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ON SOME TENSOR CONDITIONS OF NEARLY KENMOTSU f-MANIFOLDS

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Abstract. In this paper, we continue to study on nearly Kenmotsu f-manifolds motivated by previous study. In this time, we prove that a second-order symmetric closed recurrent tensor is a multiple of the associated metric tensor on nearly Kenmotsu f-manifolds. Then, we get some necessary condition under which a vector field on a nearly Kenmotsu f-manifold will be a strict generalized contact or Killing vector field. Finally, we show that every φ-recurrent nearly Kenmotsu f-manifold is an Einstein manifold of globally framed type and every locally φ-recurrent nearly Kenmotsu f-manifold is a manifold of constant curvature.

1. Introduction

The studies on complex manifold is initiated by Schouten and van Dantzig in 1930 [20]. In 1933, Kähler introduced an important class of complex manifolds, which is called Kähler manifold [13]. Then, Weil proved that the existence of (1, 1) tensor field J on complex manifold, which satisfies

$$J^2 = -I,$$

where I denotes the identity transformation [23]. In 1950, Ehresmann defined almost complex manifolds, using this tensor field J. He proved that every complex manifold is an almost complex manifold, but the converse is not true [7].

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In 1970, A. Gray introduced nearly Kähler manifolds which are not Kähler, using the covariant derivative of almost complex structure J with respect to any vector field on manifold [11]. Nearly Kähler manifolds satisfy

$$(\nabla_X J) X = 0,$$

for each vector field X. Then, using this definition, Blair introduced nearly cosymplectic manifold in 1971 [4] and Blair et al. defined nearly Sasakian structure in 1976 [5]. Recently, Balkan carried this notion on globally framed metric f-manifolds and he introduced and studied on nearly C manifolds [2] and nearly Kenmotsu f-manifolds [1].

The notion of globally framed manifold or globally framed f-manifold, which is generalization of complex and contact manifolds, was introduced by Nakagawa in 1966 [16]. Then, Blair defined three classes of globally framed manifolds, called K-manifold, S-manifold and C-manifold [3]. Many researchers studied on these manifolds. Falcitelli and Pastore introduced almost Kenmotsu f-manifolds in 2007 [8]. In 2014, Öztürk et al. defined almost α-cosymplectic f-manifolds, which are generalization of almost C-manifolds and almost Kenmotsu f-manifolds [18].

Tensor properties are so important in differential geometry, in particular in Riemannian geometry. Many researchers focused on many aspect of this topic. Wong studied recurrent tensor fields on a manifold endowed with a linear connection [24]. Levy proved that on a space of constant curvature, second order symmetric parallel non-singular tensors are constant multiples of the metric tensor [15]. Najafi and Hosseinpour Kashani considered this topic for nearly Kenmotsu f manifolds [17].

Now, let $(M,\ g)$ be a Riemannian manifold. If a $(0,\ 2)$-tensor field α satisfies $\nabla\alpha = \lambda \otimes \alpha$ for some 1-form λ, then it is said to be a recurrent tensor field on $(M,\ g)$. Here, the 1-form λ is called the recurrence co-vector of α. It is easy to see that every multiple of the metric tensor is a recurrent tensor. Furthermore, if α is called a closed recurrent tensor. Also we can say that the set of closed recurrent tensors contains the set of parallel tensors as a subset, for $\lambda = 0$ ([24], [25]).

In the present study, we focus on nearly Kenmotsu f-manifolds motivated by previous studies. Firstly, we prove that a second-order symmetric closed recurrent tensor is a multiple of the associated metric tensor on nearly Kenmotsu f-manifolds. Then, we get some necessary condition under which a vector field on a nearly Kenmotsu f-manifold will be a strict generalized contact or Killing vector field. Finally, we show that every φ-recurrent nearly Kenmotsu f-manifold is an Einstein manifold of globally framed type and every locally φ-recurrent nearly Kenmotsu f-manifold is a manifold of constant curvature -1.

2. Preliminaries

Let M be $(2n+s)$-dimensional manifold and φ is a non-null $(1,\ 1)$ tensor field on M. If φ satisfies

$$\varphi^3 + \varphi = 0, \tag{2.1}$$

then φ is called an f-structure and M is called f-manifold [26]. If $rank\varphi = 2n$, namely $s = 0$, φ is called almost complex structure and if $rank\varphi = 2n+1$, namely $s = 1$, then φ reduces an almost contact structure [10]. $rank\varphi$ is always constant [21].

On an f-manifold M, P_1 and P_2 operators are defined by

$$P_1 = -\varphi^2, \quad P_2 = \varphi^2 + I, \tag{2.2}$$

which satisfy

$$\begin{gathered} P_1 + P_2 = I, \qquad P_1^2 = P_1, \qquad P_2^2 = P_2, \\ \varphi P_1 = P_1\varphi = \varphi, \quad P_2\varphi = \varphi P_2 = 0. \end{gathered} \tag{2.3}$$

These properties show that P_1 and P_2 are complement projection operators. There are D and $D^\perp$ distributions with respect to P_1 and P_2 operators, respectively [27]. Also, $\dim(D) = 2n$ and $\dim\left(D^\perp\right) = s$.

Let M be $(2n+s)$-dimensional f-manifold and φ is a $(1,\ 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \le i \le s$ on M, respectively. If $\left(\varphi,\ \xi_i,\ \eta^i\right)$ satisfy

$$\eta^j(\xi_i) = \delta_i^j, \tag{2.4}$$

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.5}$$

then $\left(\varphi,\ \xi_i,\ \eta^i\right)$ is called globally framed f-structure or simply framed f-structure and M is called globally framed f-manifold or simply framed f-manifold [16]. For a framed f-manifold M, the following properties are satisfied [16]:

$$\varphi\xi_i = 0, \tag{2.6}$$

$$\eta^i \circ \varphi = 0. \tag{2.7}$$

If on a framed f-manifold M, there exists a Riemannian metric which satisfies

$$\eta^i(X) = g(X,\ \xi_i), \tag{2.8}$$

and

$$g(\varphi X,\ \varphi Y) = g(X,\ Y) - \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{2.9}$$

for all vector fields $X,\ Y$ on M, then M is called framed metric f-manifold [9]. On a framed metric f-manifold, fundamental 2-form Φ defined by

$$\Phi(X,\ Y) = g(X,\ \varphi Y), \tag{2.10}$$

for all vector fields $X,\ Y \in \chi(M)$ [9]. For a framed metric f-manifold,

$$N_\varphi + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i, \tag{2.11}$$

is satisfied, M is called normal framed metric f-manifold, where N_φ denotes the Nijenhuis torsion tensor of φ [12].

A globally framed metric f-manifold M is called Kenmotsu f-manifold if it satisfies

$$(\nabla_X \varphi)\,Y = \sum_{k=1}^{s} \left\{ g(\varphi X,\ Y)\,\xi_k - \eta^k(Y)\,\varphi X \right\}, \tag{2.12}$$

for all vector fields $X,\ Y \in \chi(M)$ [18]. Furthermore, if a globally framed metric f-manifold M satisfies

$$(\nabla_X \varphi)\,Y + (\nabla_Y \varphi)\,X = -\sum_{k=1}^{s} \left\{ \eta^k(X)\,\varphi Y + \eta^k(Y)\,\varphi X \right\} \tag{2.13}$$

then it is called nearly Kenmotsu f-manifold. It is easily seen that every Kenmotsu f-manifold is a nearly Kenmotsu f-manifold, but the converse is not true. When a normal Kenmotsu f-manifold M is normal, it is Kenmotsu f-manifold [1]. On a nearly Kenmotsu f-manifold M, the following identities hold:

$$R(\xi_i,\ X)\,Y = \sum_{k=1}^{s} \left\{ -g(X,\ Y)\,\xi_k + \eta^k(Y)\,X \right\}, \tag{2.14}$$

$$R(X,\ Y)\,\xi_i = \sum_{k=1}^{s} \left\{ \eta^k(X)\,Y - \eta^k(Y)\,X \right\}, \tag{2.15}$$

$$S(\varphi X,\ \varphi Y) = S(X,\ Y) + (2n+s-1)\sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y), \tag{2.16}$$

$$\left(\nabla_X \eta^i\right) Y = g(X,\ Y) - \sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y), \tag{2.17}$$

$$\sum_{k=1}^{s} \eta^k(R(X,\ Y)\,Z) = \sum_{k=1}^{s} \left\{ g(X,\ Z)\,\eta^k(Y) - g(Y,\ Z)\,\eta^k(X) \right\}, \tag{2.18}$$

for any vector fields $X,\ Y$ on M [1].

A vector field X on a nearly Kenmotsu f-manifold M is said to be a generalized contact vector field, if

$$L_X\eta^k(Y) = \sigma\eta^k(Y) \tag{2.19}$$

or a conformal vector field, if

$$L_X g(Y,\ Z) = \rho g(Y,\ Z), \tag{2.20}$$

for any vector fields Y and Z on M, where σ and ρ are scalar function defined on M and L_X denotes the Lie derivative along X. Moreover, X is called strict generalized contact vector field or Killing vector field if $\sigma = 0$ or $\rho = 0$.

3. Recurrent Tensor Fields of the Second Order On Nearly Kenmotsu f-manifolds

Theorem 3.1. *Let M be a nearly Kenmotsu f-manifold. Then a second-order symmetric closed recurrent tensor field whose recurrence co-vector annihilates ξ_k is a multiple of the metric tensor g for each $1 \leq k \leq s$.*

Proof. We suppose that M is a nearly Kenmotsu f-manifold and α is a closed recurrent $(0,\ 2)$-tensor on M which satisfies $\lambda(\xi_k) = 0$, for each $1 \leq k \leq s$. After a straightforward calculation, we obtain

$$\alpha(R(W,\ X)Y,\ Z) + \alpha(Y,\ R(W,\ X)Z) = \lambda(W)\alpha(\nabla_X Y,\ Z) - \lambda(X)\alpha(\nabla_W Y,\ Z), \tag{3.21}$$

for any vector fields $X,\ Y,\ Z,\ W$ on M. Putting $Y = Z = W = \xi_i$ in (3.21) and using $\nabla_X\xi_i = -\varphi^2 X$, then in view of $\lambda(\xi_i) = 0$ we have

$$\alpha(R(\xi_k,\ X)\xi_k,\ \xi_k) + \alpha(\xi_k,\ R(\xi_k,\ X)\xi_k) = 0. \tag{3.22}$$

By using (2.14) and (2.15) in (3.22), we get

$$g(X,\ \xi_i)\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i) + \alpha(\xi_i,\ \xi_k)\} - \alpha(X,\ \xi_i) - \alpha(\xi_i,\ X) = 0 \tag{3.23}$$

Differentiating (3.23) along Y and using $\nabla_{\xi_k}\xi_k = 0$, it follows that

$$\begin{aligned} &\{g(\nabla_Y X,\ \xi_i) + g(X,\ \nabla_Y\xi_i)\}\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i) + \alpha(\xi_i,\ \xi_k)\} \\ = \ &\alpha(\nabla_Y X,\ \xi_i) + \alpha(X,\ \nabla_Y\xi_i) + \alpha(\nabla_Y\xi_i,\ X) + \alpha(\xi_i,\ \nabla_Y X). \end{aligned} \tag{3.24}$$

Replacing X by $\nabla_Y X$ in (3.24), we derive

$$g\left(\nabla_Y X,\ \xi_i\right)\sum_{k=1}^{s}\left\{\alpha\left(\xi_k,\ \xi_i\right)+\alpha\left(\xi_i,\ \xi_k\right)\right\}-\alpha\left(\nabla_Y X,\ \xi_i\right)-\alpha\left(\xi_i,\ \nabla_Y X\right)=0 \tag{3.25}$$

From (3.24) and (3.25), we deduce

$$g\left(X,\ \nabla_Y \xi_i\right)\sum_{k=1}^{s}\left\{\alpha\left(\xi_k,\ \xi_i\right)+\alpha\left(\xi_i,\ \xi_k\right)\right\}=\alpha\left(X,\ \nabla_Y \xi_i\right)+\alpha\left(\nabla_Y \xi_i,\ X\right). \tag{3.26}$$

Taking in account of $\nabla_X \xi_i=-\varphi^2 X$, then we conclude that

$$g\left(X,\ Y-\sum_{k=1}^{s}\eta^k\left(Y\right)\xi_k\right)\sum_{k=1}^{s}\left\{\alpha\left(\xi_k,\ \xi_i\right)+\alpha\left(\xi_i,\ \xi_k\right)\right\} \tag{3.27}$$

$$= \alpha\left(X,\ Y-\sum_{k=1}^{s}\eta^k\left(Y\right)\xi_k\right)+\alpha\left(Y-\sum_{k=1}^{s}\eta^k\left(Y\right)\xi_k,\ X\right) \tag{3.28}$$

Using (3.23) and (3.27), we find

$$\alpha^{\circ}\left(X,\ Y\right)=\sum_{k=1}^{s}\alpha^{*}\left(\xi_k,\ \xi_i\right)g\left(X,\ Y\right). \tag{3.29}$$

Here, α° denotes the symmetric part of α defined by

$$\alpha^{\circ}\left(X,\ Y\right)=\frac{s}{2}\left\{\alpha\left(X,\ Y\right)+\left(Y,\ X\right)\right\}$$

and $\alpha^{*}\left(\xi_k,\ \xi_i\right)=\alpha\left(\xi_k,\ \xi_i\right)+\alpha\left(\xi_i,\ \xi_k\right)$. Furthermore, by using (3.23) and $\nabla\alpha=\lambda\otimes\alpha$, then we have $\nabla_X\mu=\lambda\left(X\right)\mu$, where X is an arbitrary vector field on M and

$$\mu=\sum_{k=1}^{s}\alpha^{*}\left(\xi_k,\ \xi_i\right).$$

Hence, if α is a parallel tensor or equivalently $\lambda=0$, so we can say μ is a constant function, but in general μ is not a constant function. Additionally, if α is symmetric, i.e.$\alpha=\alpha^{\circ}$, then we conclude $\alpha=\mu g$ and $\lambda=d\mu$.

4. Geometric Vector Fields on Nearly Kenmotsu f-manifolds

Theorem 4.1. *Every generalized contact vector field on a nearly Kenmotsu f-manifold leaving the Ricci tensor invariant is a generalized strict contact vector field.*

Proof. Let us suppose that a generalized contact vector field X leaves the Ricci tensor invariant, i.e.

$$L_X S\left(Y,\ Z\right)=0, \tag{4.30}$$

for any vector fields Y and Z on M. Taking $Y=\xi_i$ in (4.30), it implies that

$$L_X\left(S\left(Y,\ \xi_i\right)\right)=S\left(L_X Y,\ \xi_i\right)+S\left(Y,\ L_X \xi_i\right). \tag{4.31}$$

By using (2.16), (2.19) and (4.31), then we have

$$(1-(2n+s))\,\sigma\sum_{k=1}^{s}\eta^{k}(Y)=S(Y,\ L_{X}\xi_{i}). \tag{4.32}$$

Taking $Y=\xi_j$ in (4.32) and using (2.16), then we obtain

$$\sigma=\sum_{k=1}^{s}\eta^{k}(L_{X}\xi_{i}). \tag{4.33}$$

On the other hand, substituting ξ_i for Y in (2.19) it follows that

$$\sigma=-\sum_{k=1}^{s}\eta^{k}(L_{X}\xi_{i}), \tag{4.34}$$

which means $\sigma=0$.

Theorem 4.2. *Every vector field on a nearly Kenmotsu f-manifold leaving the curvature tensor invariant is a Killing vector field.*

Proof. For a vector field X on a nearly Kenmotsu f-manifold, we assume that $L_X R=0$. It is well-known that the curvature tensor of g satisfies

$$g(R(U,\ V)Y,\ Z)+g(R(U,\ V)Z,\ Y)=0, \tag{4.35}$$

for all vector fields $U,\ V,\ Y,\ Z$ on M. Applying L_X to (4.35), we have

$$L_{X}g(R(U,\ V)Y,\ Z)+L_{X}g(R(U,\ V)Z,\ Y)=0. \tag{4.36}$$

Setting $U=Y=Z=\xi_i$ in (4.36) and using (2.14), we derive

$$L_{X}g(V,\ \xi_{i})=\eta^{i}(V)\,L_{X}g(\xi_{i},\ \xi_{i}). \tag{4.37}$$

On the other hand, putting $U=Y=\xi_i$ in (4.36) and using (2.14), it implies that

$$\begin{aligned}0\ &=\ L_{X}g(V,\ Z)-\eta^{i}(V)\sum_{k=1}^{s}L_{X}g(\xi_{k},\ Z)\\&\quad+L_{X}g(\xi_{i},\ V)\sum_{k=1}^{s}\eta^{k}(Z)-g(V,\ Z)\,L_{X}g(\xi_{i},\ \xi_{i})\end{aligned} \tag{4.38}$$

From (4.37) and (4.38), then we get

$$L_{X}g(V,\ Z)=\rho g(V,\ Z), \tag{4.39}$$

where $\rho=g(\xi_i,\ \xi_i)$. Under the assumption $L_X R=0$, we see that $L_X S=0$. Furthermore, it is said to be

$$\rho=-2g(L_{X}\xi_{i},\ \xi_{i})=\frac{2}{2n+s-1}S(L_{X}\xi_{i},\ \xi_{i})=\frac{1}{(1-2n-s)}L_{X}S(\xi_{i},\ \xi_{i})=0. \tag{4.40}$$

5. φ-Recurrent Nearly Kenmotsu f-manifolds

Firstly, we give some basic definitions.

Definition 5.1. *A nearly Kenmotsu f-manifold M is said to be locally φ-symmetric manifold in the sense of Takahashi [22] if it satisfies*

$$\varphi^2\left(\left(\nabla_W R\right)(X,\ Y)\, Z\right) = 0, \tag{5.41}$$

for all vector fields $X,\ Y,\ Z,\ W$ orthogonal to ξ_k, for each $1 \leq k \leq s$.

Definition 5.2. *A nearly Kenmotsu f-manifold M is said to be φ-recurrent manifold in the sense of Takahashi [22] (locally φ-recurrent manifold, resp.) if there exists a nonzero 1-form B such that*

$$\varphi^2\left(\left(\nabla_W R\right)(X,\ Y)\, Z\right) = B(W)\, R(X,\ Y)\, Z, \tag{5.42}$$

for arbitrary vector fields $X,\ Y,\ Z,\ W$ (for all $X,\ Y,\ Z,\ W$ orthogonal to ξ_k, for each $1 \leq k \leq s$).

Theorem 5.1. *Let M be an η-Einstein nearly Kenmotsu f-manifold. If at least one of the coefficients is constant function, then M is an Einstein manifold.*

Proof. From (5.42), we have

$$\left(\nabla_W R\right)(X,\ Y)\, Z = \sum_{k=1}^{s} \eta^k\left(\left(\nabla_W R\right)(X,\ Y)\, Z\right)\xi_k - B(W)\, R(X,\ Y)\, Z. \tag{5.43}$$

By using (5.43) and Bianchi identity, we obtain

$$B(W)\sum_{k=1}^{s} \eta^k\left(R(X,\ Y)\, Z\right) + B(X)\sum_{k=1}^{s} \eta^k\left(R(Y,\ W)\, Z\right) + B(Y)\sum_{k=1}^{s} \eta^k\left(R(W,\ X)\, Z\right) = 0. \tag{5.44}$$

Now, let $\{e_i\}$, $1 \leq i \leq 2n+s$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $Y = Z = e_i$ in (5.44) and taking summation over i, in view of (2.14) and (2.15), then we conclude that

$$B(W)\sum_{k=1}^{s} \eta^k(X) = B(X)\sum_{k=1}^{s} \eta^k(W), \tag{5.45}$$

for any vector fields $X,\ W$. Replacing X by ξ_i in (5.45), it implies that

$$B(W) = \eta^i\left(\widehat{B}\right)\sum_{k=1}^{s} \eta^k(W), \tag{5.46}$$

where $B(\xi_i) = g\left(\xi_i,\ \widehat{B}\right) = \eta^i\left(\widehat{B}\right)$. Now, let us suppose that M is η-Einstein, the we can write

$$S(X,\ Y) = ag(X,\ Y) + b\sum_{k=1}^{s}\eta^k(X)\,\eta^k(Y)\,, \tag{5.47}$$

where a and b are scalar functions on M. Taking $Y = \xi_i$ in (2.17), from (5.47) we deduce

$$a + b = 1 - 2n - s. \tag{5.48}$$

Using local coordinate, we can rewrite (5.47) as follows:

$$R_{ij} = ag_{ij} + b\sum_{k=1}^{s}\eta_i^k\eta_j^k, \tag{5.49}$$

which implies

$$r = (2n + s)\,a + sb. \tag{5.50}$$

Taking the covariant derivative with respect to g from (5.49), we derive

$$R_{ij,m} = a_{,m}g_{ij} + \sum_{k=1}^{s}\left\{b_{,m}\eta_i^k\eta_j^k + b\eta_{i,m}^k\eta_j^k + b\eta_i^k\eta_{j,m}^k\right\}. \tag{5.51}$$

By contracting (5.51) with g^{im}, we get

$$R_{j,m}^m = a_{,j} + \sum_{k=1}^{s}\left\{b_{,m}\xi^m\eta_j^k + b\eta_{i,m}^k g^{im}\eta_j^k + b\eta_i^k\eta_{j,m}^k g^{im}\right\}. \tag{5.52}$$

We know that $R_{j,m}^m = \dfrac{1}{2}r_{,j}$. Thus we have

$$r_{,j} = 2\left\{a_{,j} + \sum_{k=1}^{s}\left[b_{,m}\xi^m + 2nb\right]\eta_j^k\right\}. \tag{5.53}$$

Here, we use (2.17) and $\eta_{i,m}g^{im} = \left\{g_{im} - \sum_{k=1}^{s}\eta_i^k\eta_m^k\right\}g^{im} = 2n$. Moreover, taking the covariant derivative of (5.48) and from (5.50), then we obtain

$$r_{,j} = 2na_{,j}. \tag{5.54}$$

Substituting (5.54) into (5.53), it follows that

$$na_{,j} = a_{,j} + \sum_{k=1}^{s}\left[b_{,m}\xi^m + 2nb\right]\eta_j^k. \tag{5.55}$$

By contracting (5.55) with ξ^j and using (5.48), we deduce

$$b_{,m}\xi^m = -2b. \tag{5.56}$$

Moreover, if b or a is a constant function, then (5.56) implies that $b = 0$. Hence, M is an Einstein manifold.

Theorem 5.2. *Every φ-recurrent nearly Kenmotsu f-manifold is an Einstein manifold.*

Proof. By using (5.43), we obtain

$$-g\left(\left(\nabla_W R\right)(X,\ Y)Z,\ U\right)+\sum_{k=1}^{s}\eta^k\left(\left(\nabla_W R\right)(X,\ Y)Z\right)\eta^k(U)=B(W)\,g\left(R(X,\ Y)Z,\ U\right). \tag{5.57}$$

Let $\{e_i\}$, $1\leq i\leq 2n+s$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X=U=e_i$ in (5.57) and taking summation over i, then we deduce that

$$-\left(\nabla_W S\right)(Y,\ Z)+\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)Z\right)\eta^i(e_i)=B(W)\,S(Y,\ Z). \tag{5.58}$$

Replacing Z by ξ_k in (5.58), we have

$$-\left(\nabla_W S\right)(Y,\ \xi_k)+\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i)=B(W)\,S(Y,\ \xi_k). \tag{5.59}$$

Now, we will show that $\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i)$ vanishes identically. Firstly, we recall

$$\begin{aligned}\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i) &= \sum_{k=1}^{s}\eta^k\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right) \\ &= \sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right),\end{aligned} \tag{5.60}$$

where we use $\eta^i(e_i)=0$ for $i=1,\ \ldots,2n$. From the properties, we find

$$\begin{aligned}&\sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right) \\ =\ &\sum_{k=1}^{s}\{g\left(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k\right)-g\left(R(\nabla_W e_k,\ Y)\xi_k,\ \xi_k\right) \\ &-g\left(R(e_k,\ \nabla_W Y)\xi_k,\ \xi_k\right)-g\left(R(e_k,\ Y)\nabla_W\xi_k,\ \xi_k\right)\}.\end{aligned} \tag{5.61}$$

Making use of (5.61) at $p\in M$ and using $g_{ij}(p)=\delta_{ij}$, we conclude that $\nabla_W e_k(p)=0$. On the other hand, we get

$$\sum_{k=1}^{s}g\left(R(e_k,\ \nabla_W Y)\xi_k,\ \xi_k\right)=-\sum_{k=1}^{s}g\left(R(\xi_k,\ \xi_k)\nabla_W Y,\ e_k\right)=0, \tag{5.62}$$

since R skew-symmetric. By virtue of (5.62) and $\nabla_W e_k(p)=0$ in (5.61), we derive

$$\begin{aligned}\sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right) &= \sum_{k=1}^{s}\{g\left(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k\right) \\ &\quad -g\left(R(e_k,\ Y)\nabla_W\xi_k,\ \xi_k\right)\}.\end{aligned} \tag{5.63}$$

By using $g(R(e_k,\ Y)\xi_k,\ \xi_k) = -g(R(\xi_k,\ \xi_k)Y,\ e_k) = 0$, we find

$$\sum_{k=1}^{s} \{g(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k) - g(R(e_k,\ Y)\xi_k,\ \nabla_W \xi_k)\} = 0, \tag{5.64}$$

which implies

$$\begin{aligned} 0 &= \sum_{k=1}^{s} g((\nabla_W R)(e_k,\ Y)\xi_k,\ \xi_k) \\ &= -\sum_{k=1}^{s} \{g(R(e_k,\ Y)\xi_k,\ \nabla_W \xi_k) + g(R(e_k,\ Y)\nabla_W \xi_k,\ \xi_k)\}, \end{aligned} \tag{5.65}$$

since R skew-symmetric. Hence, we prove $\sum_{i=1}^{2n+s} \eta^i((\nabla_W R)(e_i,\ Y)\xi_k)\eta^i(e_i) = 0$ and from (5.59) we have

$$-(\nabla_W S)(Y,\ \xi_k) = B(W)S(Y,\ \xi_k). \tag{5.66}$$

Furthermore, it is well-known that

$$(\nabla_W S)(Y,\ \xi_k) = \nabla_W S(Y,\ \xi_k) - S(\nabla_W Y,\ \xi_k) - S(Y,\ \nabla_W \xi_k). \tag{5.67}$$

By applying (2.16), (2.17) and $\nabla_X \xi_i = -\varphi^2 X$ in (5.67), it follows

$$(\nabla_W S)(Y,\ \xi_k) = -(2n+s-1)g(Y,\ W) - S(Y,\ W). \tag{5.68}$$

Plugging (5.68) into (5.66) and using (5.46), we conclude that

$$S(Y,\ W) = (1-2n-s)g(Y,\ W) + (1-2n-s)\eta^i\left(\widehat{B}\right)\sum_{k=1}^{s}\eta^k(Y)(W),$$

which means the manifold η-Einstein of globally framed type with $a = (1-2n-s)$ is constant. By Theorem 4., it is said to be M is an Einstein manifold

Theorem 5.3. *A locally φ-recurrent nearly Kenmotsu f-manifold has constant curvature -1.*

Proof. Differentiating (2.15) with respect to any vector field W and taking in account of (2.17), after an easy calculation we find

$$(\nabla_W R)(X,\ Y)\xi_i = g(W,\ X)Y - g(W,\ Y)X - R(X,\ Y)W. \tag{5.69}$$

By using (2.18) and from (5.69), we get

$$\sum_{k=1}^{s} \eta^k((\nabla_W R)(X,\ Y)\xi_k) = 0. \tag{5.70}$$

From (5.69) and (5.70), we have from (5.43)

$$\sum_{k=1}^{s} (\nabla_W R)(X,\ Y)\,\xi_k = B(W) \sum_{k=1}^{s} R(X,\ Y)\,\xi_k. \tag{5.71}$$

By virtue of (5.69), it implies that

$$-\,g(W,\ X)\,Y + g(W,\ Y)\,X + R(X,\ Y)\,W = B(W) \sum_{k=1}^{s} R(X,\ Y)\,\xi_k. \tag{5.72}$$

Thus, if X and Y are orthogonal to ξ_k for each $1 \leq k \leq s$, we derive

$$\sum_{k=1}^{s} R(X,\ Y)\,\xi_k = 0. \tag{5.73}$$

Hence, for all vector fields $X,\ Y$ and W, we deduce

$$R(X,\ Y)\,W = -\left\{g(W,\ X)\,Y + g(W,\ Y)\,X\right\},$$

which gives us desired result.

6. Example

Let M be a 6-dimensional manifold given by

$$M = \left\{(x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2) \in \mathbb{R}^6 :\ z_1,\ z_2 \neq 0\right\}$$

where $(x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2)$ are standard coordinates in $\mathbb{R}^6$. We choose the vector fields as in the following:

$$\begin{aligned} e_1 &= e^{-(z_1+z_2)} \frac{\partial}{\partial x_1}, & e_2 &= e^{-(z_1+z_2)} \frac{\partial}{\partial x_2}, \\ e_3 &= e^{-(z_1+z_2)} \frac{\partial}{\partial y_1}, & e_4 &= e^{-(z_1+z_2)} \frac{\partial}{\partial y_2}, \\ e_5 &= \frac{\partial}{\partial z_1}, & e_6 &= \frac{\partial}{\partial z_2}. \end{aligned}$$

which are linearly independent at any point of M. Denote g the Riemannian metric defined by

$$g = e^{2(z_1+z_2)} \sum_{i=1}^{2} \left\{dx_i \otimes dx_i + dy_i \otimes dy_i + dz_i \otimes dz_i\right\}.$$

Let η_1 and η_2 be 1-forms given by $\eta_1(X) = g(X,\ e_5)$ and $\eta_2(X) = g(X,\ e_6)$ for any vector field on M, respectively. Thus $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6\}$ is an orthornormal basis of tangent space at any point on M. We define the $(1,1)$-tensor field φ as follows:

$$\varphi\left(\sum_{i=1}^{2}\left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} + z_i \frac{\partial}{\partial z_i}\right)\right) = \sum_{i=1}^{2}\left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\right).$$

Hence we derive

$$\varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0, \quad \varphi e_6 = 0.$$

By virtue of the linearity of g and φ, we deduce that

$$\eta_1 (e_5) = 1, \quad \eta_2 (e_6) = 1, \quad \varphi^2 X = -X + \eta_1 (X) e_5 + \eta_2 (X) e_6$$

$$g (\varphi X, \ \varphi Y) = g (X, \ Y) - \eta_1 (X) \eta_1 (Y) - \eta_2 (X) \eta_2 (Y) .$$

Then for $\xi_1 = e_5$ and $\xi_2 = e_6$, $\left(\varphi, \ \xi_i, \ \eta^i, \ g\right)$ defines a globally framed metric f-structure on M. It is clear that the 1-forms are closed. On the other hand, we get

$$\Phi \left(\frac{\partial}{\partial x_i}, \ \frac{\partial}{\partial y_i} \right) = g \left(\frac{\partial}{\partial x_i}, \ \varphi \frac{\partial}{\partial y_i} \right) = g \left(\frac{\partial}{\partial x_i}, \ -\frac{\partial}{\partial x_i} \right) = e^{-2(z_1+z_2)}$$

which means that $\Phi = -e^{2(z_1+z_2)}$. Therefore, we obtain

$$d\Phi = -2e^{2(z_1+z_2)} (dz_1 + dz_2) \wedge dx \wedge dy = 2 (\eta_1 + \eta_2) \wedge \Phi$$

which gives us M is an almost Kenmotsu f-manifold. After some easy computations, it is clearly seen that the Nijenhuis tensor field vanishes identically, that is, M is normal. So M is a Kenmotsu f-manifold. It is well-known that every Kenmotsu f-manifold is a nearly Kenmotsu f-manifold (see [2]). Thus we conclude that M is a nearly Kenmotsu f-manifold

Furthermore we have

$$[e_1, \ e_5] = [e_1, \ e_6] = e_1,$$
$$[e_2, \ e_5] = [e_2, \ e_6] = e_2,$$
$$[e_3, \ e_5] = [e_3, \ e_6] = e_3,$$
$$[e_4, \ e_5] = [e_4, \ e_6] = e_3$$

and remaning terms $[e_i, \ e_j] = 0$ for all $1 \leq i, \ j \leq 6$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is defined by

$$\begin{aligned} 2g (\nabla_X Y, \ Z) &= Xg (Y, \ Z) + Yg (Z, \ X) - Zg (X, \ Y) \\ &\quad -g (X, \ [Y, \ Z]) - g (Y, \ [X, \ Z]) + g (Z, \ [X, \ Y]) . \end{aligned}$$

By using this Koszul's formula, then we obtain

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = - (e_5 + e_6)$$

and the other terms $\nabla_{e_i} e_j = 0$ for all $1 \leq i,\ j \leq 6$. It is welknown that Riemannian curvature tensor is defined by

$$R(X,\ Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,\ Y]} Z \tag{6.74}$$

for any vector fields on M. By the above results, we can easily get the non-vanishing components of the Riemannian curvature tensors as in the following:

$$\begin{aligned} R(e_1,\ e_5)\, e_1 = R(e_1,\ e_6)\, e_1 = e_5 + e_6, \\ R(e_2,\ e_5)\, e_2 = R(e_2,\ e_6)\, e_2 = e_5 + e_6, \\ R(e_3,\ e_5)\, e_3 = R(e_3,\ e_6)\, e_3 = e_5 + e_6, \\ R(e_4,\ e_5)\, e_4 = R(e_4,\ e_6)\, e_4 = e_5 + e_6. \end{aligned} \tag{6.75}$$

Now, let $X,\ Y$ and Z be three vector fields given by

$$\begin{aligned} X &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6, \\ Y &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6, \\ Z &= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6 \end{aligned}$$

where $a_i,\ b_i$ and c_i are all non-zero real numbers for all $i = 1,\ \ldots,\ 6$. By taking into account of (6.75) in (6.74), then we get

$$R(X,\ Y)Z = \{a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4\}(b_5 + b_6)(e_5 + e_6).$$

Again by using (6.75), then we obtain the scalar curvature $r = 8$. By these considerations, it is said that the 6-dimensional manifold M satisfies Theorem 2 and Theorem 3.

7. Conclusion

In this paper, we study some tensor conditions on nearly Kenmotsu f-manifold and we generalize some previous results obtain by Najafi and Hosseinpour in [17] since a nearly Kenmotsu f-manifold is a nice generalization of nearly Kenmotsu one. Additonally, we construct an example satisfying some corresponding results.

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