

VARIATION-OF-PARAMETERS AND LIPSCHITZ-TYPE STABILITY IN UNPERTURBED DIFFERENTIAL SYSTEMS WITH INITIAL TIME DIFFERENCE

COŞKUN YAKAR  AND DILARA KARSLIOĞLU  *

Abstract. In this study, we explore the connection between unperturbed differential systems that exhibit variations in both their initial conditions and the starting time. Through the application of variation-of-parameters methods, we construct integral representations that reveal this relationship. To formulate Lipschitz-type stability criteria in the presence of an initial time shift in nonlinear differential systems, it is essential to employ the variational framework derived from the unperturbed model.

Keywords: differential equations, initial time difference, Lipschitz stability, variation-of-parameters

MSC (2020): 34D10, 34D20, 34D99.

1. INTRODUCTION

The variation-of-parameters formula (VPF) is a classical and powerful tool in the qualitative analysis of nonlinear differential equations, particularly in describing the structure and behavior of solutions. In recent years, VPF-based approaches have been extended to investigate nonlinear systems in the absence of an initial time difference (ITD), as well as to develop associated differential inequality frameworks (see [2, 4, 5]). Motivated by these developments, the present paper establishes VPF-based representations that relate unperturbed systems with different initial data and incorporates the effects of initial time shifts.

The qualitative theory of dynamical systems has a long history, especially in the study of stability properties of differential equations. Classical results mainly consider perturbations in the initial state while keeping the initial time fixed, as discussed in [2, 4, 5, 17, 19]. More recently, attention has shifted toward models where both the initial state and the initial time are perturbed. This leads to the notion of initial time difference (ITD) stability. In [10, 14], variation-of-parameters techniques were used to analyze relationships between solutions corresponding to different initial times and initial conditions, leading to the concept of ITD-based stability structures.

Received: 2025.10.15

Revised: 2026.04.22

Accepted: 2026.08.03

* Corresponding author

Coşkun Yakar (cyakar@gtu.edu.tr)  ORCID:0000-0002-7759-7939

Dilara Karşoğlu (dilara.karslioglu@fbu.edu.tr)  ORCID:0000-0001-7141-7346

A key feature distinguishing ITD stability from classical Lipschitz stability is the reference framework. Classical Lipschitz stability is formulated with respect to the zero solution, whereas ITD-based stability is defined relative to a nontrivial unperturbed trajectory that incorporates both initial state perturbations and initial time shifts. As shown in Section 4, standard transformations used in classical stability theory do not directly carry over to the ITD setting, which highlights a structural difference between the two frameworks.

To develop stability criteria in this generalized setting, we employ the variational system associated with the underlying unperturbed dynamics. Within this framework, we introduce several notions of ITD-Lipschitz stability, including variation-based and time-shifted formulations. Our results demonstrate that classical stability theory is insufficient to fully characterize dynamics under ITD perturbations, and that ITD stability requires a distinct analytical treatment.

Furthermore, we show that ITD-Lipschitz stability implies ITD stability, while the converse implication does not hold in general. Additional results are obtained for exponential asymptotic ITD-Lipschitz stability and uniform ITD stability, providing a unified framework for analyzing stability under simultaneous perturbations in state and initial time.

The study of Lipschitz and exponential stability for nonlinear differential systems has been extensively developed in the literature. Classical works such as [3, 7, 11] focus on stability with respect to the zero solution under fixed initial time assumptions. These approaches typically rely on variation-of-parameters formulas, Lyapunov techniques, and differential inequalities.

More recent contributions, including [1, 13, 16], extend these ideas to systems with time-dependent perturbations and employ Gronwall-type inequalities to derive stability results. However, these studies still evaluate stability relative to a fixed reference solution or the zero equilibrium.

In contrast, the present paper introduces Lipschitz stability under Initial Time Difference (ITD), where stability is measured with respect to an unperturbed trajectory defined at a shifted initial time. This framework generalizes classical stability notions by simultaneously incorporating spatial perturbations and initial time shifts. Unlike previous approaches, we provide a unified treatment of variation-based, uniform, and exponential asymptotic ITD-stability. In particular, we establish that ITD-Lipschitz stability implies ITD stability, while the reverse implication does not generally hold.

2. PRELIMINARIES

We consider the following nonlinear differential models characterized by different initial conditions:

$$x' = f(t, x), \quad x(t_0) = x_0 \text{ for } t \geq t_0, \quad t_0 \in \mathbb{R}_+ \quad (2.1)$$

$$x' = f(t, x), \quad x(\rho_0) = y_0 \text{ for } t \geq \rho_0, \quad \rho_0 \in \mathbb{R}_+ \quad (2.2)$$

Assume that the function f is continuous and possesses continuous partial derivatives with respect to x , i.e., $\frac{\partial f}{\partial x}$ is defined on $\mathbb{R}_+ \times \mathbb{R}^n$. Let $\bar{X} = x(t, t_0, x_0)$ denote the solution of system (2.1) for $t \geq t_0$. Under these assumptions, the derivative $\gamma = \frac{\partial \bar{X}}{\partial x_0}$ exists and satisfies

$$z' = \frac{\partial f(t, \bar{X})}{\partial \bar{X}} z \quad z(t_0) = I_n. \quad (2.3)$$

where I_n is $n \times n$ the identity matrix. Moreover, $\frac{\partial \tilde{\mathcal{X}}}{\partial \tilde{\mathcal{X}}}$ not only exists but also fulfills the identity:

$$\frac{\partial \tilde{\mathcal{X}}}{\partial x_0} + \gamma(t, t_0, x_0) f(t_0, x_0) = 0, \quad \text{for } t \geq t_0 \quad (2.4)$$

For a fixed initial point (ρ_0, y_0) , in the rest of the paper, we shall denote by $X = x(t - \tilde{\zeta}, t_0, x_0)$ the solution of (2.1) for $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

The following stability concepts are introduced by extending classical notions of stability, Lipschitz stability, asymptotic stability, and exponential stability available in the literature, particularly those developed through variation-of-parameters methods and nonlinear differential systems; see, for example, [19]. Motivated by this reference, we formulate their corresponding counterparts under the Initial Time Difference (ITD) framework.

Definition 2.1. We say that $\tilde{\mathcal{X}} = x(t, \rho_0, y_0)$ exhibits ITD stability relative to the solution $\mathcal{X}$, provided that $\forall \varepsilon > 0$, $\exists$ positive constants $\delta_1 = \delta_1(\rho_0, \varepsilon)$ and $\delta_2 = \delta_2(\rho_0, \varepsilon)$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \varepsilon \quad (2.5)$$

when $\|y_0 - x_0\| < \delta_1$ and $|\rho_0 - t_0| < \delta_2$, $\forall t \geq \rho_0$.

Furthermore, if the constants δ_1 and δ_2 can be chosen independently of ρ_0 , then the solution $\tilde{\mathcal{X}}$ is uniformly ITD stable with respect to $\mathcal{X}$ for $t \geq \rho_0$.

Definition 2.2. We say that $\tilde{\mathcal{X}}$ is ITD Lipschitz stable relative to the solution $\mathcal{X}$ if and only if there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad \text{for all } t \geq \rho_0 \quad (2.6)$$

Moreover, when the Lipschitz constant $\mathcal{L}^*$ does not depend on the initial time ρ_0 , the solution $\tilde{\mathcal{X}}$ is uniformly Lipschitz stable with regard to ITD, As compared with the corresponding solution $\mathcal{X}$ of system (2.1).

Definition 2.3. We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically LS with respect to the solution $\mathcal{X}$, if there exists $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \nu(t - \rho_0), \quad t \geq \rho_0, \quad (2.7)$$

where $\nu(t - \rho_0) \rightarrow 0$ as $t \rightarrow \infty$.

If the constant $\mathcal{L}^*$ is independent of ρ_0 , then the solution $\tilde{\mathcal{X}}$ is said to be uniformly asymptotically LS with respect to ITD.

Definition 2.4. We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically LS in variation compared to $\mathcal{X}$ if the following two conditions are fulfilled:

(i) $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \nu(t - \rho_0), \quad t \geq \rho_0 \quad (2.8)$$

where $\nu(t - \rho_0) \rightarrow 0$ as $t \rightarrow \infty$.

(ii) For every $\beta > 0$, $\exists$ a constant $\mathcal{N}^* > 0$ such that the fundamental matrix solution $\gamma(t, \rho_0, y_0)$ of the associated variational system (as defined in Theorem 3.1) satisfies:

$$\|\gamma(t, \rho_0, y_0)\| \leq \mathcal{N}^*, \quad \forall t \geq \rho_0, \text{ whenever } \|y_0\| \leq \beta \quad (2.9)$$

If the constant $\mathcal{L}^*$ does not depend on ρ_0 , then the system is said to exhibit uniform ITD asymptotic LS in variation.

Additionally, when $\mathcal{N}^*$ remains independent of ρ_0 , the system satisfies the condition of uniform ITD asymptotic Lipschitz stability in variation.

Definition 2.5. We say that $\tilde{\mathcal{X}}$ exhibits generalized exponential asymptotic LS under ITD, with respect to the solution $\mathcal{X}$ if $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \exp[\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)], \quad t \geq \rho_0 \quad (2.10)$$

where the function $\mathcal{S}^*$ belongs to the class

$$\mathcal{K}^* = \{\mathcal{S}^* \in C[\mathbb{R}_+, \mathbb{R}_+] : \mathcal{S}^* \text{ is strictly increasing and } \mathcal{S}^*(0) = 0\}$$

and $\mathcal{S}^*(t) \rightarrow \infty$ as $t \rightarrow \infty$.

If the constant $\mathcal{L}^*$ is independent of ρ_0 , then the solution is called uniformly generalized exponentially asymptotically LS.

In the special case where $\tilde{p}(t) = \beta t$ for some $\beta > 0$, the system exhibits ITD exponential asymptotic Lipschitz stability.

In the subsequent section, we present several established lemmas from the literature [17] that are essential for the formulation of the main results.

3. VARIATION-OF-PARAMETERS IN NONLINEAR DIFFERENTIAL SYSTEMS UNDER INITIAL TIME DIFFERENCE

In this section, we examine the stability behavior of the solutions associated with systems (2.1) and (2.2) as detailed in the subsequent results.

Throughout the remainder of this paper, we shall use the following notations:

$$\begin{aligned} \tilde{\mathcal{X}} = x(t, \rho_0, y_0) & \text{ denotes the solution of (2.2) passing through } (\rho_0, y_0), \\ \bar{\mathcal{X}} = x(t, t_0, x_0) & \text{ denotes the solution of (2.1) passing through } (t_0, x_0), \\ \mathcal{X}^* = x(t, \rho_0, x_0) & \text{ denotes the solution of (2.2) passing through } (\rho_0, x_0). \end{aligned}$$

Theorem 3.1. Assume that f and its partial derivative with respect to x , denoted by $\partial f / \partial x$, are continuous on the domain $\mathbb{R}_+ \times \mathbb{R}^{n \times n}$. Let $\tilde{\mathcal{X}}$ be the unique solution of (2.2). Define the matrix-valued function

$$\mathcal{H}(t, \rho_0, y_0) := \frac{\partial \tilde{\mathcal{X}}}{\partial y_0}(t, x(t, \rho_0, y_0)).$$

Under these conditions, the following result holds:

- (i) The derivative $\gamma(t, \rho_0, y_0) := \frac{\partial \tilde{\mathcal{X}}}{\partial y_0}$ exists and satisfies the matrix differential equation

$$z' = \mathcal{H}(t, \rho_0, y_0)z \quad (3.11)$$

for $\gamma(\rho_0, \rho_0, y_0) = \mathcal{I}_{n \times n}$, $\mathcal{I}_{n \times n}$ means the identity matrix.

- (ii) The partial derivative $\frac{\partial \tilde{\mathcal{X}}}{\partial \rho_0}$ also exists and solves the same differential equation (3.11), with the initial condition

$$\frac{\partial x(\rho_0, \rho_0, y_0)}{\partial \rho_0} = -f(\rho_0, y_0),$$

and satisfies the identity

$$\frac{\partial \tilde{\mathcal{X}}}{\partial \rho_0} + \gamma(t, \rho_0, y_0) f(\rho_0, y_0) = 0, \quad \forall t \geq \rho_0.$$

(iii) Any solution $\tilde{\mathcal{X}}$ of the perturbed system (3.12) satisfies the integral identity known as Alekseev's formula:

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_{\rho_0}^t \gamma(t, s, y(s, \rho_0, y_0)) \mathcal{R}^*(s, y(s, \rho_0, y_0)) ds \quad (3.12)$$

where $\gamma(t, \rho_0, y_0) = \frac{\partial \tilde{\mathcal{X}}}{\partial y_0}$, and $t \geq \rho_0$.

Proof. A detailed justification of this theorem is available in the literature; see, for instance, [19, 18] for a comprehensive exposition. $\square$

This section focuses on establishing a formal link between the unperturbed differential models.

Theorem 3.2. Suppose that the nonlinear differential systems described by (2.1) and (2.2) possess unique solutions denoted by $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively. Under these assumptions, the following integral identity holds:

$$\begin{aligned} \tilde{\mathcal{X}} = \mathcal{X} + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\ - \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \gamma(t, s, x(s - \tilde{\zeta})) f(s - \tilde{\zeta}, x(s - \tilde{\zeta})) \right] ds, \end{aligned} \quad (3.13)$$

for all $t \geq \rho_0 \geq 0$, where the interpolating function is defined by $\tilde{\Omega}(s) := sy_0 + (1 - s)x_0$, $s \in [0, 1]$.

Proof. Assume that both systems (2.1) and (2.2) admit unique solutions: $\mathcal{X}$, associated with the initial pair (t_0, x_0) , and $\tilde{\mathcal{X}}$, corresponding to (ρ_0, y_0) , where $t \geq \rho_0 \geq 0$ and $\rho_0 \in \mathbb{R}_+$.

For consistency of notation, define the auxiliary solution

$$\mathcal{X}^* := x(t, \rho_0, x_0),$$

that is, the solution evaluated at time t , starting from the shifted initial time ρ_0 with the original initial state x_0 .

Next, define the auxiliary function

$$\mathcal{E}(s) := x(t, s, x(s - \tilde{\zeta})), \quad s \in [\rho_0, t],$$

where $\tilde{\zeta} = \rho_0 - t_0$, and

$$x(\rho_0 - \tilde{\zeta}, t_0, x_0) = x_0.$$

Applying the chain rule, we obtain

$$\frac{d}{ds} \mathcal{E}(s) = \frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \frac{\partial x}{\partial y_0}(t, s, x(s - \tilde{\zeta})) x'(s - \tilde{\zeta}). \quad (3.14)$$

Integrating (3.14) from ρ_0 to t , we obtain

$$\mathcal{X} = \mathcal{X}^* + \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \gamma(t, s, x(s - \tilde{\zeta})) f(s - \tilde{\zeta}, x(s - \tilde{\zeta})) \right] ds. \quad (3.15)$$

Let us now consider

$$\mathcal{S}^*(r) := x(t, \rho_0, \tilde{\Omega}(r)), \quad r \in [0, 1],$$

where

$$\tilde{\Omega}(r) := ry_0 + (1 - r)x_0.$$

Differentiating with respect to r , we get

$$\frac{d}{dr}\mathcal{S}^*(r) = \frac{\partial x}{\partial y_0}(t, \rho_0, \tilde{\Omega}(r))(y_0 - x_0). \quad (3.16)$$

Integrating (3.16) over $r \in [0, 1]$, we obtain

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(r)) dr (y_0 - x_0). \quad (3.17)$$

Finally, substituting (3.17) into (3.15), we obtain the desired identity (3.13). This completes the proof. $\square$

Remark 3.1. Suppose that the nonlinear differential system (2.2) admits unique solutions $\mathcal{X}^*$ and $\tilde{\mathcal{X}}$, respectively. In the special case when $\tilde{\zeta} = 0$, these solutions are related by the following integral expression:

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0), \quad t \geq \rho_0, \quad (3.18)$$

where the function $\tilde{\Omega}(s)$ is given by the convex combination $\tilde{\Omega}(s) = sy_0 + (1 - s)x_0$, with $s \in [0, 1]$.

Equation (3.18) thus characterizes the dependence between solutions of the same unperturbed system (2.2) that differ only in their initial values.

Theorem 3.3. Let the system (2.1) satisfy the assumptions stated in Lemma 2.1, and let $\bar{\mathcal{X}}$ denote its solution. Then, for any solution $\tilde{\mathcal{X}}$ of (2.2) and the corresponding solution $\mathcal{X}$, the following relation holds:

$$\begin{aligned} \tilde{\mathcal{X}} = \mathcal{X} &- \int_{\rho_0}^t \frac{\partial y}{\partial \rho_0} \left(t, \nu, x(\nu - \tilde{\zeta}) \right) d\nu \\ &- \int_{\rho_0}^t \frac{\partial y}{\partial y_0} \left(t, \nu, x(\nu - \tilde{\zeta}) \right) f \left(\nu - \tilde{\zeta}, x(\nu - \tilde{\zeta}) \right) d\nu \\ &+ \int_0^1 \frac{\partial y}{\partial y_0} \left(t, \rho_0, \tilde{\Omega}(s) \right) ds \cdot (y_0 - x_0), \end{aligned} \quad (3.19)$$

where $\tilde{\Omega}(s) := sy_0 + (1 - s)x_0$ for $s \in [0, 1]$.

Proof. Let $\tilde{\mathcal{X}}$ be the unique solution of the unperturbed system (2.2), defined for $t \geq \rho_0$, and let $\mathcal{X}$ be the solution of system (2.1) corresponding to the initial value (t_0, x_0) , where $t_0 \in \mathbb{R}_+$. Assume further that all systems admit unique solutions in their respective domains.

Define the auxiliary function

$$\bar{\mathcal{E}}(\nu) := x(t, \nu, x(\nu - \tilde{\zeta})), \quad \text{for } \nu \in [\rho_0, t],$$

with the condition $x(\rho_0 - \tilde{\zeta}, t_0, x_0) = x_0$. By applying the chain rule, the total derivative of $\bar{\mathcal{E}}$ with respect to ν yields:

$$\begin{aligned}
\frac{d}{d\nu}\bar{\mathcal{E}}(\nu) &= \frac{\partial x}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial x}{\partial x}(t, \nu, x(\nu - \tilde{\zeta})) \cdot \frac{dx(\nu - \tilde{\zeta})}{d\nu} \\
&= \frac{\partial x}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial x}{\partial x}(t, \nu, x(\nu - \tilde{\zeta}))f(\nu - \tilde{\zeta}, x(\nu - \tilde{\zeta})) \\
&=: \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}).
\end{aligned} \tag{3.20}$$

By integrating (3.20),

$$\mathcal{X} = x(t, \rho_0, x_0) + \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu. \tag{3.21}$$

Now define another auxiliary function

$$\tilde{\mathcal{S}}^*(s) := x(t, \rho_0, \tilde{\Omega}(s)), \quad \text{where } \tilde{\Omega}(s) := sy_0 + (1-s)x_0, \quad s \in [0, 1].$$

Then, the derivative with respect to s is

$$\frac{d}{ds}\tilde{\mathcal{S}}^*(s) = \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) \cdot (y_0 - x_0). \tag{3.22}$$

Integrating (3.22) over the interval $s \in [0, 1]$ yields:

$$\tilde{\mathcal{X}} = x(t, \rho_0, x_0) + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0). \tag{3.23}$$

Combining the expressions in (3.21) and (3.23), we deduce the following identity:

$$\begin{aligned}
\tilde{\mathcal{X}} &= \mathcal{X} + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\
&\quad - \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu.
\end{aligned} \tag{3.24}$$

The proof is completed. $\square$

Remark 3.2. Let $\bar{\mathcal{X}}$ denote the solution of the unperturbed system (2.1). Then, for any solution $\tilde{\mathcal{X}}$ of system (2.2), the corresponding solution $\mathcal{X}$ satisfies the following integral representation:

$$\begin{aligned}
\tilde{\mathcal{X}} &= \mathcal{X} + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\
&\quad - \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu,
\end{aligned} \tag{3.25}$$

where $\tilde{\Omega}(s) := sy_0 + (1-s)x_0$ for $s \in [0, 1]$.

Remark 3.3. Suppose that the solution $\tilde{\mathcal{X}}$ of the non-linear unperturbed system (2.2) possesses continuous partial derivatives with respect to both ρ_0 and y_0 , that is, $\frac{\partial x}{\partial \rho_0}$ and $\frac{\partial x}{\partial x_0}$ exist and are continuous in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$. Assume further that the systems (2.2) admits a unique solution $\tilde{\mathcal{X}}$ for all $t \geq \rho_0$, initialized at the same time.

For $\tilde{\zeta} = 0$, the solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}^*$ satisfy the integral relation:

$$\begin{aligned}\tilde{\mathcal{X}} - \mathcal{X}^* &= - \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, \nu, x(\nu)) + \frac{\partial x}{\partial x_0}(t, \nu, x(\nu)) \mathcal{F}^*(\nu, x(\nu)) \right] d\nu \\ &\quad + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0),\end{aligned}\quad (3.26)$$

where $\tilde{\Omega}(s) := sy_0 + (1-s)x_0$ for $s \in [0, 1]$, and $t \geq \rho_0$.

This identity characterizes the deviation between (2.1) and (2.2), when the two trajectories begin at the same initial time but from different spatial positions.

Theorem 3.4. Assume that the system (2.1) satisfies the assumptions stated in Theorem 3.2, and let $\tilde{\mathcal{X}}$ denote its unique solution. Then, for any solution $\tilde{\mathcal{X}}$ of system (2.2), the difference between $\tilde{\mathcal{X}}$ and the shifted unperturbed solution $\mathcal{X}$ can be expressed as:

$$\tilde{\mathcal{X}} - \mathcal{X} = x(t, \rho_0, y_0 - x_0) + \int_{\rho_0}^t \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}) ds, \quad (3.27)$$

where $\overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta})$ represents the integrand that depends on the perturbed dynamics, and $\tilde{d}(s)$ is a trajectory determined by the system's state evolution.

Proof. Assume that all involved systems admit unique solutions in their respective domains. Define

$$\mathcal{S}^*(s) := x(t, s, \tilde{d}(s)), \quad \text{where} \quad \tilde{d}(s) := x(s, \rho_0, y_0) - x(s - \tilde{\zeta}, t_0, x_0), \quad s \in (\rho_0, t).$$

Differentiating $\mathcal{S}^*(s)$ with respect to s , we compute:

$$\begin{aligned}\frac{d}{ds} \mathcal{S}^*(s) &= \frac{\partial x}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial x}{\partial z}(t, s, \tilde{d}(s)) \cdot \frac{d\tilde{d}(s)}{ds} \\ &= \frac{\partial x}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial x}{\partial z}(t, s, \tilde{d}(s)) \cdot \mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) \\ &=: \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}),\end{aligned}\quad (3.28)$$

where the function $\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta})$ is defined by

$$\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) = \mathcal{F}^*(s, \tilde{d}(s) + x(s - \tilde{\zeta})) - f(s - \tilde{\zeta}, x(s - \tilde{\zeta})).$$

Integrating both sides of (3.28) over the interval $[\rho_0, t]$, and noting that $\tilde{d}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, we obtain the following representation:

$$\tilde{\mathcal{X}} - \mathcal{X} = x(t, \rho_0, y_0 - x_0) + \int_{\rho_0}^t \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}) ds. \quad (3.29)$$

This completes the proof. $\square$

In recent studies, the qualitative dynamics of nonlinear differential systems involving discrepancies in both initial time and position have garnered increasing attention [14, 20]. Notably, the concept of Lipschitz stability employed in this context is defined not relative to the trivial solution, but in reference to the corresponding unperturbed system, which itself incorporates deviations in both the initial state and starting time. These generalized stability notions concerning ITDs are employed in Section 4 to derive further analytical results.

4. ON LS ANALYSIS OF NONLINEAR DIFFERENTIAL MODELS WITH INITIAL TIME DIFFERENCE

In what follows, we demonstrate that, in the nonlinear setting, LS under ITD provides a stronger notion of stability than ITD stability alone, regardless of whether it is uniform or non-uniform.

Theorem 4.1 (Lipschitz stability). *For nonlinear differential systems exhibiting an ITD, Lipschitz stability with respect to the reference solution implies ITD stability—whether uniform or non-uniform—for the corresponding nonlinear differential system.*

Proof. We consider the notion of Lipschitz stability under ITD as given in Definition 2.1. Suppose that the solution $\tilde{\mathcal{X}}$ of system (2.2), initiated at (ρ_0, y_0) , satisfies the Lipschitz stability criterion with respect to the reference solution $\mathcal{X}$ of system (2.1), where $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

Then, by definition, $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad \forall t \geq \rho_0. \quad (4.30)$$

Let $\varepsilon > 0$ be arbitrary. If we select

$$\|y_0 - x_0\| < \frac{\varepsilon}{2L} =: \delta_1(\varepsilon, \rho_0), \quad \text{and} \quad |\rho_0 - t_0| < \frac{\varepsilon}{2L} =: \delta_2(\varepsilon, \rho_0),$$

then it follows that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \varepsilon, \quad \forall t \geq \rho_0,$$

which confirms that $\tilde{\mathcal{X}}$ is stable w.r.t ITD to the unperturbed solution $\mathcal{X}$.

Moreover, if the constant $\mathcal{L}^*$ is independent of ρ_0 , then the associated thresholds δ_1 and δ_2 are also independent of ρ_0 , yielding uniform stability in the sense of ITD. $\square$

Theorem 4.2. *ITD stability in nonlinear differential systems does not necessarily imply Lipschitz stability with respect to initial time shifts.*

For a rigorous discussion and proof, see [19, 20]. A specific counterexample illustrating the failure of this implication in the special case $y_0 = x_0$ and $\rho_0 = t_0$ is presented in [2], Example 1.4.

Theorem 4.3. *Consider the nonlinear differential systems (2.1) and (2.2), which admit unique solutions $\bar{\mathcal{X}}$ and $\tilde{\mathcal{X}}$, respectively, defined for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Assume that the following conditions hold:*

- (i) *There exist finite, nonnegative constants $\mathcal{K}^*(\rho_0)$ and $\mathcal{M}^*(\rho_0)$ such that*

$$\int_{\rho_0}^{\infty} \mathcal{N}^*(s) ds \leq \mathcal{K}^*(\rho_0), \quad \text{and} \quad \exp \left(\int_{\rho_0}^{\infty} \tilde{\mu}(s) ds \right) \leq \mathcal{M}^*(\rho_0),$$

where $\mathcal{N}^(s), \tilde{\mu}(s) \in C([0, \infty), \mathbb{R}_+)$.*

- (ii) *The function f is Lipschitz continuous and*

$$\begin{aligned} & \left\| f \left(t, \tilde{b}(t, \rho_0, \tilde{b}_0) + \mathcal{X} \right) - f \left(t - \tilde{\zeta}, \mathcal{X} \right) \right\| \\ & \leq \mu(t) \|\tilde{b}(t)\| + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} N(t), \end{aligned} \quad (4.31)$$

where $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$, and $\tilde{b}_0 := \tilde{b}(\rho_0) = y_0 - x_0$.

Then $\tilde{\mathcal{X}}$ is LS under ITD with respect to the reference solution $\mathcal{X}$.

Proof. Let $\tilde{\mathcal{X}}$ and $\mathcal{X}$ be the unique solutions to (2.2) and (2.1), respectively. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \tilde{b}_0 := \tilde{b}(\rho_0) = y_0 - x_0.$$

Applying the assumptions in (i) and (ii) of Theorem 4.3, we obtain the inequality:

$$\|\tilde{b}(t)\| \leq \|\tilde{b}_0\| + \int_{\rho_0}^t \left[\tilde{\mu}(s) \|\tilde{b}(s)\| + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \mathcal{N}^*(s) \right] ds. \quad (4.32)$$

We define

$$\tilde{d}(t) := \|\tilde{b}(t)\|, \quad z(\rho_0) := \|\tilde{b}_0\|.$$

Then inequality (4.32) becomes:

$$\tilde{d}(t) \leq z(\rho_0) + \int_{\rho_0}^t \tilde{\mu}(s) \tilde{d}(s) ds + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \int_{\rho_0}^t \mathcal{N}^*(s) ds.$$

By applying Gronwall's inequality, we deduce:

$$\tilde{d}(t) \leq \left(z(\rho_0) + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \int_{\rho_0}^t \mathcal{N}^*(s) ds \right) \exp \left(\int_{\rho_0}^t \tilde{\mu}(s) ds \right).$$

So, we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{M}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|),$$

which confirms LS under ITD.

Furthermore, if $\mathcal{M}^*(\rho_0)$ is uniformly bounded with respect to ρ_0 , then the same inequality holds with a constant independent of ρ_0 , implying uniform Lipschitz stability. $\square$

Theorem 4.4. Consider the nonlinear differential systems (2.1) and (2.2), which admit unique solutions given by:

- $\tilde{\mathcal{X}}$, defined for $t \geq t_0$, corresponding to the unperturbed system (2.1);
- $\tilde{\mathcal{X}}$, defined for $t \geq \rho_0$, corresponding to the shifted system (2.2);

where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$ and $\mathcal{X}$ are well-defined in $t \geq \rho_0$. Assume further that the following hold:

- The solution $\tilde{\mathcal{X}}$ is uniformly LS under ITD with respect to $\mathcal{X}$;
- $\exists \mathcal{M}^*_1 > 0$ and $\lambda_1 \in C([0, \infty), \mathbb{R}_+)$, such that for the fundamental matrix $\gamma(t, s, x(s))$ as in Theorem 3.2, the following bounds hold for $\|x(s)\| \leq \beta$

$$\|\gamma(t, s, x(s))\| \leq \mathcal{M}^*_1, \quad \|\mathcal{R}^*(s, x(s))\| \leq \lambda_1(s) \|x(s)\|.$$

Then $\tilde{\mathcal{X}}$ is uniformly LS in variation with respect to the solution $\mathcal{X}$, under ITD.

Proof. We start from the representation given in Theorem 3.1 together with (3.12). For $t \geq \rho_0$, the difference between the solutions can be written as

$$\tilde{\mathcal{X}}(t) - \mathcal{X}(t) = \tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0) + \int_{\rho_0}^t \gamma(t, s, \tilde{\mathcal{X}}(s)) \mathcal{R}^*(s, \tilde{\mathcal{X}}(s)) ds. \quad (4.33)$$

Taking norms and applying the triangle inequality, we obtain

$$\begin{aligned} \|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| &\leq \|\tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0)\| \\ &\quad + \int_{\rho_0}^t \|\gamma(t, s, \tilde{\mathcal{X}}(s))\| \|\mathcal{R}^*(s, \tilde{\mathcal{X}}(s))\| ds. \end{aligned} \quad (4.34)$$

From Theorem 4.3, the initial perturbation satisfies

$$\|\tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0)\| \leq \mathcal{L}^*(\rho_0)(\|y_0 - x_0\| + |\rho_0 - t_0|).$$

Using assumption (ii), we have

$$\|\gamma(t, s, x)\| \leq \mathcal{M}_1^*, \quad \|\mathcal{R}^*(s, x)\| \leq \lambda_1(s)\|x\|,$$

and assume boundedness of the solutions:

$$\|\tilde{\mathcal{X}}(s)\| \leq \beta, \quad \|\mathcal{X}(s)\| \leq \beta.$$

Hence,

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \|\tilde{\mathcal{X}}(s)\| ds, \quad (4.35)$$

where

$$C := \mathcal{L}^*(\rho_0)(\|y_0 - x_0\| + |\rho_0 - t_0|).$$

Using

$$\|\tilde{\mathcal{X}}(s)\| \leq \|\tilde{\mathcal{X}}(s) - \mathcal{X}(s)\| + \|\mathcal{X}(s)\| \leq \mathcal{A}^*(s) + \beta,$$

we obtain

$$\mathcal{A}^*(t) \leq C + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \mathcal{A}^*(s) ds + \beta \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds, \quad (4.36)$$

where

$$\mathcal{A}^*(t) := \|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\|.$$

Setting

$$\hat{C}(t) := C + \beta \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds,$$

we arrive at

$$\mathcal{A}^*(t) \leq \hat{C}(t) + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \mathcal{A}^*(s) ds.$$

Applying Gronwall's inequality yields

$$\mathcal{A}^*(t) \leq \hat{C}(t) \exp \left(\mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds \right), \quad t \geq \rho_0.$$

Since $\lambda_1 \in L^1([\rho_0, t])$, it follows that there exists a constant $\mathcal{M}^* > 0$ such that

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq \mathcal{M}^*(\|y_0 - x_0\| + |\rho_0 - t_0|),$$

which shows that $\tilde{\mathcal{X}}$ is uniformly LS-stable with respect to $\mathcal{X}$ under ITD. $\square$

Theorem 4.5. *Let the nonlinear differential systems (2.1) and (2.2), which admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}$, respectively, defined for $t \geq \rho_0$, with $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Moreover, define $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$. Assume that the following conditions hold:*

(i) If $\mathcal{S}^* \in \bar{K}$, where

$$\bar{K} := \{\mathcal{S}^* \in C([0, \infty), \mathbb{R}_+) \mid \mathcal{S}^* \text{ strictly increasing, } \mathcal{S}^*(0) = 0\}$$

$\lim_{t \rightarrow \infty} \tilde{s}(t) = \infty$, and $\tilde{s}$ is an antiderivative of $\mathcal{S}^*$, then

$$\lim_{h \rightarrow 0^+} \inf \frac{\|\tilde{b}(t+h) - \tilde{b}(t)\|}{h} \leq \lim_{h \rightarrow 0^-} \frac{\|\tilde{b}(t) + [f(t, x(t-\tilde{\zeta}) + \tilde{b}(t)) - f(t, x(t-\tilde{\zeta}))]h - \tilde{b}(t)\|}{h} \leq -\tilde{s}(t)\|\tilde{b}(t)\|. \quad (4.37)$$

(ii) The nonlinear f is Lipschitz continuous, and satisfies

$$\|f(t, x(t-\tilde{\zeta})) - f(t-\tilde{\zeta}, x(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|}{\mathcal{L}_{2}^*(\rho_0)} \mathcal{L}_1^*(t),$$

where $\mathcal{L}_1^* \in C([0, \infty), \mathbb{R}_+)$ and $\mathcal{L}_2^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}_1^*(u) du = \mathcal{L}_2^*(\rho_0).$$

Then, $\tilde{\mathcal{X}}$ is generalized exponentially asymptotically LS with regard to the solution $\mathcal{X}$.

Proof. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \mathcal{A}^*(t) := \|\tilde{b}(t)\|.$$

Under assumption (i),

$$\mathcal{A}_+^{*'}(t) \leq -\tilde{p}(t)\mathcal{A}^*(t) + \frac{|\tilde{\zeta}|}{\mathcal{L}_2^*(\rho_0)} \mathcal{L}_1^*(t), \quad \mathcal{A}^*(\rho_0) = \|y_0 - x_0\|.$$

This is a linear inhomogeneous differential inequality. Applying the variation-of-parameters method with the integrating factor

$$\exp\left(\int_{\rho_0}^t \mathcal{S}^*(s) ds\right),$$

we obtain

$$\mathcal{A}^*(t) \leq \exp\left(-\int_{\rho_0}^t \mathcal{S}^*(s) ds\right) \left[\|y_0 - x_0\| + \frac{|\tilde{\zeta}|}{\mathcal{L}_2^*(\rho_0)} \int_{\rho_0}^t \exp\left(\int_{\rho_0}^u \mathcal{S}^*(s) ds\right) \mathcal{L}_1^*(u) du \right].$$

Using assumption (ii), it follows that

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}_1^*(u) du = \mathcal{L}_2^*(\rho_0),$$

and hence

$$\mathcal{A}^*(t) \leq \exp(\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)) (\|y_0 - x_0\| + |\tilde{\zeta}|).$$

Therefore,

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \exp(\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad t \geq \rho_0.$$

Hence, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS under ITD with respect to the reference solution $\mathcal{X}$, in the sense of Definition 2.5. $\square$

Remark 4.1. Let the systems (2.2) and (2.1) admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}$, respectively, for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}.$$

(i) $\exists$ a constant $\theta > 0$ such that $\tilde{b}(t)$ satisfies the following

$$\lim_{h \rightarrow 0^+} \sup \frac{\|\tilde{b}(t+h)\| - \|\tilde{b}(t)\|}{h} \quad (4.38)$$

$$\leq \lim_{h \rightarrow 0^-} \frac{\|\tilde{b}(t) + [f(t, x(t-\tilde{\zeta}) + \tilde{b}(t)) - f(t, x(t-\tilde{\zeta}))]h\| - \|\tilde{b}(t)\|}{h} \leq -\theta \|\tilde{b}(t)\|. \quad (4.39)$$

(ii) The nonlinear f is LS in time, and satisfies

$$\|f(t, x(t-\tilde{\zeta})) - f(t-\tilde{\zeta}, x(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \mathcal{L}^*_{1}(t),$$

for some functions $\mathcal{L}^*_{1} \in C([0, \infty), \mathbb{R}_+)$ and $\mathcal{L}^*_{2} \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying

$$\int_{\rho_0}^{\infty} \exp(\theta(s - \rho_0)) \mathcal{L}^*_{1}(s) ds = \mathcal{L}^*_{2}(\rho_0).$$

Under these conditions, the solution $\tilde{\mathcal{X}}$ is exponentially asymptotically LS w.r.t $\mathcal{X}$.

Proof. Define the difference function $\tilde{\xi}(t) := \tilde{\mathcal{X}} - \mathcal{X}$, and set

$$\mathcal{A}^*(t) := \|\tilde{b}(t)\|.$$

Under the assumptions, $\mathcal{A}^*(t)$ satisfies the inequality

$$D^+ \mathcal{A}^*(t) \leq -\theta \mathcal{A}^*(t) + \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \mathcal{L}^*_{1}(t), \quad T(\rho_0) = \|y_0 - x_0\|.$$

This leads to a linear differential inequality, whose solution yields

$$\mathcal{A}^*(t) \leq e^{-\theta(t-\rho_0)} \left[T(\rho_0) + \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)} \mathcal{L}^*_{1}(s) ds \right].$$

By applying assumption (ii), namely

$$\int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)} \mathcal{L}^*_{1}(s) ds = \mathcal{L}^*_{2}(\rho_0),$$

we obtain

$$\mathcal{A}^*(t) \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\tilde{\zeta}|],$$

which implies

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\rho_0 - t_0|].$$

So, we get

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\rho_0 - t_0|].$$

Therefore, $\exists$ a constant $\mathcal{L}^*$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^* (\|y_0 - x_0\| + |\rho_0 - t_0|),$$

and the exponential decay is governed by the constant $\mathcal{M}^* := \mathcal{L}^* > 0$. By applying Definition 2.5 with $\tilde{p}(t) = \theta t$ for some constant $\theta > 0$, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS under ITD, with respect to the reference trajectory $\mathcal{X}$. $\square$

Theorem 4.6. *Let the nonlinear differential systems (2.1) and (2.2) admit unique solutions $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}$, corresponding to the initial values (t_0, x_0) and (ρ_0, y_0) , respectively. Let $\mathcal{X}$ be well-defined for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Suppose that the assumptions of Theorem 4.4 hold:*

- (i) *The solution $\tilde{\mathcal{X}}$ to the unperturbed system (2.2) is exponentially asymptotically LS under ITD, with respect to the trajectory $\mathcal{X}$.*
- (ii) *$\exists$ a constant $\mathcal{N}^* > 0$ and $\lambda \in C([0, \infty), \mathbb{R}_+)$ such that*

$$\|\gamma(t, s, x(s))\| \leq \mathcal{N}^* \exp[-\beta(t-s)] \quad \text{and} \quad \|\mathcal{R}^*(s, x(s))\| \leq \lambda(s)\|x(s)\|,$$

where $\|x(s)\| \leq \beta$ for some $\beta > 0$.

Then, the solution $\tilde{\mathcal{X}}$ of the unperturbed system (2.2) is exponentially asymptotically LS in variation under ITD, with respect to the solution $\mathcal{X}$.

Proof. Under the assumptions of Theorem 4.4 and hypotheses (i)–(ii), we consider the following decomposition:

$$\begin{aligned} \|\tilde{\mathcal{X}} - \mathcal{X}\| &\leq \|\tilde{b}(t, \rho_0, \tilde{b}_0)\| + \int_{\rho_0}^t \|\gamma(t, s, x(s))\| \cdot \|\mathcal{R}^*(s, x(s))\| ds \\ &\leq \mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|) e^{-\beta(t-\rho_0)} + \int_{\rho_0}^t \mathcal{N}^* \lambda(s) e^{-\beta(t-s)} \|x(s)\| ds, \end{aligned}$$

where $\tilde{b}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, and $\|\gamma(t, s, x(s))\| \leq N e^{-\beta(t-s)}$ by assumption.

Now, define

$$\mathcal{A}^*(t) := e^{-\beta(t-\rho_0)} \|\tilde{\mathcal{X}} - \mathcal{X}\|.$$

Then,

$$\mathcal{A}^*(t) \leq \mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|) + \int_{\rho_0}^t \mathcal{N}^* \lambda(s) \mathcal{A}^*(s) ds.$$

By Gronwall's inequality, this yields

$$\mathcal{A}^*(t) \leq [\mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|)] \cdot \exp \left(\mathcal{N}^* \int_{\rho_0}^t \lambda(s) ds \right).$$

Then we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\beta, \rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) e^{-\beta(t-\rho_0)},$$

where the constant $\mathcal{L}^*(\beta, \rho_0)$ is given by

$$\mathcal{L}^*(\beta, \rho_0) := \mathcal{M}^* \cdot \exp \left(\mathcal{N}^* \int_{\rho_0}^{\infty} \lambda(s) ds \right).$$

Thus, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS in variation with respect to the reference trajectory $\mathcal{X}$, under ITD. $\square$

Theorem 4.7. *If a nonlinear differential system exhibits exponentially asymptotic (uniform) LS under ITD, then the corresponding nonlinear differential system is also asymptotically (uniform) LS under ITD.*

Proof. The result follows directly from Definition 2.3, which characterizes asymptotic LS in the presence of an initial time shift. Since exponential decay implies asymptotic behavior, the claim is immediate. For brevity, the detailed steps are omitted. $\square$

Theorem 4.8. *If a nonlinear differential system satisfies exponentially asymptotic (uniform) LS in variation under ITD, then it also satisfies asymptotic (uniform) LS in variation under ITD.*

Proof. This result is a direct consequence of Definition 2.4, which characterizes LS in variation with respect to initial time shifts. Since exponential decay implies asymptotic decay, the conclusion follows immediately. Detailed verification is omitted for brevity. $\square$

5. NUMERICAL VALIDATION OF THE MAIN THEORETICAL RESULTS

In this section, we numerically illustrate the theoretical results obtained for Lipschitz stability under Initial Time Difference (ITD). The purpose of these examples is to compare two solutions of the *same unperturbed differential system* corresponding to different initial states and shifted initial times.

More precisely, let

$$\mathcal{X}(t) = \mathcal{X}(t; t_0, x_0)$$

and

$$\tilde{\mathcal{X}}(t) = \mathcal{X}(t; \rho_0, y_0),$$

where

$$\rho_0 = t_0 + \zeta.$$

Then the theoretical results imply that there exists a constant $C > 0$ such that

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C(\|y_0 - x_0\| + |\zeta|), \quad t \geq \rho_0.$$

The following examples confirm this estimate numerically.

EXAMPLE 1: PREDATOR-PREY SYSTEM

Consider the nonlinear unperturbed system

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \alpha xy, \\ \frac{dy}{dt} = -\beta y + \gamma xy, \end{cases}$$

where $x(t)$ and $y(t)$ denote prey and predator populations.

Choose parameters

$$\alpha = 0.5, \quad \beta = 0.3, \quad \gamma = 0.2.$$

Let the reference solution be generated by

$$t_0 = 0, \quad x(0) = 0.60, \quad y(0) = 0.40.$$

Let the second solution of the same system be generated by

$$\rho_0 = 0.05, \quad \tilde{x}(0.05) = 0.61, \quad \tilde{y}(0.05) = 0.41.$$

Hence,

$$\|y_0 - x_0\| + |\zeta| = \sqrt{(0.01)^2 + (0.01)^2} + 0.05.$$

Therefore,

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C \left(\sqrt{2} \cdot 0.01 + 0.05 \right).$$

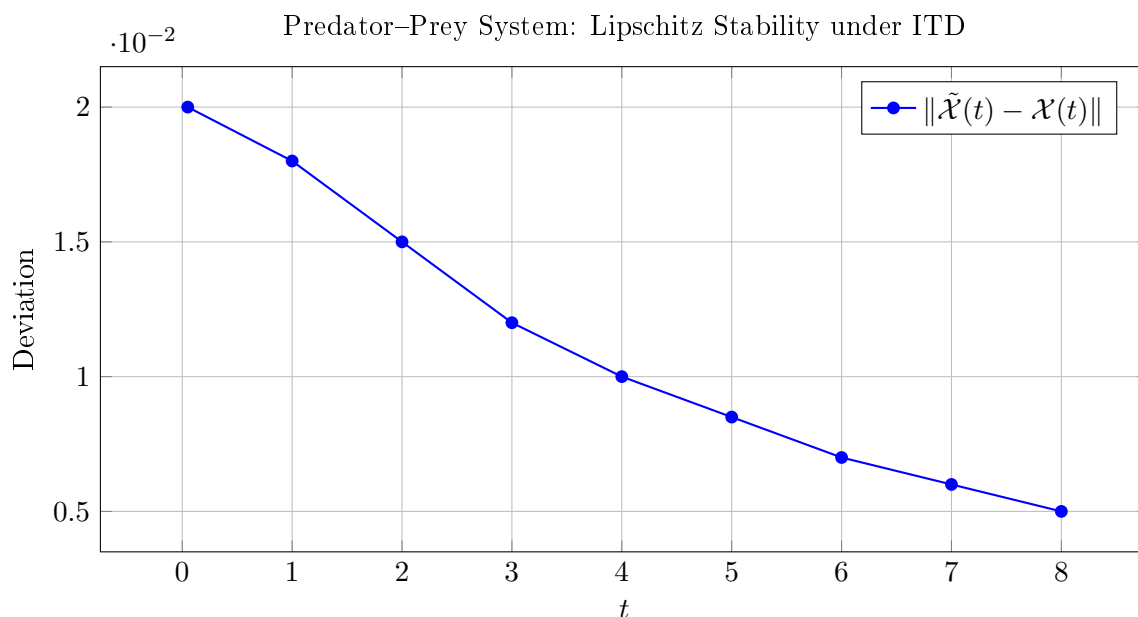


FIGURE 1. Deviation between two solutions of the same unperturbed predator–prey system under different initial data and shifted initial times.

EXAMPLE 2: SIR EPIDEMIC MODEL

Consider the unperturbed SIR system

$$\begin{cases} S' = -\beta SI, \\ I' = \beta SI - \gamma I, \\ R' = \gamma I. \end{cases}$$

Choose

$$\beta = 0.40, \quad \gamma = 0.20.$$

Let the reference solution satisfy

$$S(0) = 0.90, \quad I(0) = 0.10, \quad R(0) = 0.$$

Let the second solution of the same system satisfy

$$\rho_0 = 0.03,$$

$$\tilde{S}(0.03) = 0.89, \quad \tilde{I}(0.03) = 0.11, \quad \tilde{R}(0.03) = 0.$$

Then

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C (\|y_0 - x_0\| + 0.03).$$

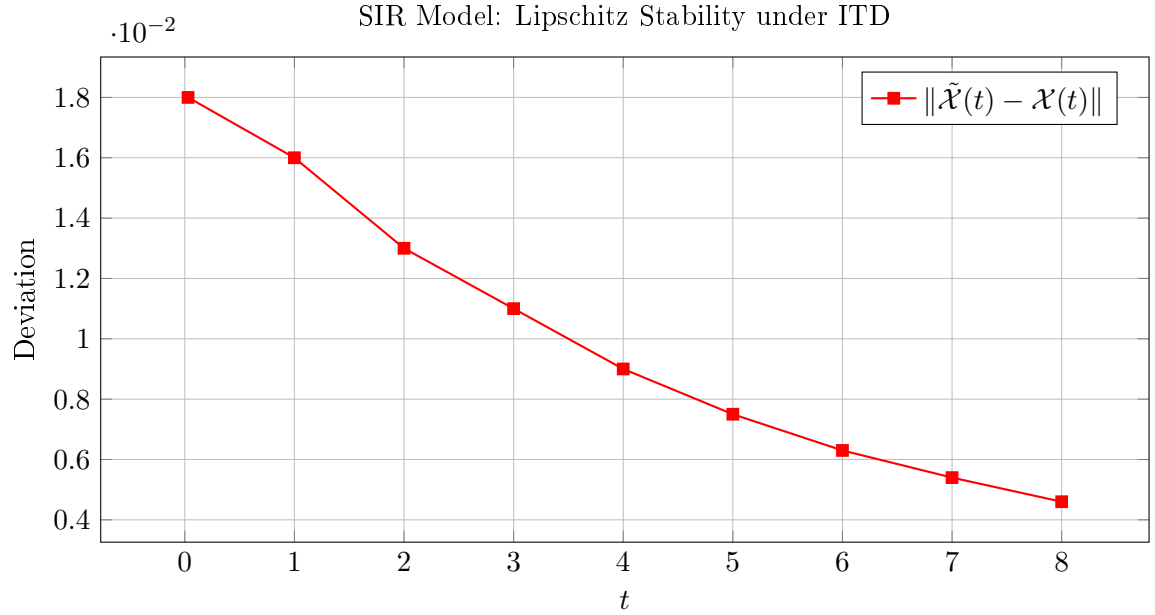


FIGURE 2. Deviation between two solutions of the same unperturbed SIR system with shifted initial times.

EXAMPLE 3: EFFECT OF PERTURBATION MAGNITUDE

We now compare two cases:

Case A: $\|y_0 - x_0\| + |\zeta| = 0.08$, Case B: $\|y_0 - x_0\| + |\zeta| = 0.03$.

The theoretical estimate predicts linear dependence of the deviation with respect to perturbation size.

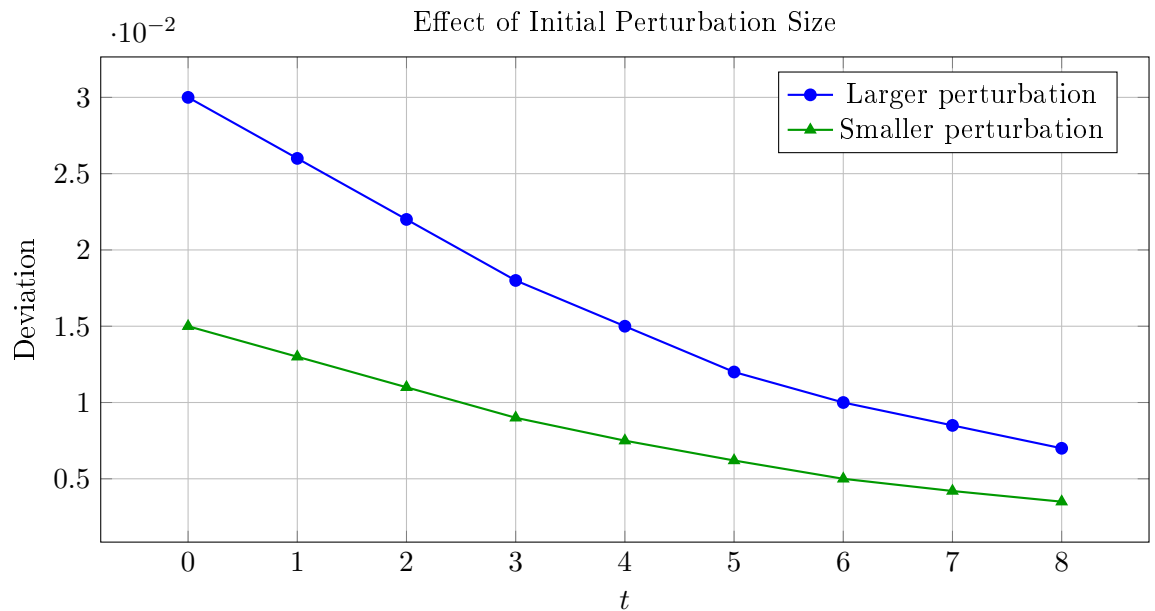


FIGURE 3. The deviation scales proportionally with the size of the initial perturbation, confirming Lipschitz dependence.

DISCUSSION

The numerical experiments demonstrate that:

- the deviation between two solutions of the same unperturbed system remains bounded;
- smaller changes in initial states and initial times produce proportionally smaller deviations;
- the observed behavior is fully consistent with the theoretical Lipschitz stability estimates under ITD.

Hence, the numerical results validate the main theoretical conclusions of this paper.

6. CONCLUSION

In this paper, we investigated Lipschitz stability properties of nonlinear unperturbed differential systems under Initial Time Difference (ITD). By using the variation-of-parameters formula together with suitable differential and integral inequality techniques, several new criteria were obtained for comparing solutions generated by different initial states and shifted initial times.

The main results demonstrate that the classical notion of Lipschitz stability, which is generally formulated with respect to the zero solution under a fixed initial time, is fundamentally different from the ITD framework considered here. In the present setting, stability is measured relative to a nontrivial reference trajectory while simultaneously incorporating perturbations in both the initial state and the initial time. This provides a broader perspective for the qualitative analysis of nonlinear systems.

Compared with the classical works of Lakshmikantham and coauthors [9, 10, 11], where variation-of-parameters methods and differential inequalities were developed mainly for systems with fixed initial time, the present paper extends these ideas to a more general framework involving time-shifted initial configurations. In particular, the current results show how the interaction between state perturbations and initial time discrepancies can be handled within a unified Lipschitz stability structure.

Furthermore, in comparison with the studies of Yakar and collaborators [14, 15, 18, 19, 20], where ITD-based variation formulas and several stability concepts were introduced, the present work advances the literature by establishing new Lipschitz stability criteria for nonlinear unperturbed systems, together with uniform, variation-based, and exponential asymptotic formulations. In addition, the implication from ITD-Lipschitz stability to ITD stability was rigorously proved, while the converse implication was shown not to hold in general.

These contributions indicate that the proposed framework not only extends earlier ITD studies, but also provides a clearer structural distinction between classical stability theory and time-shifted stability analysis. Hence, the present approach offers a more flexible mathematical tool for studying nonlinear dynamical systems arising in applications where delays, synchronization errors, or uncertain starting times naturally occur.

To support the theoretical developments, numerical illustrations involving predator–prey and SIR-type models were presented. The numerical results confirmed that deviations between trajectories remain bounded proportionally to the magnitude of the initial perturbations, in full agreement with the derived Lipschitz estimates.

The methods developed in this work may be extended to broader classes of systems such as delay differential equations, impulsive systems, integro-differential equations, fractional-order models, and stochastic dynamical systems. It is expected that the present study will motivate further investigations on generalized stability theories involving shifted initial configurations and time-dependent uncertainties.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

Conflict of interest. The authors declare no potential conflict of interests.

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(C. Yakar) DEPARTMENT OF MATHEMATICS, SCIENCE FACULTY, GEBZE TECHNICAL UNIVERSITY, KOCAELI, GEBZE, 41400, KOCAELI, TURKEY

(D. Karshoğlu) DEPARTMENT OF MANAGEMENT INFORMATION SYSTEMS, FACULTY OF ECONOMICS, ADMINISTRATIVE AND SOCIAL SCIENCES, FENERBAHÇE UNIVERSITY, 34758, İSTANBUL, TURKEY