



RELATIVE UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS RELATED TO ℓ_p - SPACE

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Abstract. The concept of relative uniform convergence of sequence of Sargent type functions spaces $m(\phi, ru)$ and $n(\phi, ru)$ are defined in this article. Here we established a bounded, p -absolutely summable, convergent, null sequence of functions using the concept of relative uniform convergent for the scale function and these are denoted by $\ell_\infty(ru)$, $\ell_p(ru)$, $c(ru)$, $c_0(ru)$ respectively. We established a relationship between the new notion $m(\phi, ru)$, $n(\phi, ru)$, and other known concepts. Also, we established some algebraic and geometrical characteristics. Many geometric concepts and difficulties may be solved through an understanding of the characteristics of these special spaces.

Keywords: Relative uniform convergent, BK-space, Convergence free, Solidness, modular, convexity

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1. INTRODUCTION

Sequence spaces have undergone several types of research over the years. The symbols ω , ℓ_∞ , c , c_0 , and ℓ_p are used to indicate the classes of all, bounded, convergent, null, and p -absolutely summable sequences of real or complex terms, respectively. Throughout the article ω_f , Z_f represent the space of all sequences of functions and the subspace of sequences of functions space respectively. Several researchers have contributed significantly to the study of sequence spaces and summability theory, including Cai et al. [2], Ibrahim et al. [10], Ibrahim et al. [11], Ibrahim et al. [12], and Bayram et al. [1], among others.

The study of different modes of convergence plays a central role in functional analysis and sequence space theory. Beyond classical notions such as pointwise, uniform, and norm convergence, relative uniform convergence has proved to be an important intermediate concept, particularly in spaces of functions and sequences. Introduced to capture uniform behavior with respect to a controlling function, relative uniform convergence provides a flexible framework that is weaker than uniform convergence but stronger than pointwise convergence. This notion has found applications in summability theory, approximation theory, and the study of locally convex and modular spaces.

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Inspired by this idea, several sequence spaces related to relative uniform convergence have been studied. These spaces enable the analysis of convergence qualities that are not sufficiently characterized by classical norms alone, as well as the behavior of sequences under scaling by a positive control function. In this context, sequence spaces defined via relative uniform convergence offer a natural setting for extending results from classical spaces such as ℓ_p and ℓ_∞ .

The idea of relative uniform convergence in relation to a scale function was first proposed by Moore [14]. The concept of relative uniform convergence of a sequence of functions was further developed by Chittenden [3] in his formulation. Several other researchers conducted additional research on the concept, such as Demirci et al. [5], Demirci and Orhan [6], Sahin and Dirik [16], Devi and Tripathy ([7], [8]), Debbarma et al. [4], Gurdal et al. [9] and so on. The space $m(\phi)$ generalizes several well-known sequence spaces and provides a unified framework for studying boundedness, summability, and geometric properties. By appropriate choices of the sequence ϕ , the space $m(\phi)$ reduces to familiar spaces, while in the general case it exhibits richer structural and topological behaviour.

The space $m(\phi)$, which is related to ℓ_p space, was made in 1960 by Sargent [17], he also investigated a few $m(\phi)$ space properties. Later, it was examined from the perspective of sequence space, and Rath and Tripathy [15], Tripathy [18], and Tripathy and Sen [19] characterized a number of matrix classes with one member as $m(\phi)$.

The purpose of the present work is to study the structural and topological properties of the space $m(\phi, ru)$. We specifically investigate its normed structure, convergence behavior under relative uniform convergence, and embedding connections with classical sequence spaces. Our findings deepen our knowledge of relative uniform convergence in contemporary functional analysis and build upon previous research on generalized sequence spaces.

2. PRELIMINARIES

Definition 2.1. *Given a compact domain D , a sequence of functions (ζ_n) converges relatively uniformly to a function ζ , if for every positive number C , there exists an integer n_C such that, for every $n \geq n_C$, the following inequality holds:*

$$|\zeta(x) - \zeta_n(x)| \leq C|\mu(x)|.$$

The function $\mu(x)$ of the definition above is called scale function which is on the same compact domain D .

According to Moore [14], Chittenden developed the concept of relative uniform convergence sequence of functions that was shown above.

Example 2.1. *Let $[0, 1]$ be a compact domain in $\mathbb{R}$ and consider the sequence of functions $\zeta_n : [0, 1] \rightarrow \mathbb{R}$, for $n \in \mathbb{N}$ defined by*

$$\zeta_n(x) = \begin{cases} \frac{1}{n^2 x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Consequently, it is evident that (ζ_n) does not converge uniformly, but rather converges somewhat uniformly to the zero function in relation to a scale function $\mu(x)$ defined by

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Kizmaz [13] proposed the idea of difference sequence space, he also explored several properties of the sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The difference sequence, $\Delta x = (\Delta x_n)$, is defined as $\Delta x_n = x_n - x_{n+1}, n \in \mathbb{N}$.

Throughout, the space ξ_s will represent the class of all subsets of $\mathbb{N}$, those contain a maximum of s elements and the space Φ consists of all non-decreasing sequences $\phi = (\phi_n)$ of real numbers, such that for every positive integer n , $0 < \phi_1 \leq \phi_n \leq \phi_{n+1} < \infty$ and $(n+1)\phi_n \geq n\phi_{n+1}$. i.e.,

$$\frac{\phi_n}{n} \geq \frac{\phi_{n+1}}{n+1}, \text{ for all } n \in \mathbb{N}. \quad (2.1)$$

Example 2.2. (1) Let $\phi_n = 1$, for all $n \in \mathbb{N}$, then $(\phi_n) \in \Phi$.

(2) Let $\phi_n = n$, for all $n \in \mathbb{N}$, then $(\phi_n) \in \Phi$.

According to Sargent [17], the spaces $m(\phi)$ and $n(\phi)$ are defined as follows,

Definition 2.2. The class of sequence $m(\phi)$ is defined by

$$m(\phi) = \left\{ (x_n) \in \omega : \sup_{s \geq 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} |x_n| < \infty \right\}.$$

The class of sequence $n(\phi)$ is defined by,

$$n(\phi) = \left\{ (x_n) \in \omega : \sup_{u \in \Lambda(x_n)} \sum_{n=1}^{\infty} |u_n| \Delta \phi_n < \infty \right\},$$

where $\Delta \phi_n = \phi_n - \phi_{n-1}$, for all $n \in \mathbb{N}$.

Definition 2.3. Let $\phi = (\phi_n)$ be a sequence of real numbers satisfying equation (2.1). Then the following inequality holds if a sequence of functions (ζ_n) constructed on a compact domain D converge relative uniformly to a limit function ζ for each tiny positive value $C > 0$.

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x) - \zeta(x)| < C|\mu(x)|,$$

where $\mu(x)$ represents the scale function defined in $D/\mu^{-1}(0)$.

The classes of all relative uniform convergence sequences of Sargent-type function are denoted by $m(\phi, ru)$ and $n(\phi, ru)$, respectively are defined as,

$$m(\phi, ru) = \left\{ (\zeta_n) \in \omega_f : \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C|\mu(x)| \right\},$$

we also introduce another class of sequence of functions:

$$n(\phi, ru) = \left\{ (\zeta_n) \in \omega_f : \sup_{(\eta_n) \in S(\zeta)} \sum_{n=1}^{\infty} |\eta_n(x)| \Delta \phi_n < C|\mu(x)| \right\},$$

where, $S(\zeta)$ represent the rearrangement of (ζ_n) .

Example 2.3. let $\phi_n = n$, for all $n \in \mathbb{N}$. Consider a sequence of function (ζ_n) defined on the range $[0, 1]$ by,

$$\zeta_n(x) = \begin{cases} \frac{n^2 x}{1+n^2 x^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Now,

$$m(\phi, ru) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| \geq \frac{x}{s(1+x^2)},$$

which is not convergent uniformly. However, it is convergent relative uniformly if we consider a scale function defined by,

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, $(\zeta_n) \in m(\phi, ru)$ w.r.t. the scale function $\mu(x)$.

Example 2.4. Consider, $\phi_n = n$, for all $n \in \mathbb{N}$. The function (ζ_n) defined on the interval $[0, 1]$ is considered by,

$$\zeta_n(x) = \begin{cases} \frac{1}{n^2 x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Now,

$$\sup_{(\eta_n) \in S(\zeta)} \sum_{n=1}^{\infty} |\eta_n(x)| \Delta \phi_n \geq \frac{1}{x}.$$

Where, (η_n) is the rearrangement of (ζ_n) , which does not exist, as $x \rightarrow 0$. So, it is not uniformly convergent. However, it is relatively uniformly convergent for a scaling function that is specified by

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, $(\zeta_n) \in n(\phi, ru)$.

Definition 2.4. The norm of the space $m(\phi, ru)$ is defined as follows,

$$\|\zeta\|_{m(\phi, ru)} = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|}.$$

The following well-known relative uniform convergence sequence of function spaces will be used in this article:

$$\begin{aligned} \ell_p(ru) &= \left\{ (\zeta_n) \in \omega_f : \sum_{n=1}^{\infty} |\zeta_n(x)|^p < C|\mu(x)| \right\} \\ \ell_1(ru) &= \left\{ (\zeta_n) \in \omega_f : \sum_{n=1}^{\infty} |\zeta_n(x)| < C|\mu(x)| \right\}, \text{ for } p = 1 \\ \ell_{\infty}(ru) &= \left\{ (\zeta_n) \in \omega_f : \sup_{n \geq 1} |\zeta_n(x)| < C|\mu(x)| \right\} \\ c(ru) &= \left\{ (\zeta_n) \in \omega_f : \frac{|\zeta_n(x)| - L}{|\mu(x)|} \rightarrow 0, \text{ as } n \rightarrow \infty \right\} \\ c_0(ru) &= \left\{ (\zeta_n) \in \omega_f : \frac{|\zeta_n(x)|}{|\mu(x)|} \rightarrow 0, \text{ as } n \rightarrow \infty \right\}. \end{aligned}$$

Note: If $\phi_n = 1$, $n \in \mathbb{N}$, then $m(\phi, ru) = \ell_1(ru)$; if $\phi_n = n$, $n \in \mathbb{N}$, then $m(\phi, ru) = \ell_{\infty}(ru)$.

Definition 2.5. A subset $Z_f \subset \omega_f$ is considered solid or normal if for every $n \in \mathbb{N}$, $|\eta_n(x)| \leq |\zeta_n(x)|$, implies $(\eta_n) \in Z_f$.

Definition 2.6. Convergence-free subsets are defined as $Z_f \subset \omega_f$ if $(\zeta_n) \in Z_f$, then $\zeta_n(x) = 0 \implies \eta_n(x) = 0$, on $x \in D$ and $(\eta_n) \in Z_f$ if $\zeta_n(x) \neq 0 \implies \eta_n(x)$ may be anything.

Definition 2.7. Suppose $Z_f \subset \omega_f$ is a linear domain. Then $\mathcal{M} : Z_f \rightarrow [0, \infty)$ is a modular in a real sequence of functions space if for $(\zeta_n), (\eta_n) \in Z_f$, the following conditions are hold.

- (1) $\mathcal{M}_{Z_f}(\zeta_n(x)) = 0$ iff $\zeta_n(x) = \bar{\theta}$, where $\bar{\theta}$ represents null sequence of function, for all $x \in Z \subseteq \mathbb{R}$.
- (2) $\mathcal{M}_{Z_f}(-\zeta_n(x)) = \mathcal{M}_{Z_f}(\zeta_n(x))$, for all $x \in Z \subseteq \mathbb{R}$.
- (3) $\mathcal{M}_{Z_f}(\lambda \zeta_n(x)) = \mathcal{M}_{Z_f}(\zeta_n(x))$, for all the scalar $\lambda = 1$.
- (4) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \mathcal{M}_{Z_f}(\zeta_n(x)) + \mathcal{M}_{Z_f}(\eta_n(x))$,
for all $(\zeta_n), (\eta_n) \in Z_f$ and $\lambda_1, \lambda_2 \geq 0, |\lambda_1| + |\lambda_2| = 1$.
Farther the modular $\mathcal{M}$ is convex if
- (5) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \lambda_1 \mathcal{M}_{Z_f}(\zeta_n(x)) + \lambda_2 \mathcal{M}_{Z_f}(\eta_n(x))$,
for all $(\zeta_n), (\eta_n) \in Z_f$ and $\lambda_1, \lambda_2 \geq 0, |\lambda_1| + |\lambda_2| = 1$. The modular $\mathcal{M}$ is called $\tilde{s}$ -convex if
- (6) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \lambda_1^{\tilde{s}} \mathcal{M}_{Z_f}(\zeta_n(x)) + \lambda_2^{\tilde{s}} \mathcal{M}_{Z_f}(\eta_n(x))$, for all $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1, 0 < \tilde{s} \leq 1$.

Definition 2.8. A sequence of functions $(\zeta_n) \in \mathcal{M}_{Z_f}$ is modular convergent to $\zeta \in \mathcal{M}_{Z_f}$ occurs if there is a $\lambda > 0$ such that $\mathcal{M}(\lambda \zeta_n(x)) - \zeta(x) \rightarrow 0$ as $n \rightarrow \infty$, for all $x \in D$.

Definition 2.9. In the normed space $m(\phi, ru)$, the unit ball and the unit sphere are defined by

$$\begin{aligned} B_{m(\phi, ru)} &= \left\{ (\zeta_n) \in m(\phi, ru) : \|\zeta_n\|_{m(\phi, ru)} = 1 \right\} \\ S_{m(\phi, ru)} &= \left\{ (\zeta_n) \in m(\phi, ru) : \|\zeta_n\|_{m(\phi, ru)} = 1 \right\}. \end{aligned}$$

Definition 2.10. A Banach space is considered uniformly convex if, for any $\varepsilon > 0$, such that for $(\zeta_n), (\eta_n) \in S_{Z_f}$, where S_{Z_f} represents the unit sphere. There exists a $\mathfrak{U}(\varepsilon) > 0$ such that,

$$\|\zeta_n\|_{Z_f} = \|\eta_n\|_{Z_f} = 1, \|\zeta_n - \eta_n\|_{Z_f} \geq \varepsilon \implies \left\| \frac{\zeta_n + \eta_n}{2} \right\|_{Z_f} \leq 1 - \mathfrak{U}(\varepsilon).$$

The purpose of this research is to investigate this form of convergence of a sequence of the Sargent type-related function and investigate some of its topological and geometrical characteristics.

3. MAIN RESULTS

This section establishes the article's results.

Theorem 3.1. The relative uniform convergent sequence space of functions $m(\phi, ru)$ is a liner space.

Proof. Let (ζ_n) and $(\eta_n) \in m(\phi, ru)$, then for $C_1, C_2 > 0$, we have

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| &< C_1 |\mu(x)|, \text{ for all } x \in D/\mu^{-1}(0), \\ \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| &< C_2 |\mu(x)|, \text{ for all } x \in D/\mu^{-1}(0). \end{aligned}$$

For both the sequences of functions (ζ_n) and (η_n) , we can take into consideration the same scale function μ without losing generality. Consider two scalars λ_1 and λ_2 , We have

$$\begin{aligned} &\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)| \\ &\leq \lambda_1 \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| + \lambda_2 \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| \\ &\leq \lambda_1 C_1 |\mu(x)| + \lambda_2 C_2 |\mu(x)| \\ &\leq (\lambda_1 C_1 + \lambda_2 C_2) |\mu(x)| \\ &= C |\mu(x)|, \text{ where } C = \lambda_1 C_1 + \lambda_2 C_2. \end{aligned}$$

Hence, $(\lambda_1 \zeta_n + \lambda_2 \eta_n) \in m(\phi, ru)$. Therefore, $m(\phi, ru)$ is a linear space. $\square$

Theorem 3.2. With the norm $\|\cdot\|_{m(\phi, ru)}$, the space $m(\phi, ru)$ is a BK-space, as defined by

$$\|\zeta_n\|_{m(\phi, ru)} = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|}.$$

Proof. Assuming throughout that D is compact domain and that $\mu : D \rightarrow (0, \infty)$ is continuous. We first show that $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space and then establish its completeness. The proof of completeness proceeds by showing pointwise convergence of each coordinate of a Cauchy sequence, defining the corresponding candidate limit sequence, and finally using uniform boundedness to justify passing the limit under the supremum involved

in the norm. Under the same assumptions, the continuity of the coordinate projections follows immediately by choosing $s = 1$ and $\sigma = \{n\}$ in the defining inequality of the norm.

Firstly we show that $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space.

- (1) Clearly $\|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = 0$ if and only if $\zeta_n = \bar{\theta}$, for all $n \in \mathbb{N}$, where $(\zeta_n) = \bar{\theta} = (\theta, \theta, \theta, \dots)$, the null sequence of functions.
- (2) Let λ be a scalar, then we have

$$\begin{aligned} \|(\lambda\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda\zeta_n(x)|}{|\mu(x)|} \\ &= |\lambda| \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

- (3) Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$, then

$$\begin{aligned} &\|(\zeta_n + \eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x) + \eta_n(x)|}{|\mu(x)|} \\ &\leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \\ &= \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} + \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

Therefore,

$$\|(\zeta_n + \eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} + \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$$

Hence, $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space.

We now demonstrate the completeness of $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$.

Let $\left\{\frac{\zeta_n^i}{\mu^i}\right\}$ be a Cauchy sequence in $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$. Such that $\sup_i \mu^i(x)$ exists over $x \in D/\mu^{-1}(0)$.

Let $\sup_i \mu^i(x) = \mu(x)$, for all $x \in D/\mu^{-1}(0)$. Where, $\frac{\zeta_n^i}{\mu} = \left\{\frac{\zeta_n^i(x)}{\mu(x)}\right\}_{n=1}^\infty$, for each fixed $i \in \mathbb{N}$.

Then for a given $C > 0$, there exist a positive integer $n_0(C) > 0$ such that,

$$\begin{aligned} &\left\|\frac{\zeta_n^i - \zeta_n^j}{\mu}\right\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n^i(x) - \zeta_n^j(x)|}{|\mu(x)|} < C, \text{ for all } i, j > n_0(C). \end{aligned}$$

Taking $s = 1$ varying $\sigma \in \xi_s$,

$$\frac{|\zeta_n^i(x) - \zeta_n^j(x)|}{|\mu(x)|} < C\phi_1, \text{ for each fixed } n \in \mathbb{N} \text{ and all } i, j \geq n_0(C).$$

Therefore,

$$\frac{|\zeta_n^i(x)|}{|\mu(x)|} - \frac{|\zeta_n^j(x)|}{|\mu(x)|} < C\phi_1, \text{ for all } i, j \geq n_0(C).$$

i.e., $\left\{\left|\frac{\zeta_n^i(x)}{\mu(x)}\right|\right\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $n \in \mathbb{N}$.

Since, $\mathbb{R}$ is complete it converges for each $n \in \mathbb{N}$ and can find a function ξ_n such that

$$\lim_{i \rightarrow \infty} \left| \frac{\zeta_n^i(x)}{\mu(x)} \right| = \left| \frac{\zeta_n(x)}{\mu(x)} \right|.$$

Then, we have

$$\lim_{i \rightarrow \infty} \left\| \frac{\zeta_n^i - \zeta_n^j}{\mu} \right\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = 0.$$

Using uniform boundedness and passing to the limit under the supremum, it follows that $\zeta = (\zeta_n) \in m(\phi, ru)$.

Hence $m(\phi, ru)$ is complete.

Now we prove that, $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ has continuous coordinate projections p_n , where

$$p_k : \omega_f \rightarrow K \text{ and } p_k(\zeta_n(x)) = \zeta_n(x), \quad x \in D/\mu^{-1}(0).$$

Indeed by taking, $s = 1$ and $\sigma = \{n\}$ in the defining inequality of the norm, we obtain

$$|\zeta_n(x)| \leq \sup_{x \in D/\mu^{-1}(0), \sigma \in \xi_s, s \geq 1} \phi_s \|\zeta\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} |\mu(x)|, \text{ for each } n \in \mathbb{N},$$

which proves the continuity of the coordinate projections. $\square$

In light of Theorem 3.2, we propose the following theorem without providing evidence.

Theorem 3.3. *The sequence space of functions $n(\phi, ru)$ is a BK-space.*

Theorem 3.4. *The sequence space of functions $m(\phi, ru)$ is solid.*

Proof. Let $(\zeta_n) \in m(\phi, ru)$. Therefore,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C |\mu(x)|. \quad (3.2)$$

Consider another sequence of functions $(\eta_n) \in \omega_f$ such that,

$$|\eta_n(x)| \leq |\zeta_n(x)|. \quad (3.3)$$

Now from (3.2) and (3.3) we can get,

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| &\leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C |\mu(x)| \\ \implies (\eta_n) &\in m(\phi, ru). \end{aligned}$$

Hence, $m(\phi, ru)$ is solid. $\square$

Theorem 3.5. *The sequence space of functions $m(\phi, ru)$ is a continuous embedding between the sequence space of functions $\ell_p(ru)$, where $1 \leq p < \infty$.*

Proof. We consider a transformation defined by $\mathcal{T} : m(\phi, ru) \rightarrow \ell_p(ru)$, such that

$$\mathcal{T}(\zeta_n(x)) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}}.$$

(1) Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$

The linearity of $\mathcal{T}$ is obvious here. Now, it's clear that, $(\zeta_n(x)) = \theta$, whenever $\mathcal{T}(\zeta_n(x)) = \theta$, where $x \in D/\mu^{-1}(0)$, (a compact domain), $\theta = (0, 0, 0, \dots)$ and hence $\mathcal{T}$ is injective.

Let us take any $(\eta_n) \in \ell_p(ru)$ such that

$$\begin{aligned} \mathcal{T}(\zeta_n(x)) &= \eta_n(x), \\ \Rightarrow \frac{1}{\phi_s} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}} &= \eta_n(x) \\ \Rightarrow \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}} &= \sum_{n \in \sigma} \phi_s \eta_n(x) \\ \Rightarrow \zeta_n(x) &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} (\phi_s \eta_n(x))^p |\mu(x)| \right)^{\frac{1}{p}} < C |\mu'(x)|. \end{aligned}$$

Where $\mu'(x)$ is the new scale function w.r.t. the sequence of function (η_n) .

Hence, the sequence space of functions $m(\phi, ru)$ is a continuous embedding with the sequence space of function $\ell_p(ru)$, where $1 \leq p < \infty$. $\square$

Given the explanation above, we assert the following corollary.

Corollary 3.1. *The sequence space of function $m(\phi, ru)$ is continuous embedded with the sequence space of function $\ell_\infty(ru)$, if $\phi_n = n$.*

Theorem 3.6. *The space $m(\phi, ru)$ is isomorphic to the space $\ell_1(ru)$, i.e., $m(\phi, ru) \cong \ell_1(ru)$, if $\phi_n = 1$.*

Proof. The proof follows directly from the definition of isomorphism by verifying linearity, bijectivity, and norm equivalence. $\square$

Result 3.1. *Convergence is not free in the sequence space of functions $m(\phi, ru)$.*

The following example leads to the stated conclusion.

Proof. Let us take a sequence of functions on a compact domain $[0, 1]$ defined by,

$$\zeta_n(x) = \begin{cases} \frac{1}{nx}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

With this established, it becomes clear that (ζ_n) does not converge uniformly, but rather converges relative uniformly to the zero function with regard to a scale function μ defined by,

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Consider the sequence (ϕ_n) defined by $\phi_n = 1$, for all $n \in \mathbb{N}$.

Then,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\frac{1}{nx}|}{|\frac{1}{x}|} < C.$$

Hence, $(\zeta_n) \in m(\phi, ru)$.

Now we consider another sequence of functions on the same compact domain

$[0, 1]$, defined by

$$\eta_n(x) = \begin{cases} nx, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0, \end{cases}$$

using $\mu(x)$ as the scaling function. But in that case,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|nx|}{|x|} \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

Therefore, $(\eta_n) \notin m(\phi, ru)$.

Hence, $m(\phi, ru)$ is not convergence-free in general. $\square$

Theorem 3.7. (1) $m(\phi, ru)$ is a symmetric space. If $(\zeta_n) \in m(\phi, ru)$ and $(\eta_n) \in m(\phi, ru)$, then

$$\|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$$

(2) Assuming $(\zeta_n) \in m(\phi, ru)$ and $|(\eta_n)| \leq |(\zeta_n)|$; for each positive number n , then $(\eta_n) \in m(\phi, ru)$ and $\|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$

Proof. (1) The proof of this lemma can be established by using Lemma 5(a) of Sargent[17].

(2) Let, $(\zeta_n) \in m(\phi, ru)$ and also, consider a sequence of function (η_n) such that $|\eta_n(x)| \leq |\zeta_n(x)|$.

Now,

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| \\ & \leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| \leq C|\mu(x)|, \text{ (since } (\zeta_n) \in m(\phi, ru) \text{)} \\ & \implies \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

Hence, $(\eta_n) \in m(\phi, ru)$. $\square$

Theorem 3.8. $m(\phi, ru) \subseteq m(\psi, ru)$ iff $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty$.

Proof. Let,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = H(< \infty).$$

Then,

$$\phi_s \leq H\psi_s, \text{ for all } s \in \mathbb{N}. \quad (3.4)$$

Let, $(\zeta_n) \in m(\phi, ru)$. Then,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{H\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C, \text{ [from 3.4]} \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < HC \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < H', \text{ where } H' = HC \\ \implies & (\zeta_n) \in m(\phi, ru). \end{aligned}$$

Therefore, The expression $m(\phi, ru) \subseteq m(\psi, ru)$.

Assume, on the other hand, that $m(\phi, ru) \subseteq m(\psi, ru)$.

Next consider $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = \infty$.

Let,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = \sup_{s \geq 1} \delta_s, \text{ and } (\zeta_n) \in m(\phi, ru).$$

Therefore,

$$\sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C.$$

Now,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} \\ &= \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \delta_s \cdot \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} = \infty. \end{aligned}$$

i.e, $(\zeta_n) \notin m(\phi, ru)$. Here we arrive at a contradiction.

Hence,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

□

Theorem 3.9. $\ell_p(ru) \subseteq m(\phi, ru) \subseteq \ell_\infty(ru)$.

Proof. Assuming $\phi_n = 1$, for all $n \in \mathbb{N}$, we have $\ell_p(ru) = m(\phi, ru)$.

Next we suppose that $(\zeta_n) \in m(\phi, ru)$.

Then,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{|\zeta_n(x)|}{|\mu(x)|} < C\phi_s, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Thus, $(\zeta_n) \in \ell_\infty(ru)$.

Hence, $\ell_p(ru) \subseteq m(\phi, ru) \subseteq \ell_\infty(ru)$

□

Theorem 3.10. $m(\phi, ru) = \ell(\phi, ru)$ if and only if $\lim_{n \rightarrow \infty} \phi_s < \infty$.

Proof. The theorem follow from lemma 5(c) of Sargent[17].

□

4. GEOMETRIC PROPERTIES

The geometric characteristics of the sequence space $m(\phi, ru)$ and $n(\phi, ru)$ of functions are the subject of certain theorems in this section.

Theorem 4.1. *The function $\mathcal{M}_{m(\phi, ru)}$ denotes the modular function of the space $m(\phi, ru)$, with modular,*

$$\mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\zeta_n(x)|}{|\mu(x)|} \right).$$

Proof. Consider a function $\mathcal{M}_{m(\phi, ru)}$ defined by,

$\mathcal{M}_{m(\phi, ru)} : m(\phi, ru) \rightarrow [0, \infty)$. Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$.

- (1) Clearly, $\mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) = 0 \implies \zeta_n(x) = \bar{\theta}$, where $x \in D/\mu^{-1}(0)$ and $\bar{\theta} = (\theta, \theta, \theta, \dots)$.
- (2) $\mathcal{M}_{m(\phi, ru)}(-\zeta_n(x)) = \mathcal{M}_{m(\phi, ru)}(\zeta_n(x))$, for all $(\zeta_n) \in m(\phi, ru), x \in D/\mu^{-1}(0)$.
- (3) For λ a scalar with $|\lambda| = 1$, we have

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda \zeta_n(x)|}{|\mu(x)|} \\ &= \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for } x \in D/\mu^{-1}(0). \end{aligned} \tag{4.5}$$

- (4) Now for any two scalar $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ we get

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)|}{|\mu(x)|} \\ &\leq |\lambda_1| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + |\lambda_2| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \\ &\leq \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) + \mathcal{M}_{m(\phi, ru)}(\eta_n(x)) \\ & \text{(Since, } \lambda_1 + \lambda_2 = 1, \text{ thus either both the scalars are less than 1).} \end{aligned}$$

□

The preceding theorem's consequence is the proof of the ensuing findings.

Theorem 4.2. *The function $\mathcal{M}_{n(\phi, ru)}$ denotes the modular function of the space $n(\phi, ru)$, with modular,*

$$\mathcal{M}_{n(\phi, ru)} = \sup_{(\eta_n) \in \Lambda(\zeta_n), x \in D/\mu^{-1}(0)} \sum_{n=1}^{\infty} \mathcal{M}_n \left(\frac{|\eta_n(x)| \Delta \phi_n}{|\mu(x)|} \right), \text{ for all } x \in D/\mu^{-1}(0).$$

Proof. The theorem follows from the Theorem 4.2. □

In light of the previous debate, we assert the following corollary.

Corollary 4.1. *The modular $\mathcal{M}_m(\phi, ru)$ and $\mathcal{M}_n(\phi, ru)$ are 1-convex or a convex modular.*

Theorem 4.3. *The Modular function $\mathcal{M}_{m(\phi, ru)}$ on the space $m(\phi, ru)$ is $\tilde{s}$ -convex.*

Proof. Consider $(\zeta_n), (\eta_n) \in \mathcal{M}_{m(\phi, ru)}$ and $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1$, where $\tilde{s} \in (0, 1]$. Now,

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)|}{|\mu(x)|} \right) \\ &\leq |\lambda_1| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\zeta_n(x)|}{|\mu(x)|} \right) + |\lambda_2| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\eta_n(x)|}{|\mu(x)|} \right) \\ &\leq |\lambda_1|^{\tilde{s}} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + |\lambda_2|^{\tilde{s}} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \end{aligned}$$

(Since, $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1$, where $\tilde{s} \in (0, 1]$, so λ_1, λ_2 must belongs to $[0, 1]$).

Hence the space $\mathcal{M}_{m(\phi, ru)}$ is $\tilde{s}$ -modular. $\square$

We present the following theorem in light of the explanation above.

Theorem 4.4. *The Modular function $\mathcal{M}_{n(\phi, ru)}$ on the space $n(\phi, ru)$ is $\tilde{s}$ -convex.*

Proof. The theorem 4.4 is a consequence of the theorem's proof. $\square$

Theorem 4.5. *The modular spaces $\mathcal{M}_{m(\phi, ru)}$ and $\mathcal{M}_{n(\phi, ru)}$ are continuous modular on the space $m(\phi, ru)$ and $n(\phi, ru)$ respectively.*

Proof. Since the function $\mathcal{M}_{m(\phi, ru)}$ is modular on the space $m(\phi, ru)$, thus we can take limit as $\lambda \rightarrow 1^+$ in (4.5) we observe that,

$$\lim_{\lambda \rightarrow 1^+} \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) = \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for all } (\zeta_n) \in \mathcal{M}_{m(\phi, ru)},$$

and for all $x \in D/\mu^{-1}(0)$. Hence, $\mathcal{M}_{m(\phi, ru)}$ is right continuous. Similarly taking limit as $\lambda \rightarrow 1^-$ in (4.5), we observe that

$$\begin{aligned} & \lim_{\lambda \rightarrow 1^-} \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) \\ &= \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for all } (\zeta_n) \in \mathcal{M}_{m(\phi, ru)} \text{ and for all } x \in D/\mu^{-1}(0). \end{aligned}$$

These two consequences give us the desired fact that the function $\mathcal{M}_{m(\phi, ru)}$ is continuous modular.

Similarly, we can establish for the modular $\mathcal{M}_{n(\phi, ru)}$ on the space $n(\phi, ru)$. $\square$

The following corollary is obvious for the rest of the proof and is the consequence of the previous theorem.

Corollary 4.2. *The modular $\mathcal{M}_{m(\phi, ru)}$ is modular convergent to $\zeta \in \mathcal{M}_{m(\phi, ru)}$.*

5. APPLICATION & FUTURE WORK

The framework developed in this paper for the sequence space $m(\phi, ru)$ provides a useful tool for studying convergence and boundedness under scale function. Such spaces naturally arise in summability theory, where relative uniform convergence offers a finer mode of convergence than pointwise convergence. The results obtained here may be applied to the study of

generalised summability methods, approximation of function sequences, dual sequence space and the analysis of operators acting between weighted sequence spaces.

6. CONCLUSION

In this article we define relative uniform convergence of Sargent type sequence of functions concerning the scale function $\mu(x)$ defined on a compact domain $D/\mu^{-1}(0) \subseteq \mathbb{R}$, we have introduced the classes of sequences of functions $m(\phi, ru)$ and $n(\phi, ru)$. We investigated their fundamental topological and geometric properties, including completeness, embedding relations, norm-related characteristics and also studied some geometric properties of convexity. Several illustrative examples were provided to clarify the theoretical developments. The results obtained not only extend and unify a number of known sequence spaces but also open avenues for further research. We believe that the framework developed here can serve as a foundation for further investigations in the theory of generalised sequence spaces.

7. OUTLINE OF THE ARTICLE

The sequence space $m(\phi, ru)$ associated with relative uniform convergence is studied in this paper, and its fundamental structural properties are established. The main results include the characterization of the normed structure of $m(\phi, ru)$, embedding relations with classical sequence spaces, and convergence properties under relative uniform control. The paper is structured as follows: Section 2 provides the necessary definitions and preliminaries; Section 3 presents the main theorems and their proofs; Section 4 discusses related consequences and further observations; and the final section concludes the paper and suggests future research directions.

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