



## $\eta$ -RICCI-YAMABE SOLITONS ON HYPERBOLIC KENMOTSU MANIFOLDS

GAZALA , ABDUL HASEEB \*, MOBIN AHMAD , AND OĞUZHAN BAHADIR 

**Abstract.** In this article, we characterize hyperbolic Kenmotsu manifolds endowed with  $\eta$ -Ricci-Yamabe solitons. Finally, the existence of  $\eta$ -Ricci-Yamabe solitons in a hyperbolic Kenmotsu manifold has been proved by constructing a non-trivial example.

**Keywords:** Ricci-Yamabe solitons, hyperbolic Kenmotsu manifolds, conharmonic curvature tensor, Einstein manifolds.

**MSC (2020):** 53C25, 53C44, 53D15.

### 1. Introduction

During 1988-1989, Hamilton proposed the notion of Ricci and Yamabe flows [9, 10]. These flows on a smooth Riemannian (or semi-Riemannian) manifold are respectively defined by

$$\frac{\partial}{\partial t}g(t) + 2S(g(t)) = 0, \quad \text{and} \quad \frac{\partial}{\partial t}g(t) + r(t)g(t) = 0, \quad g(0) = g_0, \quad (1.1)$$

where  $S$  and  $r(t)$  are the Ricci tensor and scalar curvature of the metric  $g(t)$ , respectively.

In two dimensional case, the Yamabe and Ricci flows are equivalent. However, for more than two dimension, there is not such an equivalence, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not, in general.

On a Riemannian manifold  $M$  admitting a vector field  $\mathcal{V}$ , the Ricci and Yamabe solitons are respectively defined by [2, 3]

$$\mathcal{L}_{\mathcal{V}}g + 2S + 2\lambda g = 0, \quad (1.2)$$

$$\mathcal{L}_{\mathcal{V}}g + 2(r - \lambda)g = 0, \quad (1.3)$$

here  $\mathcal{L}_{\mathcal{V}}$  is the Lie derivative operator along the vector field  $\mathcal{V}$  and  $\lambda$  is a real number.

A new class of geometric flows named "Ricci-Yamabe flow" of the type  $(p, q)$  was introduced in [8], and is defined by

$$\frac{\partial}{\partial t}g(t) = -2pS(g(t)) + qr(t)g(t), \quad g(0) = g_0, \quad (1.4)$$

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\* Corresponding author

Gazala (gazala.math@gmail.com)  $\diamond$  ORCID:0000-0002-1290-0730

Abdul Haseeb (malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa)  $\diamond$  ORCID:0000-0002-1175-6423

Mobin Ahmad (mobinahmad68@gmail.com)  $\diamond$  ORCID:0000-0002-4131-3391

Oğuzhan Bahadır (oguzbaha@gmail.com)  $\diamond$  ORCID:0000-0001-5054-8865

where  $p$  and  $q$  are scalars.

A solution of (1.4) is called a Ricci-Yamabe soliton and satisfies the following equation [6]:

$$\mathcal{L}_V g + 2pS + (2\lambda - qr)g = 0. \quad (1.5)$$

Ricci-Yamabe soliton of types  $(p, 0)$  and  $(0, q)$  are called  $p$ -Ricci and  $q$ -Yamabe solitons, respectively. The Ricci-Yamabe solitons has been studied by many others in several ways to a different extent in [1, 11, 12, 13, 15, 19]. Recently, the authors Traore et al. [21] studied almost  $\eta$ -Ricci-Bourguignon solitons and obtain certain interesting results.

As a generalization of Ricci-Yamabe soliton, the  $\eta$ -Ricci Yamabe soliton of type  $(p, q)$  on  $\mathbb{M}$  is defined by [23]

$$\mathcal{L}_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0, \quad (1.6)$$

here  $\mu$  is a constant. The soliton is called expanding, steady or shrinking according to  $\lambda > 0$ ,  $\lambda = 0$  or  $\lambda < 0$ , respectively. An  $\eta$ -Ricci-Yamabe soliton of type  $(p, 0)$  and  $(0, q)$  is called  $p$ - $\eta$ -Ricci soliton and  $q$ - $\eta$ -Yamabe soliton, respectively.

The outline of this article is as follows: In section 2, we present some basic results and definitions on hyperbolic Kenmotsu manifolds. In Section 3, we explore hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons. Section 4 deals with the study of Ricci symmetric hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons. In Section 5, hyperbolic Kenmotsu manifolds with the  $\eta$ -Recurrent Ricci tensor admitting  $\eta$ -Ricci-Yamabe solitons have been studied. Ricci semi-symmetric and Einstein semi-symmetric hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons have been studied in Section 6 and Section 7, respectively. In Section 8, we study conharmonic curvature tensor satisfying certain curvature conditions on hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons. Finally, we construct a 3-dimensional non-trivial example of a hyperbolic Kenmotsu manifold to prove the existence of an  $\eta$ -Ricci-Yamabe soliton on such manifold.

## 2. Preliminaries

A  $(2n + 1)$ -dimensional smooth manifold  $\mathbb{M}$  is named an almost hyperbolic contact metric manifold, if  $\mathbb{M}$  admits a  $(1, 1)$  type tensor field  $\phi$ , a timelike vector field  $\xi$ , a 1-form  $\eta$  and a semi-Riemannian metric  $g$  satisfying [22]

$$\phi^2 K = K + \eta(K)\xi, \quad (2.7)$$

$$\phi\xi = 0, \quad \eta(\xi) = -1, \quad (2.8)$$

$$\eta(\phi K) = 0, \quad \text{rank}(\phi) = 2n, \quad (2.9)$$

$$g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L), \quad (2.10)$$

$$\eta(K) = g(K, \xi), \quad g(\phi K, L) = -g(K, \phi L), \quad (2.11)$$

for all smooth vector fields  $K, L$  on  $\mathbb{M}$ . Then, the structure  $(\phi, \xi, \eta, g)$  on  $\mathbb{M}$  is known as almost hyperbolic contact metric structure.

If an almost hyperbolic contact metric manifold  $\mathbb{M}$  fulfills the following condition:

$$(\nabla_K \phi)L = g(\phi K, L)\xi - \eta(K)\phi L, \quad (2.12)$$

here  $\nabla$  indicates the Levi-Civita connection of  $g$ . Then,  $\mathbb{M}$  is named a hyperbolic Kenmotsu manifold [4, 5, 17].

From the above equation, it is clear that

$$\nabla_K \xi = -K - \eta(K)\xi, \quad (2.13)$$

$$(\nabla_K \eta)L = g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L). \quad (2.14)$$

Also, the following relations hold on  $\mathbb{M}$ :

$$\eta(R(K, L)U) = g(L, U)\eta(K) - g(K, U)\eta(L), \quad (2.15)$$

$$R(K, L)\xi = \eta(L)K - \eta(K)L, \quad (2.16)$$

$$R(K, \xi)\xi = -K - \eta(K)\xi, \quad (2.17)$$

$$R(\xi, K)L = g(K, L)\xi - \eta(L)K, \quad (2.18)$$

$$S(K, \xi) = 2n\eta(K), \quad (2.19)$$

$$S(\xi, \xi) = -2n \iff Q\xi = 2n\xi, \quad (2.20)$$

$$(\mathcal{L}_\xi g)(K, L) = -2[g(K, L) + \eta(K)\eta(L)], \quad (2.21)$$

here  $S$  and  $R$  are the Ricci tensor and the Riemannian curvature tensor of  $\mathbb{M}$ , respectively; and the Ricci operator  $Q$  is defined by  $S(K, L) = g(QK, L)$ .

**Definition 2.1.** A hyperbolic Kenmotsu manifold  $\mathbb{M}$  is named an  $\eta$ -Einstein manifold if its Ricci tensor  $S(\neq 0)$  is of the form

$$S(K, L) = a_1 g(K, L) + a_2 \eta(K)\eta(L), \quad (2.22)$$

where  $a_1$  and  $a_2$  are smooth functions on  $\mathbb{M}$ . In particular, if  $a_2 = 0$ , then  $\mathbb{M}$  is termed as an Einstein manifold.

Replacing  $K$  and  $U$  by  $\xi$  in (2.22) and using the identities (2.8), (2.11) and (2.20), we have

$$-2n = -a_1 + a_2. \quad (2.23)$$

Now contracting (2.22) over  $K$  and  $L$ , we have

$$r = (2n + 1)a_1 - a_2. \quad (2.24)$$

From (2.23) and (2.24), we find  $a_1 = (\frac{r}{2n} - 1)$  and  $a_2 = (\frac{r}{2n} - 2n - 1)$ . Thus, (2.22) leads to

$$S(K, L) = (\frac{r}{2n} - 1)g(K, L) + (\frac{r}{2n} - 2n - 1)\eta(K)\eta(L). \quad (2.25)$$

This gives

$$QK = \left(\frac{r}{2n} - 1\right)K + \left(\frac{r}{2n} - 2n - 1\right)\eta(K)\xi. \quad (2.26)$$

Taking the covariant derivative of (2.26) along  $L$  and using (2.13), (2.14) and (2.26), we have

$$(\nabla_L Q)K = \frac{L(r)}{2n}(K + \eta(K)\xi) - \left(\frac{r}{2n} - 2n - 1\right)[g(L, K)\xi + 2\eta(K)\eta(L)\xi + \eta(K)L],$$

which by contracting over  $L$  and using the formula  $(\text{div} Q)K = \frac{1}{2}K(r)$  gives

$$K(r) = \frac{\xi(r)}{n-1}\eta(K) - \frac{2n(r - 2n(2n+1))}{n-1}\eta(K). \quad (2.27)$$

Putting  $K = \xi$  in (2.27) and using (2.8), we infer

$$\xi(r) = 2(r - 2n(2n + 1)).$$

From the last two equations, it follows that

$$K(r) = -2(r - 2n(2n + 1))\eta(K).$$

Now, we have the following proposition:

**Proposition 2.1.** *In a hyperbolic Kenmotsu manifold  $\mathbb{M}$ , we have*

$$\xi(r) = 2(r - 2n(2n + 1)), \quad (2.28)$$

$$K(r) = -2(r - 2n(2n + 1))\eta(K), \quad (2.29)$$

for any  $K$  on  $\mathbb{M}$ .

**Remark:** If the scalar curvature  $r$  of a  $(2n + 1)$ -dimensional hyperbolic Kenmotsu manifold is constant, then  $r = 2n(2n + 1)$ .

**Definition 2.2.** *The Ricci tensor  $S$  of a hyperbolic Kenmotsu manifold  $\mathbb{M}$  is called  $\eta$ -recurrent if*

$$(\nabla_K S)(L, U) = \eta(K)S(L, U), \quad (2.30)$$

for all smooth vector fields  $K, L, U$  on  $\mathbb{M}$ .

**Definition 2.3.** *A hyperbolic Kenmotsu manifold  $\mathbb{M}$  is said to be Ricci symmetric if*

$$(\nabla_K S)(L, U) = 0, \quad (2.31)$$

for all smooth vector fields  $K, L, U$  on  $\mathbb{M}$ .

**Definition 2.4.** *A hyperbolic Kenmotsu manifold  $\mathbb{M}$  is called a manifold of quasi-constant curvature if its curvature tensor  $R$  is expressed as*

$$\begin{aligned} R(K, L, U, W) = & a[g(L, U)g(K, W) - g(K, U)g(L, W)] \\ & + b[g(K, W)T(L)T(U) - g(K, U)T(L)T(W) \\ & + g(L, U)T(K)T(W) - g(L, W)T(L)T(U)], \end{aligned} \quad (2.32)$$

here  $R(K, L, U, W) = g(R(K, L)U, W)$ ;  $a, b$  are the scalar functions and  $\rho$  is a unit vector field defined by

$$g(K, \rho) = T(K), \quad (2.33)$$

for any  $K, L, U, W$  on  $\mathbb{M}$ .

### 3. $\eta$ -Ricci-Yamabe solitons on hyperbolic Kenmotsu manifolds

Let a hyperbolic manifold  $\mathbb{M}$  admit an  $\eta$ -Ricci-Yamabe soliton  $(g, \xi, p, q, \lambda, \mu)$ . Then, we have

$$(\mathcal{L}_\xi g)(K, L) + 2pS(K, L) + (2\lambda - qr)g(K, L) + 2\mu\eta(K)\eta(L) = 0. \quad (3.34)$$

Using (2.21) in (3.34), we find

$$S(K, L) = -\frac{1}{p}\left(\lambda - \frac{qr}{2} - 1\right)g(K, L) - \frac{(\mu - 1)}{p}\eta(K)\eta(L), \quad p \neq 0. \quad (3.35)$$

Putting  $L = \xi$  in (3.35), then using (2.8) and (2.11), we have

$$S(K, \xi) = A\eta(K), \quad (3.36)$$

where  $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$ . Now, from (2.19) and (3.36), it follows that

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (3.37)$$

Thus, we state the following theorem:

**Theorem 3.1.** *An  $\mathbb{M}$  admitting  $\eta$ -Ricci-Yamabe solitons  $(g, \xi, p, q, \lambda, \mu)$  is an  $\eta$ -Einstein manifold of the form (3.35). Moreover, the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - 2np$ .*

In particular, if  $\mu = 0$ , then (3.37) turns to  $\lambda = \frac{qr}{2} - 2np$ .

**Corollary 3.1.** *Let  $\mathbb{M}$  be a hyperbolic Kenmotsu manifold with the constant scalar curvature and admit Ricci-Yamabe solitons. Then, the soliton on  $\mathbb{M}$  is shrinking, steady and expanding if  $q < \frac{2p}{2n+1}$ ,  $q = \frac{2p}{2n+1}$  and  $q > \frac{2p}{2n+1}$ , respectively.*

#### 4. Ricci symmetric hyperbolic Kenmotsu manifolds admitting $\eta$ -Ricci-Yamabe solitons

Let a hyperbolic Kenmotsu manifold  $\mathbb{M}$  admit  $\eta$ -Ricci-Yamabe solitons. Then, (3.35) holds, i.e.,

$$S(K, L) = -\frac{1}{p}(\lambda - \frac{qr}{2} - 1)g(K, L) - \frac{(\mu - 1)}{p}\eta(K)\eta(L), \quad p \neq 0. \quad (4.38)$$

Now, taking the covariant differentiation of (4.38) with respect to  $U$ , we have

$$(\nabla_U S)(K, L) = -\frac{(\mu - 1)}{p}\{\eta(L)(\nabla_U \eta)K + \eta(K)(\nabla_U \eta)L\},$$

which by using (2.14) takes the form

$$(\nabla_U S)(K, L) = -\frac{(\mu - 1)}{p}\{g(U, K)\eta(L) + g(U, L)\eta(K) + 2\eta(K)\eta(L)\eta(U)\}. \quad (4.39)$$

In view of (2.31), (4.39) turns to

$$\frac{(\mu - 1)}{p}\{g(U, K)\eta(L) + g(U, L)\eta(K) + 2\eta(K)\eta(L)\eta(U)\} = 0. \quad (4.40)$$

Now, putting  $L = \xi$  in (4.40), then using (2.8), (2.10) and (2.11), we obtain

$$\frac{(\mu - 1)}{p}g(\phi U, \phi K) = 0. \quad (4.41)$$

Since,  $g(\phi U, \phi K) \neq 0$  and  $p \neq 0$ , then it follows that  $\mu = 1$ . Thus, (4.38) turns to

$$S(K, L) = -\frac{1}{p}(\lambda - \frac{qr}{2} - 1)g(K, L). \quad (4.42)$$

Putting  $L = \xi$  in (4.42) then using (2.11) and (2.19), we find

$$\lambda = \frac{qr}{2} - 2np + 1. \quad (4.43)$$

Thus, we have the following theorem:

**Theorem 4.1.** *A Ricci symmetric hyperbolic Kenmotsu manifold  $\mathbb{M}$  admitting  $\eta$ -Ricci-Yamabe solitons is an Einstein manifold of the form (4.42). Moreover, the scalar  $\lambda$  is given by  $\lambda = \frac{qr}{2} - 2np + 1$ .*

Next, putting  $U = \xi$  in (4.39), then using (2.8) and (2.11), it follows that

$$(\nabla_{\xi} S)(K, L) = 0. \quad (4.44)$$

This relation shows that the Ricci tensor  $S$  is parallel to the vector field  $\xi$ . Now, we state the following corollary:

**Corollary 4.1.** *The Ricci tensor  $S$  on  $\mathbb{M}$  admitting  $\eta$ -Ricci-Yamabe solitons is parallel to the vector field  $\xi$ .*

## 5. $\eta$ -Ricci-Yamabe solitons on hyperbolic Kenmotsu manifolds with $\eta$ -Recurrent Ricci tensor

Let  $\mathbb{M}$  be a hyperbolic Kenmotsu manifold with the  $\eta$ -Recurrent Ricci tensor, and admit  $\eta$ -Ricci-Yamabe solitons. Then, from (2.30) and (4.39), we have

$$-\frac{(\mu - 1)}{p} \{g(U, K)\eta(L) + g(K, L)\eta(U) + 2\eta(K)\eta(L)\eta(U)\} = \eta(K)S(L, U). \quad (5.45)$$

Substituting  $L = U = \xi$  in equation (5.45), then using (2.8), (2.11) and (3.36), we get  $A\eta(K) = 0 \implies$  either  $A = 0$ , or  $\eta(K) = 0$ . For the first case  $A = 0$ , we have  $\lambda - \mu = \frac{qr}{2}$ . Next, for the second case  $\eta(K) = 0$ , (2.30) turns to  $(\nabla_K S)(L, U) = 0$ . Thus, we state the following result:

**Theorem 5.1.** *In a hyperbolic Kenmotsu manifold  $\mathbb{M}$  with  $\eta$ -Recurrent Ricci tensor admitting  $\eta$ -Ricci-Yamabe solitons either  $\lambda - \mu = \frac{qr}{2}$ , or  $\mathbb{M}$  reduces to the Ricci symmetric hyperbolic Kenmotsu manifold.*

## 6. Ricci semi-symmetric hyperbolic Kenmotsu manifolds admitting $\eta$ -Ricci-Yamabe solitons

The notion of Ricci semi-symmetric Riemannian spaces was introduced by Mirzoyan [16] in 1992. In this section we consider a hyperbolic Kenmotsu manifold  $\mathbb{M}$  admitting  $\eta$ -Ricci-Yamabe solitons which is Ricci semi-symmetric, i.e.,  $R(K, L) \cdot S = 0$ . This leads to

$$S(R(K, L)W, U) + S(W, R(K, L)U) = 0, \quad (6.46)$$

for  $K, L, U, W$  on  $\mathbb{M}$ .

Putting  $K = \xi$  in (6.46), we have

$$S(R(\xi, L)W, U) + S(W, R(\xi, L)U) = 0. \quad (6.47)$$

By using (2.18) in (6.47), we have

$$g(L, W)S(\xi, U) - \eta(W)S(L, U) + g(L, U)S(W, \xi) - \eta(U)S(L, W) = 0.$$

Again putting  $W = \xi$ , then using (2.11) and (3.36), the foregoing equation turns to

$$S(L, U) = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)g(L, U). \quad (6.48)$$

By replacing  $U = \xi$  in (6.48) then using (2.11) and (2.19), we find

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (6.49)$$

**Theorem 6.1.** *Let  $\mathbb{M}$  be a Ricci semi-symmetric hyperbolic Kenmotsu manifold and admit  $\eta$ -Ricci-Yamabe solitons, then  $\mathbb{M}$  is an Einstein manifold of the form (6.48). Moreover, the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - 2np$ .*

## 7. Einstein semi-symmetric hyperbolic Kenmotsu manifolds admitting $\eta$ -Ricci-Yamabe solitons

**Definition 7.1.** *A hyperbolic Kenmotsu manifold  $\mathbb{M}$  is called Einstein semi-symmetric if  $R \cdot E = 0$ , where  $E$  is the Einstein tensor given by*

$$E(K, L) = S(K, L) - \frac{r}{n}g(K, L), \quad (7.50)$$

for any  $K, L$  on  $\mathbb{M}$ .

Let a hyperbolic Kenmotsu manifold  $\mathbb{M}$  admit  $\eta$ -Ricci-Yamabe solitons, and is Einstein semi-symmetric, i. e.,  $R \cdot E = 0$ . Thus, we have

$$E(R(K, L)U, W) + E(U, R(K, L)W) = 0,$$

which in view of (7.50) takes the form

$$\begin{aligned} S(R(K, L)U, W) + S(U, R(K, L)W) &= \frac{r}{n}\{g(R(K, L)U, W) \\ &+ g(U, R(K, L)W)\}. \end{aligned} \quad (7.51)$$

By putting  $K = U = \xi$  in (7.51), we have

$$S(R(\xi, L)\xi, W) + S(\xi, R(\xi, L)W) = \frac{r}{n}\{g(R(\xi, L)\xi, W) + g(\xi, R(\xi, L)W)\}.$$

Making use of (2.17), (2.18) and (3.36), the foregoing equation leads to

$$S(L, W) = -\frac{1}{p}\left(\lambda - \frac{qr}{2} - \mu\right)g(L, W). \quad (7.52)$$

Putting  $W = \xi$  in (7.52) and using (2.11), (2.19), we find

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (7.53)$$

**Theorem 7.1.** *Let  $\mathbb{M}$  be an Einstein semi-symmetric hyperbolic Kenmotsu manifold and admit  $\eta$ -Ricci-Yamabe solitons, then  $\mathbb{M}$  is an Einstein manifold. Moreover, the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - 2np$ .*

## 8. Conharmonically flat hyperbolic Kenmotsu manifolds admitting $\eta$ -Ricci-Yamabe solitons

The conharmonic curvature tensor  $C$  on a (pseudo)-Riemannian manifold  $\mathbb{M}$  is defined by [14]

$$\begin{aligned} C(K, L)U &= R(K, L)U - \frac{1}{(n-2)}[S(L, U)K - S(K, U)L \\ &+ g(L, U)QK - g(K, U)QL], \end{aligned} \quad (8.54)$$

for all  $K, L, U$  on  $\mathbb{M}$ .

The conharmonic curvature tensor has also been studied by the authors in [7, 18, 20].

In this section, first we study conharmonically flat hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons, i.e.,  $C(K, L)U = 0$ . Then, (8.54) gives

$$R(K, L)U = \frac{1}{(n-2)}[S(L, U)K - S(K, U)L + g(L, U)QK - g(K, U)QL]. \quad (8.55)$$

Taking the inner product of (8.55) with  $W$ , we have

$$\begin{aligned} g(R(K, L)U, W) &= \frac{1}{(n-2)}[S(L, U)g(K, W) - S(K, U)g(L, W) \\ &\quad + g(L, U)S(K, W) - g(K, U)S(L, W)], \end{aligned} \quad (8.56)$$

which by using (3.35) leads to

$$\begin{aligned} g(R(K, L)U, W) &= \frac{2A}{(n-2)}[g(L, U)g(K, W) - g(K, U)g(L, W)] \\ &\quad + \frac{B}{(n-2)}[g(K, W)\eta(L)\eta(U) - g(L, W)\eta(K)\eta(U) \\ &\quad + g(L, U)\eta(K)\eta(W) - g(K, U)\eta(L)\eta(W)], \end{aligned} \quad (8.57)$$

where  $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$  and  $B = -\frac{1}{p}(\mu - 1)$ .

By putting  $W = \xi$  in (8.56), then using (2.11), (2.15) and (3.36), we find

$$S(L, U)\eta(K) - S(K, U)\eta(L) = (n-2-A)[g(L, U)\eta(K) - g(K, U)\eta(L)]. \quad (8.58)$$

Putting  $K = \xi$  in (8.58), then using (2.8), (2.11) and (3.36), we find

$$S(L, U) = (n-2-A)g(L, U) + (n-2-2A)\eta(U)\eta(L). \quad (8.59)$$

By putting  $U = \xi$  in (8.59) and using (3.36), we get

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (8.60)$$

**Theorem 8.1.** *A conharmonically flat hyperbolic Kenmotsu manifold admitting  $\eta$ -Ricci-Yamabe solitons is an  $\eta$ -Einstein manifold of quasi-constant curvature of the form 8.57. Moreover, the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - 2np$ .*

Next, we study  $\xi$ -conharmonically flat hyperbolic Kenmotsu manifolds admitting  $\eta$ -Ricci-Yamabe solitons, i.e.,  $C(K, L)\xi = 0$ . Then, from (8.54) it follows that

$$R(K, L)\xi = \frac{1}{(n-2)}[S(L, \xi)K - S(K, \xi)L + \eta(L)QK - \eta(K)QL],$$

which by using (2.16) and (3.36) takes the form

$$\eta(L)K - \eta(K)L = \frac{1}{(n-2)}[A\eta(L)K - A\eta(K)L + \eta(L)QK - \eta(K)QL], \quad (8.61)$$

where  $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$ .

Equation (8.61) can be written as

$$[(n-2) - A](\eta(L)K - \eta(K)L) = \eta(L)QK - \eta(K)QL,$$

which by putting  $L = \xi$ , then using (2.8), (2.20) and the value of  $A$ , we find

$$[(n-2) + \frac{1}{p}(\lambda - \frac{qr}{2} - \mu)]K + [(n-2) + \frac{2}{p}(\lambda - \frac{qr}{2} - \mu)]\eta(K)\xi = QK. \quad (8.62)$$

Taking the inner product of (8.62) with  $L$  and using (2.11), we find

$$S(K, L) = [(n-2) + \frac{1}{p}(\lambda - \frac{qr}{2} - \mu)]g(K, L) + [(n-2) + \frac{2}{p}(\lambda - \frac{qr}{2} - \mu)]\eta(K)\eta(L). \quad (8.63)$$

Putting  $L = \xi$  in (8.63) and using (2.11), we find

$$S(K, \xi) = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)\eta(K). \quad (8.64)$$

By using (2.19) in (8.64), we get

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (8.65)$$

**Theorem 8.2.** *A  $\xi$ -conharmonically flat hyperbolic Kenmotsu manifold admitting  $\eta$ -Ricci-Yamabe solitons is an  $\eta$ -Einstein manifold of the form (8.63). Moreover, the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - 2np$ .*

Next, we consider a conharmonically  $\phi$ -semisymmetric hyperbolic Kenmotsu manifold admitting  $\eta$ -Ricci-Yamabe solitons, i.e.,  $C(K, L) \cdot \phi = 0$ . This implies that

$$C(K, L)\phi U - \phi C(K, L)U = 0, \quad (8.66)$$

which by putting  $K = \xi$  takes the form

$$C(\xi, L)\phi U - \phi C(\xi, L)U = 0. \quad (8.67)$$

From (8.54), we find

$$\begin{aligned} C(\xi, L)\phi U &= R(\xi, L)\phi U - \frac{1}{(n-2)}[S(L, \phi U)\xi - S(\xi, \phi U)L \\ &\quad + g(L, \phi U)Q\xi - g(\xi, \phi U)QL]. \end{aligned} \quad (8.68)$$

By using (2.9), (2.18), (3.35) and (3.36), (8.68) reduces to

$$C(\xi, L)\phi U = (1 - \frac{2A}{n-2})g(L, \phi U). \quad (8.69)$$

Also from (8.54), we find

$$C(\xi, L)U = [1 - \frac{2A}{n-2}](g(L, U)\xi - \eta(U)L) - \frac{B}{n-2}\eta(U)\eta(L)\xi,$$

which gives

$$\phi C(\xi, L)U = -[1 - \frac{2A}{n-2}]\eta(U)\phi L. \quad (8.70)$$

From (8.67), (8.69) and (8.70), we arrive at

$$[1 - \frac{2A}{n-2}](g(L, \phi U)\xi - \eta(U)\phi L) = 0. \quad (8.71)$$

By putting  $L = \xi$  in (8.71), it follows that

$$\lambda - \mu = \frac{qr}{2} - \frac{(n-2)p}{2}. \quad (8.72)$$

**Theorem 8.3.** *In a conharmonically  $\phi$ -semisymmetric hyperbolic Kenmotsu manifold admitting  $\eta$ -Ricci-Yamabe solitons the scalars  $\lambda$  and  $\mu$  are connected by  $\lambda - \mu = \frac{qr}{2} - \frac{(n-2)p}{2}$ .*

### 9. Example

Let us consider the 3-dimensional manifold  $\mathbb{M}^3 = \{(k, l, u) \in \mathbb{R}^3 : u \neq 0\}$ , where  $(k, l, u)$  are the standard coordinates in  $\mathbb{R}^3$ . Let  $\{\varrho_1, \varrho_2, \varrho_3\}$  be a set of vector fields on  $\mathbb{M}^3$  given by

$$\varrho_1 = u \frac{\partial}{\partial k}, \quad \varrho_2 = u \frac{\partial}{\partial l}, \quad \varrho_3 = u \frac{\partial}{\partial u} = \xi.$$

Clearly,  $\{\varrho_1, \varrho_2, \varrho_3\}$  are linearly independent at each point of  $\mathbb{M}^3$ . Let  $g$  be the semi-Riemannian metric defined by

$$g(\varrho_i, \varrho_j) = \begin{cases} 1, & i = j = 1, \\ -1, & 2 \leq i = j \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let the 1-form  $\eta$  is defined by  $\eta(K) = g(K, \varrho_3)$  for any  $K$  on  $\mathbb{M}^3$ . Let  $\phi$  be the  $(1, 1)$  tensor field defined by

$$\phi(\varrho_1) = \varrho_2, \quad \phi(\varrho_2) = -\varrho_1, \quad \phi(\varrho_3) = 0.$$

Then, using the linearity of  $\phi$  and  $g$ , we have

$$\phi^2(K) = K + \rho(K)\varrho_3, \quad \eta(\varrho_3) = -1, \quad g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L),$$

for all smooth vector fields  $K, L$  on  $\mathbb{M}^3$ . Thus, for  $\varrho_3 = \xi$ ,  $(\phi, \xi, \rho, g)$  defines an almost hyperbolic contact metric structure on  $\mathbb{M}^3$ .

Let  $\nabla$  be the Levi-Civita connection with respect to the metric  $g$ . Then, we have

$$[\varrho_1, \varrho_2] = 0, \quad [\varrho_1, \varrho_3] = -\varrho_1, \quad [\varrho_2, \varrho_3] = -\varrho_2.$$

By using Koszul's formula:

$$2g(\nabla_K L, U) = Kg(L, U) + Lg(U, K) - Ug(K, L) - g(K, [L, U]) - g(L, [K, U]) + g(U, [K, L]),$$

we can easily calculate

$$\begin{array}{lll} \nabla_{\varrho_1} \varrho_1 = \varrho_3, & \nabla_{\varrho_1} \varrho_2 = 0, & \nabla_{\varrho_1} \varrho_3 = \varrho_1, \\ \nabla_{\varrho_2} \varrho_1 = 0, & \nabla_{\varrho_2} \varrho_2 = \varrho_3, & \nabla_{\varrho_2} \varrho_3 = \varrho_2, \\ \nabla_{\varrho_3} \varrho_1 = 0, & \nabla_{\varrho_3} \varrho_2 = 0, & \nabla_{\varrho_3} \varrho_3 = 0. \end{array}$$

Now, we check

$$(\nabla_{\varrho_1} \phi) \varrho_2 = \nabla_{\varrho_1} \phi \varrho_2 - \phi(\nabla_{\varrho_1} \varrho_2) = \nabla_{\varrho_1} (-\varrho_1) - \phi(0) = -\varrho_3.$$

On the other hand, we have

$$g(\phi \varrho_1, \varrho_2) \varrho_3 - \eta(\varrho_2) \phi \varrho_1 = g(\varrho_2, \varrho_2) \varrho_3 - 0 = \varrho_3.$$

Hence,  $(\nabla_{\varrho_1} \phi) \varrho_2 \neq g(\phi \varrho_1, \varrho_2) \varrho_3 - \rho(\varrho_2) \phi \varrho_1$ .

Therefore, the manifold is not a Kenmotsu. But we can show that the manifold is a hyperbolic

Kenmotsu. In fact, we have

$$(\nabla_K \phi)L = g(\phi K, L)\xi - \eta(L)\phi K,$$

for all smooth vector fields  $K, L$  on  $\mathbb{M}^3$ . For example, taking  $K = \varrho_1, L = \varrho_2$ , we have

$$(\nabla_{\varrho_1} \phi)\varrho_2 = -\varrho_3, \quad g(\phi\varrho_1, \varrho_2)\varrho_3 - \rho(\varrho_2)\phi\varrho_1 = g(\varrho_2, \varrho_2)\varrho_3 - 0 = \varrho_3.$$

Hence, the manifold is not a Kenmotsu but a hyperbolic Kenmotsu.

Now, we compute the curvature tensor. The well-known expression for the curvature tensor  $R$  is

$$R(K, L)U = \nabla_K \nabla_L U - \nabla_L \nabla_K U - \nabla_{[K, L]}U.$$

By using the above results, we easily obtain the following components of the curvature tensor:

$$\begin{aligned} R(\varrho_1, \varrho_2)\varrho_1 &= -\varrho_2, & R(\varrho_2, \varrho_1)\varrho_1 &= \varrho_2, & R(\varrho_1, \varrho_3)\varrho_1 &= -\varrho_3, & R(\varrho_3, \varrho_1)\varrho_1 &= \varrho_3, \\ R(\varrho_1, \varrho_2)\varrho_2 &= -\varrho_1, & R(\varrho_2, \varrho_1)\varrho_2 &= \varrho_1, & R(\varrho_2, \varrho_3)\varrho_2 &= \varrho_3, & R(\varrho_3, \varrho_2)\varrho_2 &= -\varrho_3, \\ R(\varrho_1, \varrho_3)\varrho_3 &= -\varrho_1, & R(\varrho_1, \varrho_3)\varrho_3 &= -\varrho_1, & R(\varrho_2, \varrho_3)\varrho_3 &= -\varrho_2, & R(\varrho_3, \varrho_2)\varrho_3 &= \varrho_2. \end{aligned}$$

From these curvature tensors, we can easily calculate

$$S(\varrho_1, \varrho_1) = 2, \quad S(\varrho_2, \varrho_2) = -2, \quad S(\varrho_3, \varrho_3) = -2 \implies r = 6. \quad (9.73)$$

This verifies Remark for the three dimensional case.

Putting  $K = L = \xi$  in (3.35) and using (9.73), for a hyperbolic Kenmotsu 3-manifold, we infer

$$\lambda - \mu = \frac{qr}{2} - 2p.$$

Hence  $\lambda$  and  $\mu$  satisfies (3.35), and so  $g$  defines an  $\eta$ -Ricci-Yamabe soliton on the given hyperbolic Kenmotsu 3-manifold.

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(Gazala) DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, INTEGRAL UNIVERSITY, KURSI ROAD, LUCKNOW-226026, INDIA.

(A. Haseeb) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, JAZAN UNIVERSITY, P.O. Box 114, JAZAN- 45142, KINGDOM OF SAUDI ARABIA

(M. Ahmad) DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, INTEGRAL UNIVERSITY, KURSI ROAD, LUCKNOW-226026, INDIA.

(O. Bahadır) DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, K.S.U, KAHRAMAN-MARAS, TURKEY.