5)  $(v * (v * \omega)) * \omega = 0$ .

**Definition 2.2.** ([17]) *Let  $(\Upsilon; *, 0)$  be a  $d$ -algebra, and  $\emptyset \neq \Phi \subseteq \Upsilon$ . If for any  $v, \omega \in \Phi$ , the element  $v * \omega$  also belong to  $\Phi$ , then  $\Phi$  is called a  $d$ -subalgebra of  $\Upsilon$ . Moreover,  $\Phi$  is called a  $d$ -ideal of  $\Upsilon$  if it satisfies the following conditions:*

- (dI-1)  $v * \omega \in \Phi$  and  $\omega \in \Phi$  imply  $v \in \Phi$ ,  
 (dI-2)  $v \in \Phi$  and  $\omega \in \Upsilon$  imply  $v * \omega \in \Phi$ , i.e.,  $\Phi * \Upsilon \subseteq \Phi$ .

**Lemma 2.1.** ([17]) *If  $\Phi$  is a  $d$ -ideal of a  $d$ -algebra  $\Upsilon$ , then  $0 \in \Phi$ .*

**Definition 2.3.** ([17]) *Let  $\Upsilon$  be a  $d$ -algebra. A  $d$ -ideal  $\Phi$  of  $\Upsilon$  is called a  $d^\#$ -ideal of  $\Upsilon$ , if  $\nu * \omega \in \Phi$  and  $\omega * \eta \in \Phi$  imply  $\nu * \eta \in \Phi$  for all  $\nu, \omega, \eta \in \Upsilon$ .*

**Definition 2.4.** ([16]) *Let  $(\Upsilon; *, 0)$  be a  $d$ -algebra, and  $\nu, \omega \in \Upsilon$ . Define the operation  $\nu \wedge \omega = \omega * (\omega * \nu)$ . Furthermore, a  $d$ -algebra  $\Upsilon$  is said to be commutative if  $\nu \wedge \omega = \omega \wedge \nu$  holds for all  $\nu, \omega \in \Upsilon$ .*

**Definition 2.5.** ([22]) *A non-empty subset  $\Phi$  of a  $d$ -algebra  $\Upsilon$  is called a commutative  $d$ -ideal of  $\Upsilon$  if it satisfies  $0 \in \Phi$  and  $(\nu * \omega) * \eta \in \Phi$ ,  $\eta \in \Phi \Rightarrow \nu * (\omega \wedge \nu) \in \Phi$  for all  $\nu, \omega, \eta \in \Phi$ .*

**Definition 2.6.** ([16]) *Let  $\Upsilon$  be a  $d$ -algebra and  $v \in \Upsilon$ . Define  $v * \Upsilon = \{v * \omega \mid \omega \in \Upsilon\}$ . If for any  $v \in \Upsilon$ ,  $v * \Upsilon = \{v, 0\}$ , then  $\Upsilon$  is called an edge  $d$ -algebra.*

**Remark 2.1.** ([16]) *If  $(\Upsilon, \leq)$  is an ordered set, then the operation  $*$  on  $\Upsilon$  given by  $\nu * \omega = 0$  iff  $\nu \leq \omega$ .*

**Lemma 2.2.** ([16]) *If  $\Upsilon$  is an edge  $d$ -algebra, then  $v * 0 = v$  for any  $v \in \Upsilon$ .*

**Lemma 2.3.** ([16],[17]) *Let  $\Upsilon$  be a  $d$ -algebra. Then, for all  $\nu, \omega \in \Upsilon$ , the inequality  $(\nu * \omega) * (\eta * \omega) \leq \nu * \eta$  holds.*

Moreover, if  $\Upsilon$  is an edge  $d$ -algebra, then the following conditions hold: For all  $\nu, \omega, \eta \in \Upsilon$

- i.  $(\nu * (\nu * \omega)) * \omega = 0$ ,
- ii.  $(\nu * \omega) * \eta = (\nu * \eta) * \omega$ .

**Definition 2.7.** ([17]) *A  $d$ -algebra  $\Upsilon$  is called  $d^*$ -algebra if it satisfies the identity  $(\nu * \omega) * \nu = 0$  for all  $\nu, \omega \in \Upsilon$ .*

**Definition 2.8.** ([23]) *Let  $S$  be a universal set,  $I = [0, 1]$ , and  $\mu : S \rightarrow I$  be a function. Thus  $X = \{(s, \mu(s)) \mid s \in S\}$  is called a fuzzy set.*

**Definition 2.9.** ([5]) *Let  $S$  be a universal set,  $I = [0, 1]$ , and  $\mu, \theta : S \rightarrow I$  be the membership and non-membership functions, respectively. Thus  $X = \{(s, \mu(s), \theta(s)) \mid s \in S\}$  is called an intuitionistic fuzzy set. Here,  $\mu(s) + \theta(s) \in I$  for all  $s \in S$ .*

**Definition 2.10.** ([19]) *Let  $S$  be a non-empty reference set. A linear Diophantine fuzzy set on  $S$  is defined to be an object of the form:*

$$\mathcal{L} = \{(s; (\xi(s), \hbar(s)), (\tau, \varphi)) \mid s \in S\},$$

where  $\xi$  and  $\hbar$  are fuzzy sets in  $S$ , and  $\tau, \varphi \in [0, 1]$  such that  $0 \leq \tau + \varphi \leq 1$  and  $0 \leq \tau \xi(s) + \varphi \hbar(s) \leq 1$  for all  $s \in S$ .

We say that  $\xi(s)$  and  $\hbar(s)$  are the degrees of membership and non-membership at a point  $s \in S$ , respectively, and  $\tau, \varphi$  are reference parameters. The linear Diophantine fuzzy set  $\mathcal{L}$  simply denoted by  $\mathcal{L} := (\xi, \hbar; \tau, \varphi)$ .

**Remark 2.2.** *In the definition of a linear Diophantine fuzzy set  $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$  introduced in [19], since  $\xi(s), \bar{h}(s), \tau$ , and  $\varphi$  take values in  $[0, 1]$ , it follows that whenever  $0 \leq \tau + \varphi \leq 1$ , the condition*

$$0 \leq \tau\xi(s) + \varphi\bar{h}(s) \leq 1$$

*is automatically satisfied for all  $s \in S$ .*

*Although this observation may suggest that the reference parameters do not impose additional restrictions, their significance lies in weighting the relative influence of membership and non-membership degrees. Different choices of  $(\tau, \varphi)$  may yield different evaluations of the same fuzzy information and consequently affect the defining inequalities of LDF-subalgebras and LDF-ideals. Therefore, linear Diophantine fuzzy sets provide a parameter-dependent extension of intuitionistic and Pythagorean fuzzy frameworks.*

*Let  $S$  be a non-empty reference set,  $\tau, \varphi \in [0, 1]$  be reference parameters, and  $\xi$  and  $\bar{h}$  be functions from  $S$  to  $[0, 1]$ .*

*Consider an object of the form:*

$$\mathcal{L} = \{(s; (\xi(s), \bar{h}(s)), (\tau, \varphi)) \mid s \in S\}$$

*and it is simply denoted by  $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$ , and we say it is a Diophantine structure in  $S$ . In the definition of a linear Diophantine fuzzy set in [19], the reference parameters  $\tau, \varphi \in [0, 1]$  are chosen independently of the membership and non-membership degrees  $\xi(s)$  and  $\bar{h}(s)$ . For convenience, the parameters assigned to each  $s \in S$  will be denoted by  $\tau(s)$  and  $\varphi(s)$ .*

**Definition 2.11.** ([18]) *Let  $S$  be a non-empty reference set. A weak linear Diophantine fuzzy set on  $S$  is defined to be an object of the form:*

$$\mathcal{L} = \{(s; (\xi(s), \bar{h}(s)), (\tau(s), \varphi(s))) \mid s \in S\},$$

*where  $\xi$  and  $\bar{h}$  are fuzzy sets in  $S$ , and  $\tau(s), \varphi(s) \in [0, 1]$  are reference parameters such that  $\tau(s) \cdot \xi(s) + \varphi(s) \cdot \bar{h}(s) \in [0, 1]$  for all  $s \in S$ .*

For simplicity, the weak linear Diophantine fuzzy set is denoted by  $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$ , in the above definition.

In a weak linear Diophantine fuzzy set  $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$  on  $S$ , we consider the following mappings:

$$\begin{aligned} \tau\xi &: S \rightarrow [0, 1], s \mapsto \tau(s) \cdot \xi(s), \\ \varphi\bar{h} &: S \rightarrow [0, 1], s \mapsto \varphi(s) \cdot \bar{h}(s). \end{aligned}$$

Note that  $\tau\xi$  and  $\varphi\bar{h}$  are fuzzy sets in  $S$ . So we can establish two level sets

$$U(\tau\xi; \alpha) := \{s \in S \mid \tau\xi(s) \geq \alpha\} \text{ and } L(\varphi\bar{h}; \beta) := \{s \in S \mid \varphi\bar{h}(s) \leq \beta\},$$

of  $\tau\xi$  and  $\varphi\bar{h}$  respectively, where  $\alpha, \beta \in [0, 1]$ . We call  $U(\tau\xi; \alpha)$  and  $L(\varphi\bar{h}; \beta)$  weak linear Diophantine level sets of  $\tau\xi$  and  $\varphi\bar{h}$ , respectively (see [18]).

**Definition 2.12.** ([19]) *Let  $S$  be a universal set and  $\mathcal{L}_1, \mathcal{L}_2$  be linear Diophantine fuzzy sets on  $S$ . Thus*

(1) The intersection  $\mathcal{L}_1 \cap \mathcal{L}_2$  of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  is defined as

$$\{(s; (\xi_{\mathcal{L}_1}(s) \wedge \xi_{\mathcal{L}_2}(s), \hbar_{\mathcal{L}_1}(s) \vee \hbar_{\mathcal{L}_2}(s)), (\tau_{\mathcal{L}_1}(s) \wedge \tau_{\mathcal{L}_2}(s), \varphi_{\mathcal{L}_1}(s) \vee \varphi_{\mathcal{L}_2}(s))) \mid s \in S\}.$$

(2) The union  $\mathcal{L}_1 \cup \mathcal{L}_2$  of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  is defined as

$$\{(s; (\xi_{\mathcal{L}_1}(s) \vee \xi_{\mathcal{L}_2}(s), \hbar_{\mathcal{L}_1}(s) \wedge \hbar_{\mathcal{L}_2}(s)), (\tau_{\mathcal{L}_1}(s) \vee \tau_{\mathcal{L}_2}(s), \varphi_{\mathcal{L}_1}(s) \wedge \varphi_{\mathcal{L}_2}(s))) \mid s \in S\}.$$

(3)  $\mathcal{L}_1$  is subset of  $\mathcal{L}_2$ , denoted by  $\mathcal{L}_1 \subseteq_D \mathcal{L}_2$ , if  $\mathcal{L}_1(s) \leq \mathcal{L}_2(s)$  for all  $s \in S$ , i.e.,  $\xi_{\mathcal{L}_1}(s) \leq \xi_{\mathcal{L}_2}(s)$ ,  $\hbar_{\mathcal{L}_1}(s) \geq \hbar_{\mathcal{L}_2}(s)$ ,  $\tau_{\mathcal{L}_1}(s) \leq \tau_{\mathcal{L}_2}(s)$  and  $\varphi_{\mathcal{L}_1}(s) \geq \varphi_{\mathcal{L}_2}(s)$  for all  $s \in S$ .

(4)  $\mathcal{L}_1 = \mathcal{L}_2$  if  $\mathcal{L}_1 \subseteq_D \mathcal{L}_2$  and  $\mathcal{L}_2 \subseteq_D \mathcal{L}_1$ .

(5) The complement  $\mathcal{L}_1^c$  of  $\mathcal{L}_1$  is defined as

$$\mathcal{L}_1^c = \{(s; \hbar_{\mathcal{L}_1}(s), \xi_{\mathcal{L}_1}(s)), (\varphi_{\mathcal{L}_1}(s), \tau_{\mathcal{L}_1}(s))) \mid s \in S\}.$$

Here, “ $\vee$ ” and “ $\wedge$ ” represent the maximum and minimum, respectively.

### 3. LINEAR DIOPHANTINE FUZZY $d$ -SUBALGEBRAS AND $d$ -IDEALS

In this section, we present the definitions and illustrative examples of linear Diophantine fuzzy  $d$ -subalgebras and linear Diophantine fuzzy  $d$ -ideals. We investigate several properties satisfied by these structures and provide proofs regarding the conditions under which a transition between them is possible.

**Definition 3.1.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a linear Diophantine fuzzy set of  $\Upsilon$ . Thus  $\mathcal{L}$  is called a linear Diophantine fuzzy  $d$ -subalgebra (briefly, LDFd-sub) if

$$(LDFd\text{-sub1}) (\forall \nu, \omega \in \Upsilon) (\xi(\nu * \omega) \geq \xi(\nu) \wedge \xi(\omega), \hbar(\nu * \omega) \leq \hbar(\nu) \vee \hbar(\omega))$$

$$(LDFd\text{-sub2}) (\forall \nu, \omega \in \Upsilon) (\tau(\nu * \omega) \geq \tau(\nu) \wedge \tau(\omega), \varphi(\nu * \omega) \leq \varphi(\nu) \vee \varphi(\omega)).$$

**Example 3.1.** Consider a set  $\Upsilon = \{0, \nu, \omega, \eta\}$  with a binary operation  $*$  given in the following table:

| $*$      | 0        | $\nu$    | $\omega$ | $\eta$   |
|----------|----------|----------|----------|----------|
| 0        | 0        | 0        | 0        | 0        |
| $\nu$    | $\nu$    | 0        | 0        | $\omega$ |
| $\omega$ | $\omega$ | $\omega$ | 0        | 0        |
| $\eta$   | $\eta$   | $\eta$   | $\nu$    | 0        |

Thus  $(\Upsilon, *, 0)$  is a  $d$ -algebra. Define a linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as follows:

| $\Upsilon$ | $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ |
|------------|---------------------------------------------|
| 0          | (0.86, 0.53; 0.7, 0.4)                      |
| $\nu$      | (0.66, 0.72; 0.6, 0.5)                      |
| $\omega$   | (0.74, 0.62; 0.3, 0.5)                      |
| $\eta$     | (0.52, 0.84; 0.1, 0.7)                      |

It is routine to verify that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-sub of  $\Upsilon$ .

**Remark 3.1.** In Example 3.1, the functions  $\tau$  and  $\varphi$  play an essential role in satisfying condition (LDFd-sub2). Indeed, if the membership and non-membership functions  $\xi$  and  $\hbar$  are kept unchanged while the reference function  $\tau$  is modified by setting

$$\tau(\omega) = 0.8$$

instead of  $\tau(\omega) = 0.3$ , then

$$\tau(\omega * \omega) = \tau(0) = 0.7 < 0.8 = \tau(\omega) \wedge \tau(\omega),$$

and consequently condition (LDFd-sub2) fails. Therefore, the resulting structure is no longer a linear Diophantine fuzzy  $d$ -subalgebra. This shows that the reference functions may affect the algebraic behaviour of the structure, even when the membership and non-membership functions remain unchanged.

**Proposition 3.1.** Let  $\Upsilon$  be a  $d$ -algebra. Thus every LDFd-sub of  $\Upsilon$   $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  satisfies:

$$(\forall \nu \in \Upsilon)(\mathcal{L}(\nu) \leq \mathcal{L}(0)),$$

where the relation “ $\leq$ ” is understood in the sense of the component-wise partial order defined in Definition 2.12(3).

*Proof.* Let  $\nu \in \Upsilon$ . Thus, we have

$$\xi(0) = \xi(\nu * \nu) \geq \xi(\nu) \wedge \xi(\nu) = \xi(\nu),$$

$$\hbar(0) = \hbar(\nu * \nu) \leq \hbar(\nu) \vee \hbar(\nu) = \hbar(\nu),$$

$$\tau(0) = \tau(\nu * \nu) \geq \tau(\nu) \wedge \tau(\nu) = \tau(\nu),$$

and

$$\varphi(0) = \varphi(\nu * \nu) \leq \varphi(\nu) \vee \varphi(\nu) = \varphi(\nu).$$

Therefore, we get  $\mathcal{L}(\nu) \leq \mathcal{L}(0)$  for all  $\nu \in \Upsilon$ .  $\square$

**Definition 3.2.** Let  $\Upsilon$  be a  $d$ -algebra. A linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  is called a linear Diophantine fuzzy  $d$ -ideal (briefly, LDFd-id) if it satisfies:

$$(LDFd-id1) (\forall \nu \in \Upsilon)(\xi(0) \geq \xi(\nu), \hbar(0) \leq \hbar(\nu), \tau(0) \geq \tau(\nu), \varphi(0) \leq \varphi(\nu)),$$

$$(LDFd-id2) (\forall \nu, \omega \in \Upsilon)(\xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega), \hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega)),$$

$$(LDFd-id3) (\forall \nu, \omega \in \Upsilon)(\tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega), \varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega)).$$

**Example 3.2.** Consider a  $d$ -algebra  $\Upsilon = \{0, \nu, \omega, \eta\}$ , which is not a BCK-algebra, defined as follows.

| $*$      | 0        | $\nu$    | $\omega$ | $\eta$ |
|----------|----------|----------|----------|--------|
| 0        | 0        | 0        | 0        | 0      |
| $\nu$    | $\nu$    | 0        | 0        | $\nu$  |
| $\omega$ | $\omega$ | $\omega$ | 0        | 0      |
| $\eta$   | $\eta$   | $\eta$   | $\eta$   | 0      |

Define a linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as follows:

| $\Upsilon$ | $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ |
|------------|---------------------------------------------|
| 0          | (0.62, 0.36; 0.8, 0.1)                      |
| $\nu$      | (0.53, 0.45; 0.5, 0.4)                      |
| $\omega$   | (0.51, 0.45; 0.4, 0.5)                      |
| $\eta$     | (0.35, 0.51; 0.1, 0.7)                      |

It is easy to show that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-id of  $\Upsilon$ .

**Proposition 3.2.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be an LDFd-id of  $\Upsilon$ , and  $\nu, \omega \in \Upsilon$  such that  $\nu \leq \omega$ . Then  $\mathcal{L}(\omega) \leq \mathcal{L}(\nu)$ .

*Proof.* Let  $\nu, \omega \in \Upsilon$  such that  $\nu \leq \omega$ . Hence, we get

$$\begin{aligned} \xi(\nu) &\geq \xi(\nu * \omega) \wedge \xi(\omega) = \xi(0) \wedge \xi(\omega) = \xi(\omega), \\ \hbar(\nu) &\leq \hbar(\nu * \omega) \vee \hbar(\omega) = \hbar(0) \vee \hbar(\omega) = \hbar(\omega), \\ \tau(\nu) &\geq \tau(\nu * \omega) \wedge \tau(\omega) = \tau(0) \wedge \tau(\omega) = \tau(\omega), \\ \varphi(\nu) &\leq \varphi(\nu * \omega) \vee \varphi(\omega) = \varphi(0) \vee \varphi(\omega) = \varphi(\omega), \end{aligned}$$

and so  $\mathcal{L}(\omega) \leq \mathcal{L}(\nu)$ . □

**Proposition 3.3.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be an LDFd-id of  $\Upsilon$ , and  $\nu, \omega, \eta \in \Upsilon$  such that  $\nu * \omega \leq \eta$ . Thus, we have  $\mathcal{L}(\eta) \wedge \mathcal{L}(\omega) \leq \mathcal{L}(\nu)$ .

*Proof.* Let  $\nu, \omega, \eta \in \Upsilon$  such that  $\nu * \omega \leq \eta$ . Hence, we get

$$\begin{aligned} \xi(\nu) &\geq \xi(\nu * \omega) \wedge \xi(\omega) \geq (\xi((\nu * \omega) * \eta) \wedge \xi(\eta)) \wedge \xi(\omega) \\ &= \xi(0) \wedge \xi(\eta) \wedge \xi(\omega) \\ &= \xi(\eta) \wedge \xi(\omega), \\ \hbar(\nu) &\leq \hbar(\nu * \omega) \vee \hbar(\omega) \leq (\hbar((\nu * \omega) * \eta) \vee \hbar(\eta)) \vee \hbar(\omega) \\ &= \hbar(0) \vee \hbar(\eta) \vee \hbar(\omega) \\ &= \hbar(\eta) \vee \hbar(\omega), \\ \tau(\nu) &\geq \tau(\nu * \omega) \wedge \tau(\omega) \geq (\tau((\nu * \omega) * \eta) \wedge \tau(\eta)) \wedge \tau(\omega) \\ &= \tau(0) \wedge \tau(\eta) \wedge \tau(\omega) \\ &= \tau(\eta) \wedge \tau(\omega), \\ \varphi(\nu) &\leq \varphi(\nu * \omega) \vee \varphi(\omega) \leq (\varphi((\nu * \omega) * \eta) \vee \varphi(\eta)) \vee \varphi(\omega) \\ &= \varphi(0) \vee \varphi(\eta) \vee \varphi(\omega) \\ &= \varphi(\eta) \vee \varphi(\omega). \end{aligned}$$

Therefore, we have  $\mathcal{L}(\eta) \wedge \mathcal{L}(\omega) \leq \mathcal{L}(\nu)$ . □

**Remark 3.2.** There is no implication between as LDFd-subalgebra and LDFd-ideal in  $d$ -algebras. In Example 3.1, the LDFd-subalgebra  $\mathcal{L}$  is not an LDFd-ideal, because  $\xi(\eta) \not\geq \xi(\eta * \omega) \wedge \xi(\omega)$ . Furthermore, Example 3.3 shows that LDFd-ideals may not necessarily be LDFd-subalgebras. This situation holds in the case of  $d^*$ -algebras (see Theorem 3.1).

**Example 3.3.** Let  $\Upsilon = \{0, \nu, \omega, \eta\}$  be a  $d$ -algebra which is not a BCK-algebra with the following Cayley table:

| $*$      | 0        | $\nu$  | $\omega$ | $\eta$   |
|----------|----------|--------|----------|----------|
| 0        | 0        | 0      | 0        | 0        |
| $\nu$    | $\nu$    | 0      | 0        | $\omega$ |
| $\omega$ | $\omega$ | $\eta$ | 0        | 0        |
| $\eta$   | $\eta$   | $\eta$ | $\eta$   | 0        |

We define a linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as:

| $\Upsilon$ | $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ |
|------------|---------------------------------------------|
| 0          | (0.85, 0.12; 0.7, 0.14)                     |
| $\nu$      | (0.73, 0.32; 0.5, 0.41)                     |
| $\omega$   | (0.52, 0.16; 0.5, 0.36)                     |
| $\eta$     | (0.36, 0.44; 0.2, 0.44)                     |

It is easy to verify that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-id of  $\Upsilon$  but not an LDFd-sub of  $\Upsilon$ , because  $\xi(\omega * \nu) = \xi(\eta) \not\geq \xi(\omega) \wedge \xi(\nu)$  or  $\tau(\omega * \nu) = \tau(\eta) \not\geq \tau(\omega) \wedge \tau(\nu)$ .

**Theorem 3.1.** Let  $\Upsilon$  be a  $d^*$ -algebra. Then every LDFd-id of  $\Upsilon$  is an LDFd-sub of  $\Upsilon$ .

*Proof.* Assume that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-id of  $\Upsilon$  and  $\nu, \omega \in \Upsilon$ . Since  $\Upsilon$  is a  $d^*$ -algebra, we have  $(\nu * \omega) * \nu = 0$  and so  $\nu * \omega \leq \nu$ . Using Proposition 3.2, we get  $\mathcal{L}(\nu) \leq \mathcal{L}(\nu * \omega)$ . Hence,

(LDFd-sub1) :

$$\begin{aligned} \xi(\nu * \omega) &\geq \xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega) \geq \xi(\nu) \wedge \xi(\omega), \\ \hbar(\nu * \omega) &\leq \hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega) \leq \hbar(\nu) \vee \hbar(\omega). \end{aligned}$$

(LDFd-sub2) :

$$\begin{aligned} \tau(\nu * \omega) &\geq \tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega) \geq \tau(\nu) \wedge \tau(\omega), \\ \varphi(\nu * \omega) &\leq \varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega) \leq \varphi(\nu) \vee \varphi(\omega). \end{aligned}$$

Therefore,  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-sub of  $\Upsilon$ . □

**Theorem 3.2.** Let  $\Upsilon$  be an edge  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be an LDFd-sub of  $\Upsilon$ . Then  $\mathcal{L}$  is an LDFd-id of  $\Upsilon \Leftrightarrow$  For all  $\nu, \omega, \eta \in \Upsilon$ ,  $\nu * \omega \leq \eta$  implies  $\xi(\nu) \geq \xi(\omega) \wedge \xi(\eta)$ ,  $\hbar(\nu) \leq \hbar(\omega) \vee \hbar(\eta)$ ,  $\tau(\nu) \geq \tau(\omega) \wedge \tau(\eta)$ , and  $\varphi(\nu) \leq \varphi(\omega) \vee \varphi(\eta)$ .

*Proof.* ( $\Rightarrow$ ) : Follows from Proposition 3.3.

( $\Leftarrow$ ) : Since  $\Upsilon$  is an edge  $d$ -algebra, we have  $\nu * (\nu * \omega) \leq \omega$ . Using our hypothesis,  $\xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega)$ ,  $\hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega)$ ,  $\tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega)$  and  $\varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega)$ . Moreover, it follows from Proposition 3.1  $\mathcal{L}(\nu) \leq \mathcal{L}(0)$  for all  $\nu \in \Upsilon$ . Therefore,  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is an LDFd-id of  $\Upsilon$ . □

#### 4. TYPES AND PROPERTIES OF LINEAR DIOPHANTINE FUZZY $d$ -IDEALS

In this section, we examined two specific types of ideals in  $d$ -algebras in relation to linear Diophantine fuzzy sets, and investigated their connection with the structure of  $d$ -ideals.

**Definition 4.1.** Let  $\Upsilon$  be a  $d$ -algebra. An  $LDFd$ -id  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is called a linear Diophantine fuzzy  $d^\#$ -ideal (briefly,  $LDFd^\#$ -id) of  $\Upsilon$  if it satisfies:

$$\begin{aligned} (LDFd^\#-id1) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\xi(\nu * \eta) \geq \xi(\nu * \omega) \wedge \xi(\omega * \eta), \hbar(\nu * \eta) \leq \hbar(\nu * \omega) \vee \hbar(\omega * \eta)), \\ (LDFd^\#-id2) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\tau(\nu * \eta) \geq \tau(\nu * \omega) \wedge \tau(\omega * \eta), \varphi(\nu * \eta) \leq \varphi(\nu * \omega) \vee \varphi(\omega * \eta)). \end{aligned}$$

**Example 4.1.** Let  $\Upsilon = \{0, \nu, \omega, \eta\}$  be a set with the following Cayley table:

| $*$      | 0      | $\nu$    | $\omega$ | $\eta$ |
|----------|--------|----------|----------|--------|
| 0        | 0      | 0        | 0        | 0      |
| $\nu$    | $\nu$  | 0        | 0        | $\nu$  |
| $\omega$ | $\eta$ | $\omega$ | 0        | $\eta$ |
| $\eta$   | $\eta$ | $\omega$ | $\omega$ | 0      |

Then  $(\Upsilon, *, 0)$  is a  $d$ -algebra, but not a BCK-algebra. We define a linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as:

| $\Upsilon$ | $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ |
|------------|---------------------------------------------|
| 0          | (0.55, 0.14; 0.37, 0.28)                    |
| $\nu$      | (0.42, 0.36; 0.21, 0.63)                    |
| $\omega$   | (0.25, 0.52; 0.14, 0.85)                    |
| $\eta$     | (0.25, 0.52; 0.14, 0.85)                    |

Some computations show that  $\mathcal{L}$  is an  $LDFd^\#$ -id of  $\Upsilon$ .

**Remark 4.1.** Clearly, while every  $LDFd^\#$ -ideal is an  $LDFd$ -ideal, the reverse implication does not always apply.

**Example 4.2.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a  $LDFd$ -id of  $\Upsilon$  as in Example 3.2. We see that  $\mathcal{L}$  is not an  $LDFd^\#$ -id of  $\Upsilon$ , since  $\hbar(\nu * \eta) = \hbar(\nu) = 0.45 \not\leq \hbar(\nu * \omega) \vee \hbar(\omega * \eta) = \hbar(0) = 0.36$ .

**Definition 4.2.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a linear Diophantine fuzzy set on  $\Upsilon$ . Then  $\mathcal{L}$  is called a linear Diophantine fuzzy commutative  $d$ -ideal (briefly,  $LDFCd$ -id) if it satisfies ( $LDFd$ -id1) and the following conditions:

$$\begin{aligned} (LDFCd-id1) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\xi(\nu * (\omega * (\omega * \nu))) \geq \xi((\nu * \omega) * \eta) \wedge \xi(\eta), \hbar(\nu * (\omega * (\omega * \nu))) \leq \hbar((\nu * \omega) * \eta) \vee \hbar(\eta)), \\ (LDFCd-id2) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\tau(\nu * (\omega * (\omega * \nu))) \geq \tau((\nu * \omega) * \eta) \wedge \tau(\eta), \varphi(\nu * (\omega * (\omega * \nu))) \leq \varphi((\nu * \omega) * \eta) \vee \varphi(\eta)). \end{aligned}$$

**Example 4.3.** Consider a  $d$ -algebra  $\Upsilon$  of Example 4.1. Now define a linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as:

| $\Upsilon$ | $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ |
|------------|---------------------------------------------|
| 0          | (0.7, 0.3; 0.51, 0.41)                      |
| $\nu$      | (0.6, 0.5; 0.33, 0.63)                      |
| $\omega$   | (0.2, 0.8; 0.17, 0.74)                      |
| $\eta$     | (0.2, 0.8; 0.17, 0.74)                      |

It is easy to verify that  $\mathcal{L}$  is an LDFCd-id of  $\Upsilon$ .

**Remark 4.2.** In general, LDFd-ideals are not necessarily LDFCd-ideals.

**Example 4.4.** The LDFd-id structure given in Example 3.3 is not an LDFCd-id of  $\Upsilon$  since  $\xi(\omega * (\eta * (\eta * \omega))) = \xi(\omega) \not\geq \xi((\omega * \eta) * \nu) \wedge \xi(\nu) = \xi(\nu)$ .

**Theorem 4.1.** In edge  $d$ -algebras, the structures of LDFd-ideal and LDFCd-ideal coincide.

*Proof.* Let  $\Upsilon$  be an edge  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be an LDFd-id of  $\Upsilon$ . Then, for all  $\nu, \omega, \eta \in \Upsilon$ , we have  $((\nu * (\omega * (\omega * \nu))) * ((\nu * \omega) * \eta)) \leq \eta$ . Since  $\mathcal{L}$  is an LDFd-id of  $\Upsilon$ ,

$$\xi(\nu * (\omega * (\omega * \nu))) \geq \xi((\nu * \omega) * \eta) \wedge \xi(\eta), \hbar(\nu * (\omega * (\omega * \nu))) \leq \hbar((\nu * \omega) * \eta) \vee \hbar(\eta),$$

$$\tau(\nu * (\omega * (\omega * \nu))) \geq \tau((\nu * \omega) * \eta) \wedge \tau(\eta), \varphi(\nu * (\omega * (\omega * \nu))) \leq \varphi((\nu * \omega) * \eta) \vee \varphi(\eta)$$

by Proposition 3.3. Therefore,  $\mathcal{L}$  is an LDFCd-id of  $\Upsilon$ .

Now, we assume that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be an LDFCd-id of  $\Upsilon$ . Then, we have  $\nu = \nu * (0 * (0 * \nu))$  and so

$$\begin{aligned} \xi(\nu) &= \xi(\nu * (0 * (0 * \nu))) \geq \xi((\nu * 0) * \omega) \wedge \xi(\omega) = \xi(\nu * \omega) \wedge \xi(\omega), \\ \hbar(\nu) &= \hbar(\nu * (0 * (0 * \nu))) \leq \hbar((\nu * 0) * \omega) \vee \hbar(\omega) = \hbar(\nu * \omega) \vee \hbar(\omega), \\ \tau(\nu) &= \tau(\nu * (0 * (0 * \nu))) \geq \tau((\nu * 0) * \omega) \wedge \tau(\omega) = \tau(\nu * \omega) \wedge \tau(\omega), \\ \varphi(\nu) &= \varphi(\nu * (0 * (0 * \nu))) \leq \varphi((\nu * 0) * \omega) \vee \varphi(\omega) = \varphi(\nu * \omega) \vee \varphi(\omega). \end{aligned}$$

Thus we have  $\mathcal{L}$  is an LDFd-id of  $\Upsilon$ . □

## 5. WEAK LINEAR DIOPHANTINE FUZZY $d$ -SUBALGEBRAS AND $d$ -IDEALS

In this section, we present the definitions and illustrative examples of weak linear Diophantine fuzzy  $d$ -subalgebras and weak linear Diophantine fuzzy  $d$ -ideals.

**Definition 5.1.** Let  $\Upsilon$  be a  $d$ -algebra and  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a weak linear Diophantine fuzzy set of  $\Upsilon$ . Thus  $\mathcal{L}$  is called a weak linear Diophantine fuzzy  $d$ -subalgebra (briefly, WLDFd-sub) if

$$\begin{aligned} (WLDFd\text{-sub1}) \quad & (\forall \nu, \omega \in \Upsilon) (\tau\xi(\nu * \omega) \geq \min\{\tau(\nu), \tau(\omega)\} \cdot \min\{\xi(\nu), \xi(\omega)\}), \\ (WLDFd\text{-sub2}) \quad & (\forall \nu, \omega \in \Upsilon) (\varphi\hbar(\nu * \omega) \leq \max\{\varphi(\nu), \varphi(\omega)\} \cdot \max\{\hbar(\nu), \hbar(\omega)\}). \end{aligned}$$

**Example 5.1.** Let  $\Upsilon = \{0, \nu, \omega, \eta\}$  be a set with the following Cayley table:

| $*$      | 0        | $\nu$    | $\omega$ | $\eta$   |
|----------|----------|----------|----------|----------|
| 0        | 0        | 0        | 0        | 0        |
| $\nu$    | $\nu$    | 0        | 0        | $\omega$ |
| $\omega$ | $\omega$ | $\omega$ | 0        | 0        |
| $\eta$   | $\eta$   | $\eta$   | $\eta$   | 0        |

Then  $(\Upsilon, *, 0)$  is a  $d$ -algebra which is not a BCK-algebra. We define a weak linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as:

| $s \in \Upsilon$ | $\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$ |
|------------------|------------------------------------------------------------|
| 0                | (0.85, 0.22; 0.7, 0.5)                                     |
| $\nu$            | (0.64, 0.56; 0.5, 0.8)                                     |
| $\omega$         | (0.81, 0.35; 0.6, 0.7)                                     |
| $\eta$           | (0.78, 0.41; 0.2, 0.6)                                     |

It is routine to verify that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a WLDFd-sub of  $\Upsilon$ .

**Theorem 5.1.** Let  $\Upsilon$  be a  $d$ -algebra. Every LDFd-sub of  $\Upsilon$  is a WLDFd-sub of  $\Upsilon$ .

*Proof.* The result is straightforward from the definitions.  $\square$

The following example shows that the converse of Theorem 5.1 may not be true. That is, if  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a WLDFd-sub of  $\Upsilon$ , then there are  $\nu, \omega \in \Upsilon$  that satisfy  $\xi(\nu * \omega) \not\geq \xi(\nu) \wedge \xi(\omega)$ ,  $\hbar(\nu * \omega) \not\leq \hbar(\nu) \vee \hbar(\omega)$ ,  $\tau(\nu * \omega) \not\geq \tau(\nu) \wedge \tau(\omega)$ ,  $\varphi(\nu * \omega) \not\leq \varphi(\nu) \vee \varphi(\omega)$ .

**Example 5.2.** Consider the  $d$ -algebra given in Example 3.1. We define the following weak linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  as:

| $s \in \Upsilon$ | $\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$ |
|------------------|------------------------------------------------------------|
| 0                | (0.62, 0.51; 0.7, 0.1)                                     |
| $\nu$            | (0.41, 0.63; 0.2, 0.9)                                     |
| $\omega$         | (0.38, 0.84; 0.6, 0.8)                                     |
| $\eta$           | (0.44, 0.84; 0.2, 0.4)                                     |

It is routine to verify that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a WLDFd-sub of  $\Upsilon$  and  $\xi(\nu * \eta) = \xi(\omega) = 0.38 \not\geq 0.41 = \xi(\nu) \wedge \xi(\eta)$ . Moreover,  $\varphi(\eta * \omega) = \varphi(\nu) = 0.9 \not\leq 0.8 = \varphi(\eta) \vee \varphi(\omega)$ .

**Proposition 5.1.** Let  $\Upsilon$  be a  $d$ -algebra. Thus every WLDFd-sub  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  of  $\Upsilon$  satisfies:

$$(\forall \nu \in \Upsilon)(\tau\xi(0) \geq \tau(\nu) \cdot \xi(\nu), \varphi\hbar(0) \leq \varphi(\nu) \cdot \hbar(\nu)).$$

*Proof.* Let  $\nu \in \Upsilon$ . Thus we have

$$\begin{aligned} \tau\xi(0) &= \tau\xi(\nu * \nu) \geq \min\{\tau(\nu), \tau(\nu)\} \cdot \min\{\xi(\nu), \xi(\nu)\} = \tau(\nu) \cdot \xi(\nu), \\ \varphi\hbar(0) &= \varphi\hbar(\nu * \nu) \leq \max\{\varphi(\nu), \varphi(\nu)\} \cdot \max\{\hbar(\nu), \hbar(\nu)\} = \varphi(\nu) \cdot \hbar(\nu). \end{aligned}$$

$\square$

**Theorem 5.2.** *Let  $\Upsilon$  be a  $d$ -algebra, and let  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a WLDFd-sub of  $\Upsilon$ . If the reference parameters  $\tau$  and  $\varphi$  satisfy:*

$$\begin{aligned} \min \{ \tau(\nu), \tau(\omega) \} \cdot \min \{ \xi(\nu), \xi(\omega) \} &\geq \min \{ \tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega) \}, \\ \max \{ \varphi(\nu), \varphi(\omega) \} \cdot \max \{ \hbar(\nu), \hbar(\omega) \} &\leq \max \{ \varphi(\nu) \cdot \hbar(\nu), \varphi(\omega) \cdot \hbar(\omega) \}, \end{aligned}$$

*for all  $\nu, \omega \in \Upsilon$ , then the non-empty weak linear Diophantine level sets of  $\tau\xi$  and  $\varphi\hbar$  are  $d$ -subalgebras of  $\Upsilon$ .*

*Proof.* Let  $\nu, \omega \in \Upsilon$  and  $\alpha, \beta \in [0, 1]$  be such that  $\nu, \omega \in U(\tau\xi; \alpha) \cap L(\varphi\hbar; \beta)$ . Thus  $\tau\xi(\nu) \geq \alpha, \tau\xi(\omega) \geq \alpha, \varphi\hbar(\nu) \leq \beta$  and  $\varphi\hbar(\omega) \leq \beta$ . Consequently,

$$\begin{aligned} \tau\xi(\nu * \omega) &\geq \min \{ \tau(\nu), \tau(\omega) \} \cdot \min \{ \xi(\nu), \xi(\omega) \} \\ &\geq \min \{ \tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega) \} \\ &= \min \{ \tau\xi(\nu), \tau\xi(\omega) \} \geq \alpha \end{aligned}$$

and

$$\begin{aligned} \varphi\hbar(\nu * \omega) &\leq \max \{ \varphi(\nu), \varphi(\omega) \} \cdot \max \{ \hbar(\nu), \hbar(\omega) \} \\ &\leq \max \{ \varphi(\nu) \cdot \hbar(\nu), \varphi(\omega) \cdot \hbar(\omega) \} \\ &= \max \{ \varphi\hbar(\nu), \varphi\hbar(\omega) \} \leq \beta. \end{aligned}$$

Hence  $\nu * \omega \in U(\tau\xi; \alpha) \cap L(\varphi\hbar; \beta)$ . Therefore  $U(\tau\xi; \alpha)$  and  $L(\varphi\hbar; \beta)$  are  $d$ -subalgebras of  $\Upsilon$ .  $\square$

**Remark 5.1.** *These conditions are introduced as sufficient conditions and are not claimed to be necessary. They ensure the closure properties required for weak linear Diophantine fuzzy level sets to form  $d$ -subalgebras in a constructive manner.*

The following example illustrates Theorem 5.2.

**Example 5.3.** *Consider the  $d$ -algebra  $(\Upsilon, *, 0)$  in Example 3.2. Let  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a weak linear Diophantine fuzzy set on  $\Upsilon$ ,  $\tau\xi$  and  $\varphi\hbar$  are calculated by the following table.*

| $s \in \Upsilon$ | $\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$ | $\tau\xi(s)$ | $\varphi\hbar(s)$ |
|------------------|------------------------------------------------------------|--------------|-------------------|
| 0                | (0.83, 0.31; 0.8, 0.1)                                     | 0.664        | 0.031             |
| $\nu$            | (0.64, 0.48; 0.7, 0.4)                                     | 0.448        | 0.192             |
| $\omega$         | (0.52, 0.47; 0.4, 0.3)                                     | 0.208        | 0.141             |
| $\eta$           | (0.39, 0.58; 0.2, 0.6)                                     | 0.078        | 0.348             |

*It is routine to check that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a WLDFd-sub of  $\Upsilon$  and the reference parameters satisfy the conditions of Theorem 5.2. The weak linear Diophantine level sets of  $\tau\xi$  and  $\varphi\hbar$  are calculated as follows:*

$$U(\tau\xi; \alpha) = \begin{cases} \Upsilon & \text{if } \alpha \in [0, 0.078] \\ \{0, \nu, \omega\} & \text{if } \alpha \in (0.078, 0.208] \\ \{0, \nu\} & \text{if } \alpha \in (0.208, 0.448] \\ \{0\} & \text{if } \alpha \in (0.448, 0.664] \\ \emptyset & \text{if } \alpha \in (0.664, 1] \end{cases}$$

and

$$L(\varphi\hbar; \beta) = \begin{cases} \Upsilon & \text{if } \beta \in [0.348, 1] \\ \{0, \nu, \omega\} & \text{if } \beta \in [0.192, 0.348) \\ \{0, \omega\} & \text{if } \beta \in [0.141, 0.192) \\ \{0\} & \text{if } \beta \in [0.031, 0.141) \\ \emptyset & \text{if } \beta \in [0, 0.031) \end{cases}$$

which are  $d$ -subalgebras of  $\Upsilon$ .

The following example shows that the converse of Theorem 5.2 may not be true, that is, there exists a weak linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  such that the weak linear Diophantine level sets of  $\tau\xi$  and  $\varphi\hbar$  are  $d$ -subalgebras of  $\Upsilon$ , but the reference parameters  $\tau$  and  $\varphi$  don't satisfy the conditions of Theorem 5.2.

**Example 5.4.** Consider the  $d$ -algebra  $(\Upsilon, *, 0)$  in Example 3.2. Let  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a weak linear Diophantine fuzzy set on  $\Upsilon$ ,  $\tau\xi$  and  $\varphi\hbar$  are calculated by the following table.

| $s \in \Upsilon$ | $\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$ | $\tau\xi(s)$ | $\varphi\hbar(s)$ |
|------------------|------------------------------------------------------------|--------------|-------------------|
| 0                | (0.7, 0.4; 0.8, 0.3)                                       | 0.56         | 0.12              |
| $\nu$            | (0.8, 0.3; 0.2, 0.7)                                       | 0.16         | 0.21              |
| $\omega$         | (0.6, 0.5; 0.4, 0.5)                                       | 0.24         | 0.25              |
| $\eta$           | (0.3, 0.6; 0.5, 0.7)                                       | 0.15         | 0.42              |

Then the non-empty the weak linear Diophantine level sets of  $\tau\xi$  and  $\varphi\hbar$  are  $d$ -subalgebras of  $\Upsilon$ , and they are given as follows:

$$U(\tau\xi; \alpha) = \begin{cases} \Upsilon & \text{if } \alpha \in [0, 0.15] \\ \{0, \nu, \omega\} & \text{if } \alpha \in (0.15, 0.16] \\ \{0, \omega\} & \text{if } \alpha \in (0.16, 0.24] \\ \{0\} & \text{if } \alpha \in (0.24, 0.56] \\ \emptyset & \text{if } \alpha \in (0.56, 1] \end{cases}$$

and

$$L(\varphi\hbar; \beta) = \begin{cases} \Upsilon & \text{if } \beta \in [0.42, 1] \\ \{0, \nu, \omega\} & \text{if } \beta \in [0.25, 0.42) \\ \{0, \nu\} & \text{if } \beta \in [0.21, 0.25) \\ \{0\} & \text{if } \beta \in [0.12, 0.21) \\ \emptyset & \text{if } \beta \in [0, 0.12) \end{cases}$$

It is routine to check that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a WLDFd-sub of  $\Upsilon$ , but the reference parameters  $\tau$  and  $\varphi$  do not satisfy the conditions of Theorem 5.2, since

$$\begin{aligned} \min \{\tau(\nu), \tau(\omega)\} \cdot \min \{\xi(\nu), \xi(\omega)\} &= \min \{0.2, 0.4\} \cdot \min \{0.8, 0.6\} \\ &= 0.12 \not\geq 0.16 \\ &= \min \{0.2 \cdot 0.8, 0.4 \cdot 0.6\} = \min \{\tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega)\} \end{aligned}$$

and/or

$$\begin{aligned}
 \max \{ \varphi(\omega), \varphi(\nu) \} \cdot \max \{ \hbar(\omega), \hbar(\nu) \} &= \max \{ 0.5, 0.7 \} \cdot \max \{ 0.5, 0.3 \} \\
 &= 0.35 \not\leq 0.25 \\
 &= \max \{ 0.5 \cdot 0.5, 0.7 \cdot 0.3 \} = \max \{ \varphi(\omega) \cdot \hbar(\omega), \varphi(\nu) \cdot \hbar(\nu) \}.
 \end{aligned}$$

**Definition 5.2.** Let  $\Upsilon$  be a  $d$ -algebra. A weak linear Diophantine fuzzy set  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  on  $\Upsilon$  is called a weak linear Diophantine fuzzy  $d$ -ideal (briefly, *WLDFd-id*) if it satisfies:

$$\begin{aligned}
 (WLDFd-id1) \quad &(\forall \nu \in \Upsilon) (\tau \xi(0) \geq \tau \xi(\nu), \varphi \hbar(0) \leq \varphi \hbar(\nu)), \\
 (WLDFd-id2) \quad &(\forall \nu, \omega \in \Upsilon) (\tau \xi(\nu) \geq \min \{ \tau(\nu * \omega), \tau(\omega) \} \cdot \min \{ \xi(\nu * \omega), \xi(\omega) \}), \\
 (WLDFd-id3) \quad &(\forall \nu, \omega \in \Upsilon) (\varphi \hbar(\nu) \leq \max \{ \varphi(\nu * \omega), \varphi(\omega) \} \cdot \max \{ \hbar(\nu * \omega), \hbar(\omega) \}).
 \end{aligned}$$

**Example 5.5.** Consider the  $d$ -algebra  $(\Upsilon, *, 0)$  in Example 3.2. Let  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  be a weak linear Diophantine fuzzy set on  $\Upsilon$ ,  $\tau \xi$  and  $\varphi \hbar$  are calculated by the following table.

| $s \in \Upsilon$ | $\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$ | $\tau \xi(s)$ | $\varphi \hbar(s)$ |
|------------------|------------------------------------------------------------|---------------|--------------------|
| 0                | (0.8, 0.2; 0.6, 0.4)                                       | 0.48          | 0.08               |
| $\nu$            | (0.7, 0.6; 0.5, 0.7)                                       | 0.35          | 0.42               |
| $\omega$         | (0.3, 0.5; 0.8, 0.9)                                       | 0.24          | 0.45               |
| $\eta$           | (0.4, 0.3; 0.3, 0.5)                                       | 0.12          | 0.15               |

It is easy to show that  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is a *WLDFd-id* of  $\Upsilon$ .

Note that every *LDFd-id* of  $\Upsilon$  is a *WLDFd-id*, but the converse need not be true. In Example 5.5,  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  is not an *LDFd-id* of  $\Upsilon$ , since

$$0.5 = \tau(\nu) \not\geq \tau(\nu * \omega) \wedge \tau(\omega) = \tau(0) \wedge \tau(\omega) = 0.6$$

and/or

$$0.5 = \hbar(\omega) \not\leq \hbar(\omega * \eta) \vee \hbar(\eta) = \hbar(0) \vee \hbar(\eta) = 0.3.$$

**Remark 5.2.** It has been observed in the examples that there is no transition between the definitions of *WLDFd-sub* and *WLDFd-id*. In fact, the *WLDFd-sub*  $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$  given in Example 5.1 is not a *WLDFd-id*, because

$$\begin{aligned}
 0.32 &= \tau \xi(\nu) \not\geq \min \{ \tau(\nu * \omega), \tau(\omega) \} \cdot \min \{ \xi(\nu * \omega), \xi(\omega) \} \\
 &= \min \{ \tau(0), \tau(\omega) \} \cdot \min \{ \xi(0), \xi(\omega) \} \\
 &= \min \{ 0.7, 0.6 \} \cdot \min \{ 0.85, 0.81 \} \\
 &= 0.6 \cdot 0.81 = 0.486.
 \end{aligned}$$

Moreover, Example 5.5 shows that a *WLDFd-id* may not be a *WLDFd-sub*, because

$$\begin{aligned}
 0.35 &= \tau \xi(\nu) \not\geq \min \{ \tau(\nu * \eta), \tau(\eta) \} \cdot \min \{ \xi(\nu * \eta), \xi(\eta) \} \\
 &= \min \{ \tau(\nu), \tau(\eta) \} \cdot \min \{ \xi(\tau), \xi(\eta) \} \\
 &= \min \{ 0.5, 0.3 \} \cdot \min \{ 0.7, 0.4 \} \\
 &= 0.3 \cdot 0.4 = 0.12.
 \end{aligned}$$

**Remark 5.3.** To clarify the relationships among the introduced linear Diophantine fuzzy structures, we summarize the main implications established in this paper for  $d$ -algebras.

| <i>Structure Type</i> | <i>Relation</i>                              |
|-----------------------|----------------------------------------------|
| <i>LDFd-sub</i>       | $\Rightarrow$ <i>WLDFd-sub</i> (Theorem 5.1) |
| <i>WLDFd-sub</i>      | $\nRightarrow$ <i>LDFd-sub</i> (Example 5.2) |
| <i>WLDFd-sub</i>      | $\Leftrightarrow$ <i>WLDFd-id</i>            |

## 6. Conclusion

In this study, we investigated the structural relationship between linear Diophantine fuzzy sets and  $d$ -algebraic structures. We introduced the notions of linear Diophantine fuzzy  $d$ -subalgebras and linear Diophantine fuzzy  $d$ -ideals, and analyzed their fundamental properties and interrelations. In particular, we established conditions under which these structures are compatible and examined their behavior under different algebraic settings.

Furthermore, we studied weak linear Diophantine fuzzy  $d$ -subalgebras and weak linear Diophantine fuzzy  $d$ -ideals, and provided examples illustrating the difference between the classical and weak formulations. We also derived conditions ensuring that weak linear Diophantine fuzzy level sets generate  $d$ -subalgebras, thereby clarifying the structural role of level-based constructions in this framework.

The results presented in this paper contribute to the development of fuzzy ideal theory in  $d$ -algebras by incorporating the flexibility of linear Diophantine fuzzy sets. The proposed framework may be extended in future work to other algebraic structures such as BCI/BCK-type algebras and related logical systems, particularly in the study of uncertainty within algebraic reasoning models.

**Acknowledgments.** The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

**Conflict of interest.** The authors declare no potential conflict of interests.

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