



## TORSE-FORMING VECTOR FIELDS AND NORMALIZED NULL HYPERSURFACES

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**Abstract.** The study focuses on normalized null hypersurfaces within semi-Riemannian manifolds that possess torse-forming vector fields. We demonstrate that these hypersurfaces exhibit a triple product structure. Additionally, we present obstruction results related to both topology and specific geometric criteria concerning the rigging and rigged vector field. We also provide a result on the splitting of the hypersurface, which is endowed with its rigged Riemannian structure. Furthermore, taking the potential vector field as the screen component of a torse-forming vector field, we investigate screen distribution with Ricci solitons.

**Keywords:** Normalized null hypersurface, Rigging vector field, Ricci solitons, Torse-forming vector field, Screen quasi-conformal.

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### 1. INTRODUCTION

A torse-forming vector field is defined as a vector field  $K$  on a semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  that fulfills the following equation:

$$\bar{\nabla}_U K = \Psi U + \mu(U)K,$$

for all vector field  $U \in \Gamma(\bar{\Sigma})$ , where  $\Psi$  is a smooth function on  $\bar{\Sigma}$ ,  $\mu$  is a one-form, and  $\bar{\nabla}$  denotes the Levi-Civita connection of  $(\bar{\Sigma}, \bar{g})$ . The function  $\Psi$  is referred to as the conformal scalar, while the one-form  $\mu$  is called the generating[24].

Torse-forming vector fields are of particular interest because of their use in submanifold theory, relativity, and cosmology. These vector fields are significant in a number of differential geometry and physics topics (see [7], [23]). K. Yano examined the definition and analysis of torse-forming vector fields for the first time in [30].

In addition to these facts, the notion of a Ricci soliton is initially observed by Hamilton's. After G. Perelman [28] used Ricci solitons to solve the Poincaré conjecture, Ricci flow and Ricci solitons gained interest. A Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  with a metric tensor  $\bar{g}$  is referred

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to as a Ricci soliton if a smooth vector field  $V$  tangent to  $\bar{\Sigma}$  exists and satisfies the equation described below:

$$L_V \bar{g} + \bar{Ric} = \Lambda \bar{g},$$

where  $L_V \bar{g}$  is the Lie derivative of  $\bar{g}$  with respect to  $V$ ,  $\bar{Ric}$  represents the Ricci tensor of  $(\bar{\Sigma}, \bar{g})$ , and  $\Lambda$  is a constant. We call the vector field  $V$  the potential field. A Ricci soliton is called shrinking if  $\Lambda > 0$ , steady if  $\Lambda = 0$ , expanding if  $\Lambda < 0$ .

The geometry of Riemannian submanifolds and non-degenerate submanifolds of semi-Riemannian manifolds share a number of similarities, according to [27]. However, the geometry is quite different and difficult when the induced metric on a submanifold is degenerate, and the existing methods fail.

The authors began the general study of arbitrary null submanifolds of semi-Riemannian manifolds in [14],[13]. They avoid the anomaly resulting from the degeneracy of the induced metric by fixing geometric data consisting of a null section and a screen distribution (or, alternatively, a null section and a null transversal section) on a null hypersurface. Although this approach has been very successful, it has the disadvantage of depending on two different and arbitrary decisions, and it doesn't appear to be appropriate for examining the inherent geometry of the null hypersurfaces.

One of the novel methods for dealing with this issue, as presented in [16], is to choose a transversal vector field  $E$  defined on an open neighborhood of the null hypersurface  $\Sigma$  on semi-Riemannian manifolds  $\bar{\Sigma}$ . This is known as the rigging for  $\Sigma$ . Through this selection, a null section  $\xi$  (also known as a rigged vector field) is created, such that  $\bar{g}(E, \xi) = 1$ . This method has also been used in other studies [5, 19, 20, 22, 26, 21]

Many researchers have studied the geometry of null hypersurfaces in Lorentzian manifolds furnished with different kinds of vector fields, including parallel, conformal, closed conformal, homothetic, Killing vector fields, and affine conformal Killing (see [4, 2, 5, 8, 20, 29, 8, 6]). Nevertheless, the case with torse-forming vector fields has not been investigated thus far.

The purpose of this work is to study the geometry of normalized null hypersurfaces of semi-Riemannian manifolds admitting a torse-forming vector field, with a focus on the consequences of the existence of such a vector field on both the geometry of null hypersurfaces and the curvature of the ambient space, as well as exploring the conditions under which screen distributions can admit Ricci soliton structures that arise from screen component of a torse-forming vector field  $K$ .

The following is the structure of this paper. Section 2 includes the necessary preliminary information. We demonstrate in Section 3 that such hypersurfaces have a triple product structure (Theorem 3.1, Theorem 3.2, Theorem 3.5, and Theorem 3.6). Theorem 4.1 and Theorem 4.4 establish obstruction results involving both topology and prescribed geometric conditions on the rigging and rigged vector field, while Theorem 4.2 provides a splitting result on the hypersurface equipped with its rigged Riemannian structure. These results are found in Section 4. According to Theorem 5.1, we characterize null hypersurfaces in Section 5, whose screen distributions admit a Ricci soliton induced from the screen component of a torse-forming vector field  $K$ .

## 2. RIGGING TECHNIQUE FOR NULL HYPERSURFACES

Let  $(\bar{\Sigma}, \bar{g})$  be a  $(n + 2)$ -dimensional semi-Riemannian manifold with index  $t$ , where  $0 < t < n + 2$ , and  $(\Sigma, g)$  a null hypersurface of  $(\bar{\Sigma}, \bar{g})$ . A *rigging* for  $\Sigma$  is a vector field  $E$  defined on some open set of  $\bar{\Sigma}$  containing  $\Sigma$  such that for each  $p \in \Sigma$ ,  $\Sigma_p \notin T_p \Sigma$  [16].

For a fixed rigging  $E$  for  $\Sigma$ , we set  $\alpha = \bar{g}(E, \cdot)$ ,  $\omega = i^* \alpha$ ,  $\tilde{g} = \bar{g} + \alpha \otimes \alpha$  and  $\tilde{g} = i^* \tilde{g}$ , where  $i : \Sigma \hookrightarrow \bar{\Sigma}$  is the canonical inclusion map. It is well known that  $\tilde{g}$  is a non degenerate metric on  $\Sigma$ . The *rigged vector field* on  $\Sigma$  is the unique null vector field  $\xi$  given by  $\tilde{g}(\xi, \cdot) = \alpha$  and it satisfies

$$\bar{g}(E, \xi) = 1. \quad (2.1)$$

A null hypersurface  $\Sigma$  furnished with a rigging  $E$  is said to be normalized and is denoted  $(\Sigma, E)$ . A rigging  $E$  defines a screen distribution  $\mathcal{S}(E)$  given by  $\mathcal{S}(E) = T\Sigma \cap E^\perp = \ker \omega$ . The null transverse vector field on  $\Sigma$  is

$$N = E - \frac{1}{2} \bar{g}(E, E) \xi, \quad (2.2)$$

which is the unique null vector field such that  $\bar{g}(N, \xi) = 1$ . Moreover,  $T\bar{\Sigma}$  admits the following splitting

$$\begin{aligned} T\bar{\Sigma}|_\Sigma &= T\Sigma \oplus \text{span}(N) \\ &= \{\mathcal{S}(E) \oplus \text{span}(\xi)\} \oplus \text{span}(N). \end{aligned} \quad (2.3)$$

From the decomposition (2.3), the Gauss and Weingarten equations of  $\Sigma$  and  $\mathcal{S}(E)$  are the following [13] :

$$\bar{\nabla}_U V = \nabla_U V + B(U, V)N, \quad \bar{\nabla}_U N = -A_N U + \tau(U)N, \quad (2.4)$$

$$\nabla_U P Y = \overset{\star}{\nabla}_U P Y + C(U, P Y) \xi, \quad \nabla_U \xi = -\overset{\star}{A}_\xi U - \tau(U) \xi, \quad (2.5)$$

for any  $U, V \in \Gamma(T\Sigma)$ , where  $\bar{\nabla}$  represents the Levi-Civita connection on  $(\bar{\Sigma}, \bar{g})$ ,  $\nabla$  represents the connection on  $\Sigma$  that is induced from  $\bar{\nabla}$  via the projection along the null transverse vector field  $N$  and  $\overset{\star}{\nabla}$  indicates the connection on the screen distribution  $\mathcal{S}(E)$  that is induced from  $\nabla$  via the projection morphism  $P$  of  $\Gamma(T\Sigma)$  onto  $\Gamma(\mathcal{S}(E))$  in accordance with the decomposition (2.5). Now the  $(0, 2)$  tensors  $B$  and  $C$  are the second fundamental forms on  $T\Sigma$  and  $\mathcal{S}(E)$  respectively,  $A_N$  and  $\overset{\star}{A}_\xi$  are the shape operators on  $T\Sigma$  with respect to the rigging  $E$  and the rigged vector field  $\xi$  respectively and  $\tau$  a 1-form on  $T\Sigma$  defined by

$$\tau(U) = \bar{g}(\bar{\nabla}_U N, \xi).$$

As  $(\nabla_U g)(V, Z) = B(U, V)\omega(Z) + B(U, Z)\omega(V)$ ,  $\forall U, V, Z \in \Gamma(T\Sigma)$ , then the induced linear connection  $\nabla$  is not a metric connection. Also  $C$  is not symmetric as  $C(U, V) - C(V, U) = g(\nabla_U V - \nabla_V U, N) = \omega([U, V])$ ,  $\forall U, V \in \mathcal{S}(E)$ .

Let denote by  $\bar{R}$  and  $R$  the Riemannian curvature tensors of  $\bar{\nabla}$  and  $\nabla$ , respectively. Then the following are the Gauss-Codazzi equations [13].

$$\begin{aligned} \bar{g}(\bar{R}(U, V)Z, \xi) &= (\nabla_U B)(V, Z) - (\nabla_V B)(U, Z) \\ &\quad + \tau^N(U)B(V, Z) - \tau^N(V)B(U, Z), \\ \bar{g}(\bar{R}(U, V)Z, PT) &= \bar{g}(R(U, V)Z, PT) + B(U, Z)C^N(V, PT) \\ &\quad - B(V, Z)C^N(U, PT), \\ \bar{g}(\bar{R}(U, V)\xi, N) &= \bar{g}(R(U, V)\xi, N) = C^N(V, \overset{\star}{A}_\xi U) \\ &\quad - C^N(\overset{\star}{A}_\xi V, U) \\ &\quad - 2d\tau^N(U, V), \quad \forall U, V, Z, W \in \Gamma(TM|_{\mathcal{Q}}). \end{aligned} \tag{2.6}$$

$$\begin{aligned} \bar{g}(\bar{R}(U, V)PZ, N) &= \bar{g}((\nabla_U A_N)V, PZ) - \bar{g}((\nabla_V A_N)U, PZ) \\ &\quad + \tau^N(V)\bar{g}(A_N U, PZ) - \tau^N(U)\bar{g}(A_N V, PZ). \end{aligned}$$

for every  $U, V, Z$  and  $T$  in  $\Gamma(T\Sigma)$ .

A normalized null hypersurface  $(\Sigma, E)$  is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function  $\beta$  on  $\Sigma$  such that at each  $p \in \Sigma$  and  $\forall U, V \in T_p\Sigma$ ,

$$B(p)(U, V) = \beta(p)g(U, V) \text{ (resp. } B \text{ vanishes identically on } \Sigma \text{)}. \tag{2.7}$$

Also the screen distribution  $\mathcal{S}(E)$  is totally umbilical (resp. totally geodesic) if for all  $U, V \in \Gamma(T\Sigma)$

$$C(U, PV) = \lambda g(U, V) \text{ (resp. } A_N = 0 \text{)}. \tag{2.8}$$

The null mean curvature  $\mathbf{H}_\xi$  and the screen mean curvature  $\mathbf{H}_N$  are defined by :

$$\mathbf{H}_\xi = \frac{1}{n} \sum_{i=1}^n B(\overset{\circ}{E}_i, \overset{\circ}{E}_i), \quad \mathbf{H}_N = \frac{1}{n} \sum_{i=1}^n C(\overset{\circ}{E}_i, \overset{\circ}{E}_i),$$

where  $\{\overset{\circ}{E}_1, \dots, \overset{\circ}{E}_n\}$  is an orthonormal basis of  $\mathcal{S}(E)$ .

With regard to  $\xi$  and  $N$ , we have seen that two shape operators are supplied for any null hypersurface:  $\overset{\star}{A}_\xi$  and  $A_N$ . The authors of [3] simplify the analysis by taking into account the scenario in which these shape operators are linearly connected, in which case  $\Sigma$  is referred to as a null hypersurface with conformal screen distribution. Navarro et al. expanded on this idea in [25], defining the screen quasi-conformal hypersurfaces.

**Definition 2.1.** *We say that the normalized null hypersurface  $(\Sigma, E)$  (or the rigging  $E$ ) has a quasi-conformal screen distribution if the shape operators  $A_N$  and  $\overset{\star}{A}_\xi$  of  $\Sigma$  and  $\mathcal{S}(E)$  satisfy*

$$A_N U = \kappa \overset{\star}{A}_\xi U + \sigma P U, \tag{2.9}$$

for any  $U \in \Gamma(T\Sigma)$  and some functions  $\kappa$  and  $\sigma$ . Equivalently,

$$C(U, PV) = \kappa B(U, PV) + \sigma g(PU, PV)$$

for any  $U, V \in \Gamma(T\Sigma)$ . For  $\sigma = 0$ , we simply say that  $(\Sigma, E)$  (or the rigging  $E$ ) has a conformal screen distribution.

We conclude this part by providing an example of a null hypersurface equipped with a rigging vector field that meets the definition previously mentioned.

**Example 2.1.** Let  $\bar{\Sigma}$  be a Generalized Robertson-Walker (GRW) spacetime; that is the warped product  $\bar{\Sigma} = I \times_{\psi} H$ , where  $I$  (the base) is an open interval of the real line  $\mathbb{R}$ ,  $(H, g_H)$  the fiber is a Riemannian manifold of dimension  $n+1$  and  $\psi > 0$  is a smooth warping function defined on  $I$ . It is then endowed with the Lorentzian metric  $\bar{g} = -du^2 + \psi^2(t)g_H$ , where  $u$  stands for the natural (global) parameter on  $\mathbb{R}$ . It is established that the vector field  $E = \psi \partial t$  is a timelike closed conformal vector field with a closed conformal factor  $\psi'$ . Therefore, we can utilize it as a rigging for any null hypersurface  $\Sigma$  in GRW. Let  $\xi$  represent the associated rigged vector field. The related null transverse vector field is denoted as

$N = E - \frac{1}{2}\bar{g}(E, E)\xi = E + \frac{1}{2}\psi^2\xi$ . From this, it is worth nothing that [1, Eq.3.1]  $\tau(U) = 0$ ,  $PU \cdot \psi = 0$ ,  $A_N U = \frac{1}{2}\psi^2 \overset{\star}{A}_{\xi} U - \psi'PU$ ,  $A_N \xi = 0$ ,  $\tau(\xi) = 0$ ,  $\xi \cdot \psi = -\frac{\psi'}{\psi}, \forall U \in \Gamma(T\Sigma)$ .

**Example 2.2.** The null cone  $\Sigma^{n+1}$  of  $\mathbb{R}_1^{n+1}$  ( $n > 2$ ) is given by the equation  $-(u^0)^2 + \sum_{i=1}^{n+1}(u_i)^2 = 0, u_0 \neq 0$ . It is known that  $\Sigma^{n+1}$  is a normalized null hypersurface of  $\mathbb{R}_1^{n+2}$ . The rigging vector field is given by  $E = -\frac{1}{u_0} \frac{\partial}{\partial u_0}$ . The corresponding rigged vector field is given by  $\xi = \sum_{i=0}^{n+1} u_i \frac{\partial}{\partial u_i}$ . The associated null transversal vector field is  $N = \frac{1}{2(u^0)^2} \{-u^0 \frac{\partial}{\partial u^0} + \sum_{a=1}^{n+1} u_i \frac{\partial}{\partial u_i}\}$ . As  $\xi$  is the position vector field, we get:  $\bar{\nabla}_U \xi = \nabla_U \xi = U, \forall U \in \Gamma(T\Sigma)$ . Then,  $-\overset{\star}{A}_{\xi} U - \tau(U)\xi = U$ . Since  $\overset{\star}{A}_{\xi}$  is  $\mathcal{S}(E)$ -valued, we obtain  $\forall U \in \Gamma(\mathcal{S}(E)) \overset{\star}{A}_{\xi} U = -U$ . Next, it is known that  $A_N U = \frac{1}{2(u^0)^2} \overset{\star}{A}_{\xi} U, \forall U \in \Gamma(T\Sigma^{n+1})$ . Which means that  $\Sigma^{n+1}$  is screen conformal.

### 3. TORSE-FORMING VECTOR FIELDS

In the entire context,  $K$  represents a fixed torse-forming vector field in a semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  with potential function  $\Psi$ , that is

$$\bar{\nabla}_U K = \Psi U + \mu(U)K, \quad (3.10)$$

$\forall U$  on  $(\bar{\Sigma}, \bar{g})$ , where  $\Psi$  is a smooth function on  $\bar{\Sigma}$ ,  $\mu$  is a one-form. The function  $\Psi$  is called the conformal scalar while the one-form  $\mu$  is called the generating form [24]. When  $\mu(K) = 0$ , the vector field  $K$  is said to be a torqued vector field [9]. Let  $W$  be the dual vector field corresponding to the generating form  $\mu$ , satisfying  $\mu(U) = \bar{g}(U, W)$  for every vector field  $U$  on  $(\bar{\Sigma}, \bar{g})$ . If the one-form  $\mu$  is identically zero, then  $K$  is known as a closed conformal vector field (see Example 2.1). When  $\Psi$  is a non-zero constant and  $\mu = 0$ ,  $K$  is known as a concurrent vector field (the position vector field in Example 2.2 is concurrent), and if  $\Psi = 0$  with  $\mu$  is a non-zero,  $K$  is described as a recurrent vector field.

**Remark 3.1.** If  $K$  is null, then by Eq.(3.10) we get

$$0 = U\bar{g}(K, K) = 2\bar{g}(\bar{\nabla}_U K, K) = 2\Psi\bar{g}(U, K),$$

$\forall U \in \Gamma(T\bar{\Sigma})$ . Thus, since  $\Psi$  must be identically zero,  $K$  is a recurrent vector field.

**Example 3.1.** Let  $\bar{\Sigma} = (\mathbb{R}_1^4, \bar{g}_0)$  be the 4 dimension Minkowski space time, that is  $\bar{g}_0 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$ , with  $(t, x_1, x_2, x_3)$  being coordinates in  $\mathbb{R}^4$ . We consider the conformal

metric  $\bar{g} = e^{2t}\bar{g}_0$ . As  $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$  is an orthogonal frame basis with respect to  $\bar{g}_0$ , then is worth noting that the vectors fields

$$u_1 = e^{-t}\partial_1, \quad u_2 = e^{-t}\partial_2, \quad u_3 = e^{-t}\partial_3, \quad u_4 = e^{-t}\partial_t$$

is a frame basis with respect to  $\bar{g}$ . Let  $\bar{\nabla}$  denotes the Levi-Civita connection of the metric  $\bar{g}$  and  $\nabla^0$  the Levi-Civita connection of  $\bar{g}_0$ . It is known that

$$\bar{\nabla}_U V = \nabla_U^0 V + U(f)V - V(f)U - \bar{g}_0(U, V)\text{grad}_{\bar{g}_0}(f)$$

$\forall U, V \in \Gamma(T\bar{\Sigma})$ , where  $f = t$ . Using this known result, we obtain

$$\begin{aligned} \bar{\nabla}_{u_i} u_i &= e^{-t}u_4, & \bar{\nabla}_{u_4} u_i &= 0, & \bar{\nabla}_{u_i} u_j &= 0 \text{ with } (i \neq j) \\ \bar{\nabla}_{u_i} u_4 &= -e^{-t}u_i, & \bar{\nabla}_{u_4} u_4 &= -2e^{-t}u_4, & & \text{with } (1 \leq i, j \leq 3). \end{aligned}$$

Any vector field  $U$  on  $(\bar{\Sigma}, \bar{g})$  can be split as  $U = au_1 + bu_2 + cu_3 + du_4$ , where  $a, b, c, d$  are smooth functions on  $\bar{\Sigma}$ . It is easy to show that  $u_4$  is torse-forming vector field :

$$\begin{aligned} \bar{\nabla}_U u_4 &= a\bar{\nabla}_{u_1} u_4 + b\bar{\nabla}_{u_2} u_4 + c\bar{\nabla}_{u_3} u_4 + d\bar{\nabla}_{u_4} u_4 \\ &= -ae^{-t}u_1 - be^{-t}u_2 - ce^{-t}u_3 - 2de^{-t}u_4 \\ &= -e^{-t}(au_1 + bu_2 + cu_3 + du_4) - de^{-t}u_4 \\ &= -e^{-t}U + \mu(U)u_4. \end{aligned}$$

Thus,  $K = u_4$  is a torse-forming vector field with conformal scalar  $\Psi = -e^{-t}$  and generating form  $\mu(\cdot) = \bar{g}(\cdot, W)$ , with  $W = e^{-t}u_4$ .

In [20],[2], a starting point to describe the geometry of  $\Sigma$  through closed conformal vector fields is the following pointwise decomposition along  $\Sigma$ :

$$K = K^T + bN = K_{\mathcal{S}} + a\xi + bN, \tag{3.11}$$

where  $a$  and  $b$  are smooth functions on  $\bar{\Sigma}$  defined by  $a = \bar{g}(K, N)$ ,  $b = \bar{g}(K, \xi)$  and  $K_{\mathcal{S}} \in \Gamma(\mathcal{S}(E))$ .

The reader may compare this with [22, Proposition 3.1].

**Proposition 3.1.** *Let  $K$  be a torse-forming vector field on  $(\bar{\Sigma}, \bar{g})$  and  $(\Sigma, E)$  a normalized null hypersurface of  $\bar{\Sigma}$ . Then, for any  $U$  tangent to  $\Sigma$ , the following holds:*

$$\nabla_U^* K_{\mathcal{S}} = a \overset{\star}{A}_{\xi}(U) + bA_N(U) + \Psi PU + \mu(U)K_{\mathcal{S}}, \tag{3.12}$$

$$\Psi\omega(U) + a\mu(U) = C(U, K_{\mathcal{S}}) + U(a) - a\tau(U), \tag{3.13}$$

$$b\mu(U) = B(U, K_{\mathcal{S}}) + U(b) + b\tau(U). \tag{3.14}$$

*Proof.* Applying (3.11) along with the property that  $K$  represents torse-forming vector fields with the conformal factor  $\Psi$ ,  $\forall U \in \Gamma(T\Sigma)$ , we obtain on one side

$$\begin{aligned} \bar{\nabla}_U K &= \Psi U + \mu(U)K \\ &= \Psi PU + \mu(U)K_{\mathcal{S}} + (\Psi\omega(U) + a\mu(U))\xi + b\mu(U)N. \end{aligned} \tag{3.15}$$

On other hand, we have

$$\begin{aligned}
\bar{\nabla}_U(K_{\mathcal{S}} + a\xi + bN) &= \bar{\nabla}_U K_{\mathcal{S}} + \bar{\nabla}_U(a\xi) + \bar{\nabla}_U(bN) \\
&= \nabla_U K_{\mathcal{S}} + B(U, K_{\mathcal{S}})N + U(a)\xi + a\bar{\nabla}_U\xi + U(b)N + b\bar{\nabla}_U N \\
&= \nabla_U^* K_{\mathcal{S}} + C(U, K_{\mathcal{S}})\xi + B(U, K_{\mathcal{S}})N + U(a)\xi + a(-\overset{\star}{A}_{\xi}(U) - \tau(U)\xi) \\
&\quad + U(b)N + b(-A_N(U) + \tau(U)N) \\
&= \nabla_U^* K_{\mathcal{S}} - a\overset{\star}{A}_{\xi}(U) - bA_N(U) + \{C(U, K_{\mathcal{S}}) + U(a) - a\tau(U)\}\xi \\
&\quad + \{B(U, K_{\mathcal{S}}) + U(b) + b\tau(U)\}N.
\end{aligned} \tag{3.16}$$

Matching the screen, the radical, and the transversal components of Eq.(3.15), we derive Eq.(3.12), Eq.(3.13), and Eq.(3.14).  $\square$

As a direct result of Proposition 3.1, we can outline the scenarios in which  $K$  is completely contained within  $\text{span}(\xi)$ ,  $\text{span}(N)$ , or  $\mathcal{S}(\zeta)$ .

**Corollary 3.1.** *Let  $K$  be a torse-forming vector field on  $(\bar{\Sigma}, \bar{g})$  and  $(\Sigma, E)$  a normalized null hypersurface of  $\bar{\Sigma}$ . Given  $U \in T\Sigma$ , the following holds :*

(i) *If  $K$  is tangent to  $\text{span}(\xi)$ , then*

$$a\overset{\star}{A}_{\xi}(U) + \Psi PU = 0, \quad \Psi\omega(U) + a\mu(U) = C(U, K_{\mathcal{S}}) + U(a) - a\tau(U). \tag{3.17}$$

(ii) *If  $K$  is tangent to  $\mathcal{S}(E)$ , then*

$$\begin{aligned}
\nabla_U^* K &= \Psi PU + \mu(U)K, \\
C(U, K) &= \Psi\omega(U), \quad C(PU, K) = B(U, K) = 0.
\end{aligned}$$

(iii) *If  $K$  is tangent to  $\text{span}(N)$ , then*

$$bA_N(U) = 0, \quad b\mu(U) = U(b) + b\tau(U).$$

(iv) *If  $K$  is tangent to  $\text{span}(\xi) \oplus \text{span}(N)$ , then*

$$\begin{aligned}
a\overset{\star}{A}_{\xi}(U) + bA_N(U) + \Psi PU &= 0, \quad \Psi\omega(U) + a\mu(U) = U(a) - a\tau(U), \\
b\mu(U) &= U(b) + b\tau(U).
\end{aligned}$$

*Proof.* If  $K$  is tangent to  $\xi$ , then  $K = a\xi$ , i.e.  $K_{\mathcal{S}} = 0$  and  $b = 0$ . Using this in Eq.(3.12) and Eq.(3.13), we have  $a\overset{\star}{A}_{\xi}(U) + \Psi PU = 0$  and  $a\Psi\omega(U) + a\mu(U) = U(a) - a\tau(U)$ , respectively. Which completes the proof of item (i). Next, if  $K$  is tangent to  $\mathcal{S}(E)$ , then  $a = b = 0$ . From this, Eq.(3.12), Eq.(3.13) and Eq.(3.14) leads to  $\nabla_U^* K_{\mathcal{S}} = \Psi PU + \mu(U)K_{\mathcal{S}}$ ,  $\Psi\omega(U) = C(U, K_{\mathcal{S}})$  and  $B(U, K_{\mathcal{S}}) = 0$ , respectively. Which gives the proof of item (ii). If  $K$  is tangent to  $N$ , then  $a = 0$ ,  $K_{\mathcal{S}} = 0$  and  $K = bN$  with  $b \neq 0$ . Using this in Eq.(3.12), we obtain  $bA_N(U) + \Psi PU = 0$ . But, Eq.(3.13) gives  $\Psi\omega(U) = 0$ , so in particular for  $U = \xi$ ,  $\Psi = 0$ . Thus  $bA_N = 0$ . Also, Eq.(3.14) leads to  $b\mu(U) = U(b) + b\tau(U)$ . Finally, if  $K$  is tangent to  $\text{span}(\xi) \oplus \text{span}(N)$ , then  $K_{\mathcal{S}} = 0$ , i.e.,  $K = a\xi + bN$ . Thus, from Eq.(3.12), Eq.(3.13) and Eq.(3.14), we have  $a\overset{\star}{A}_{\xi}(U) + bA_N(U) + \Psi PU = 0$ ,  $\Psi\omega(U) + a\mu(U) = U(a) - a\tau(U) = 0$ ,  $b\mu(U) = U(b) + b\tau(U)$ .  $\square$

Based on the decomposition in (3.11) and Corollary 3.1, we obtain the subsequent Theorem :

**Theorem 3.1.** *Let  $(\Sigma, E)$  be a normalized null hypersurface of a semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  admitting a non zero torse-forming vector field  $K$ .*

- (1) *If  $K$  belongs to  $span(\xi)$ , then  $\Sigma$  is totally umbilic.*
- (2) *If  $K$  belongs to  $span(N)$ , then  $\Sigma$  is totally screen geodesic and locally a product manifold  $c_\xi \times \overset{\circ}{\Sigma}$ , where  $c_\xi$  is a null curve tangent to  $span(\xi)$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ .*

*Proof.* Since  $K$  belongs to  $span(\xi)$ , from Eq. (3.17) of Corollary 3.1, we get  $a \overset{\star}{A}_\xi(U) + \Psi PU = 0, \forall U \in \mathfrak{X}(\Sigma)$ , that is  $\overset{\star}{A}_\xi(U) = -\frac{\Psi}{a}PU$  as  $a \neq 0$ , which implies that  $\Sigma$  is totally umbilic, which gives item (1). Next, as  $K$  belongs to  $span(N)$ , then from item (iii) of Corollary 3.1 it follows that  $A_N(U) = 0$  for any  $U \in \mathfrak{X}(\Sigma)$ , since  $b \neq 0$ . Thus  $\mathcal{S}(E)$  is totally geodesic, Moreover, from the first equation of (2.5), we see that  $\mathcal{S}(E)$  is parallel with respect to  $\nabla$ , and thus an integrable distribution on  $\Sigma$ . It follows that it is locally a product manifold  $c_\xi \times \overset{\circ}{\Sigma}$  where  $c_\xi$  is a null curve tangent to  $T\Sigma^\perp$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ , which gives item (2) and completes the proof. □

**Corollary 3.2.** *Let  $(\Sigma, E)$  be a normalized null hypersurface of a semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  admitting a non zero torse-forming vector field  $K$ . If  $K$  lies in  $span(N)$ , then  $\bar{\Sigma}$  admits no proper totally screen umbilic null hypersurface.*

**Corollary 3.3.** *Let  $(\Sigma, E)$  be a normalized null hypersurface of a semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  admitting a non zero torse-forming vector field  $K$ . If  $K$  lies in  $\mathcal{S}(E)$ , then  $\Sigma$  can not be a proper totally umbilic or proper totally screen umbilic null hypersurface.*

*Proof.* Suppose that  $\Sigma$  is totally umbilic and  $K$  belongs to  $\mathcal{S}(E)$ , then item (ii) of Corollary (3.1) and Eq.(2.7) imply that  $0 = B(X, K) = \beta \bar{g}(X, K)$ , for any  $X$  tangent to  $\mathcal{S}(E)$ . Since  $\mathcal{S}(E)$  is non-degenerate, we have  $\beta = 0$ , that is  $\Sigma$  is totally geodesic. Similarly, if  $\mathcal{S}(E)$  is totally umbilic, we prove that  $\lambda = 0$  from Eq.(2.8) and item (ii) of Corollary 3.1. □

Setting  $U = \xi$  in (3.13)-(3.14) and using equation (2.9) we have the following.

**Proposition 3.2.** *Let  $(\bar{\Sigma}, \bar{g})$  be a Lorentzian manifold equipped with a torse-forming vector field  $K$  with conformal factor  $\Psi$  and  $(\Sigma, E)$  a normalized null hypersurface of  $\bar{\Sigma}$ . Then we have the following partial differential equation of first order*

$$b(\tau(\xi) - \mu(\xi)) + \xi(b) = 0. \tag{3.18}$$

Moreover, if  $(\Sigma, E)$  is screen quasi-conformal, then

$$\Psi + a(\tau(\xi) + \mu(\xi)) - \xi(a) = 0, \tag{3.19}$$

where  $a = \bar{g}(K, N)$ ,  $b = \bar{g}(K, \xi)$  and  $\mu$  is the generating form.

From Eqs.(3.18) and (3.19) the following Corollaries hold.

**Corollary 3.4.** *Let  $(\bar{\Sigma}, \bar{g})$  be a Lorentzian manifold equipped with a timelike torse-forming vector field  $K$  with conformal factor  $\Psi$  and  $(\Sigma, E)$  a normalized null hypersurface. If  $b$  is constant along the integral curves of  $\xi$ , then  $\tau(\xi) = \mu(\xi)$ . Moreover, if  $(\Sigma, E)$  is screen quasi-conformal then  $\Psi$  satisfies the following relation  $\Psi = \xi(a) - 2a\tau(\xi)$ .*

*Proof.* As  $b$  is constant along the integral curves of  $\xi$ , we have from Eq.(3.18) that  $b(\tau(\xi) - \mu(\xi)) = 0$ , which implies that  $\tau(\xi) = \mu(\xi)$ . Indeed,  $b = 0$  leads to  $\bar{g}(K, K) = \bar{g}(K_{\mathcal{S}}, K_{\mathcal{S}})$  which is also not possible since  $K_{\mathcal{S}}$  is a spacelike vector and  $K$  is a timelike. Setting  $\tau(\xi) = \mu(\xi)$  in Eq.(3.19), the last claim follows.  $\square$

At this point, it is natural to ask under which circumstances we can choose the rigging vector field  $E$  in a way that the corresponding null hypersurface  $(\Sigma, E)$  is screen quasi conformal(see Defintion 2.9). Based on Example 2.1, it is worth noting that if the ambient space admits a closed conformal rigging vector field, then it induces a screen quasi-conformal structure in any null hyperfurface. As the next result shows, there is an analogous connection between torse forming vector fields and screen quasi conformal distributions, thus generalizing the above assertion.

**Theorem 3.2.** *Let  $(\Sigma, E)$  be a normalized null hypersurface of  $(\bar{\Sigma}, \bar{g})$  and  $K \in \Gamma(T\Sigma)$  a torse forming vector field with  $\bar{g}(K, K) \neq 0$ . If  $K$  is nowhere tangent to  $\Sigma$  then there exists a normalization  $E$  such that the corresponging screen distribution  $\mathcal{S}(E)$  is screen quasi conformal. Moreover,  $\Sigma$  is locally a product manifold  $c_{\xi} \times \overset{\circ}{\Sigma}$ , where  $c_{\xi}$  is a null curve tangent to  $span(\xi)$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ .*

*Proof.* Since  $K$  is nowhere tangent to  $\Sigma$ , we use it as a rigging  $E$  for  $\Sigma$ . Setting  $K = E$ , and using Eq.(2.2), we have that  $a = \frac{1}{2}\bar{g}(E, E)$  and  $b = 1$ . Using item (iii) of Corollary 3.1, we have the following equations:

$$\begin{aligned} \frac{1}{2}\bar{g}(E, E) \overset{\star}{A}_{\xi}(U) + A_N(U) + \Psi P U &= 0, \\ \Psi \omega(U) + \frac{1}{2}\bar{g}(E, E)\mu(U) &= +\frac{1}{2}U(\bar{g}(E, E)) - \frac{1}{2}\bar{g}(E, E)\tau(U), \\ \mu(U) = \tau(U) &= \bar{g}(\bar{\nabla}_U N, \xi). \end{aligned} \tag{3.20}$$

Thus,  $A_N = -\frac{1}{2}\bar{g}(E, E) \overset{\star}{A}_{\xi} - \Psi P$ , and we have that  $\Sigma$  is screen quasi-conformal with  $\kappa = -\frac{1}{2}\bar{g}(E, E)$  and  $\sigma = -\Psi$ . Then  $\mathcal{S}(E)$  is integrable and from [25],  $\Sigma$  is locally a product manifold  $c_{\xi} \times \overset{\circ}{\Sigma}$ , where  $c_{\xi}$  is a null curve tangent to  $span(\xi)$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ . Which completes the proof  $\square$

The following Lemmas provide some basic facts about torse-forming vector fields. To performed the previously mentioned action, we set  $\theta = g(K, K)$ , and let  $\gamma$  be the metrically one-form equivalent to  $K$  i.e  $\gamma = \bar{g}(\cdot, K)$ .

**Lemma 3.1.** *Given  $U, V \in \mathfrak{X}(\bar{\Sigma})$ , we have :*

- (i)  $d\gamma(U, V) = \mu(U)\gamma(V) - \mu(V)\gamma(U)$ ,
- (ii)  $(L_K \bar{g})(U, V) = 2\Psi \bar{g}(U, V) + \mu(U)\gamma(V) + \mu(V)\gamma(U)$ ,
- (iii)  $\bar{div}(K) = \Psi(n + 2) + \mu(K)$ .

*Proof.*  $\forall U, V \in \mathfrak{X}(\bar{\Sigma})$ , we have

$$\begin{aligned} d\gamma(U, V) &= \bar{g}(\bar{\nabla}_U K, V) - \bar{g}(U, \bar{\nabla}_V K) \\ &= \bar{g}(\Psi U + \mu(U)K, V) - \bar{g}(U, \Psi V + \mu(V)K) \\ &= \mu(U)\gamma(V) - \mu(V)\gamma(U), \end{aligned} \tag{3.21}$$

also,

$$\begin{aligned}
 (L_K \bar{g})(U, V) &= \bar{g}(\bar{\nabla}_U K, V) + \bar{g}(U, \bar{\nabla}_V K) \\
 &= \bar{g}(\Psi U + \mu(U)k, V) + \bar{g}(u, \Psi V + \mu(V)K) \\
 &= 2\Psi \bar{g}(U, V) + \mu(U)\gamma(V) + \mu(V)\gamma(U).
 \end{aligned}
 \tag{3.22}$$

Which give item (i) and (ii). From Eqs.(3.21) and (3.22), we obtain item (iii).

Now, let  $\{e_1, e_2, \dots, e_{n+2}\}$  be an orthonormal frame basis of  $(\bar{\Sigma}, \bar{g})$ . We compute the divergence of  $K$  as :

$$\begin{aligned}
 \overline{div}(K) &= \sum_{i=1}^{n+2} \epsilon_i \bar{g}(\bar{\nabla}_{e_i} K, e_i) = \sum_{i=1}^{n+2} \epsilon_i \bar{g}(\Psi e_i + \mu(e_i)K, e_i) \\
 &= \Psi \sum_{i=1}^{n+2} \epsilon_i \bar{g}(e_i, e_i) + \sum_{i=1}^{n+2} \epsilon_i \mu(e_i) \bar{g}(K, e_i) = (n+2)\Psi + \sum_{i=1}^{n+2} \epsilon_i \bar{g}(W, e_i) \bar{g}(K, e_i) \\
 &= (n+2)\Psi + \bar{g}(W, K) = (n+2)\Psi + \mu(K).
 \end{aligned}$$

□

**Lemma 3.2.** *Suppose that  $K \in \mathfrak{X}(\bar{\Sigma})$  is torse-forming vector field. Then for all  $U, V \in \mathfrak{X}(\bar{\Sigma})$ , we have*

- (i)  $\bar{\nabla}\theta = 2\Psi K + 2\theta W$ .
- (ii)  $\bar{\Delta}\theta = 2K(\Psi) + 2\Psi((n+2)\Psi + 3\mu(K)) + 2\theta(2\mu(W) + \overline{div}(W))$ .
- (iii) *If  $U \in \mathfrak{X}(\bar{\Sigma})$ , with  $U \perp K$ , then  $2\theta U(\Psi) = \Psi U(\theta) + 2\theta d\mu(K, U)$ .*
- (iv)  $\bar{R}_{UV}K = (U(\Psi) - \Psi\mu(U))V - (V(\Psi) - \Psi\mu(V))U + (d\mu(U, V))K$ .  
 $\bar{Ric}(U, K) = -(n+1)(\bar{g}(\bar{\nabla}\Psi, U) - \Psi\mu(U)) - d\mu(K, U)$ .

*Proof.*  $\forall U \in \mathfrak{X}(\bar{\Sigma})$  together with Eq.(3.10), give

$$\begin{aligned}
 g(\bar{\nabla}\theta, U) &= U.\theta = U\bar{g}(K, K) \\
 &= 2\bar{g}(\bar{\nabla}_U K, K) \\
 &= 2\bar{g}(\Psi U + \mu(U)K, K) \\
 &= \bar{g}(2\Psi K, U) + 2\bar{g}(K, K)\bar{g}(W, U) \\
 &= \bar{g}(2\Psi K + 2\theta W, U).
 \end{aligned}
 \tag{3.23}$$

The first item follows using the fact that  $\bar{g}$  is non-degenerate.

By definition, we get

$$\begin{aligned}
\bar{\Delta}\theta &= \overline{div}(\bar{\nabla}\theta) \\
&= \overline{div}(2\Psi K + 2\theta W) \\
&= 2K(\Psi) + 2\Psi\overline{div}K + 2W(\theta) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 2W\bar{g}(K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\bar{g}(\bar{\nabla}_W K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\bar{g}(\Psi W + \mu(W)K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\Psi\bar{g}(W, K) + 4\mu(W)\bar{g}(K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + 3\mu(K)) + 2\theta(2\mu(W) + \overline{div}(W)),
\end{aligned}$$

which gives item (ii).

(iii) From the first item and for any  $U \in \mathfrak{X}(\bar{\Sigma})$ , if  $U \perp K$ , we have

$$\begin{aligned}
\bar{g}(\bar{\nabla}\theta, U) &= U.\theta \\
&= \bar{g}(2\Psi K + 2\theta W, U) \\
&= 2\theta\bar{g}(W, U).
\end{aligned} \tag{3.24}$$

Since  $U \perp K$ , this leads to  $K(\theta) = 2\theta(\Psi + \mu(K)) = \bar{g}(\bar{\nabla}\theta, K)$ , and  $U(\theta) = 2\theta\mu(U)$ . Now taking the derivative of  $K(\theta)$  with respect to  $U$  gives

$$\begin{aligned}
U(K(\theta)) &= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta U(\mu(K)) \\
&= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta(\bar{g}(\bar{\nabla}_U W, K) + \bar{g}(W, \bar{\nabla}_U K)) \\
&= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta\mu(U)\mu(K) \\
&\quad + 2\theta\Psi\mu(U) + 2\theta\bar{g}(\bar{\nabla}_U W, K) \\
&= 3\Psi U(\theta) + 2\theta U(\Psi) + 3\mu(K)U(\theta) + 2\theta\bar{g}(\bar{\nabla}_U W, K).
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\text{But, } U(K(\theta)) &= U\bar{g}(\bar{\nabla}\theta, K) \\
&= \bar{g}(\bar{\nabla}_U \bar{\nabla}\theta, K) + \bar{g}(\bar{\nabla}\theta, \bar{\nabla}_U K) \\
&= \bar{g}(\bar{\nabla}_K \bar{\nabla}\theta, U) + \bar{g}(2\Psi K + 2\theta W, \Psi U + \mu(U)K) \\
&= \bar{g}(\bar{\nabla}_K (2\Psi K + 2\theta W), U) + 2\Psi U(\theta) + U(\theta)\mu(K) \\
&= 2K(\theta)\mu(U) + 2\theta\bar{g}(\bar{\nabla}_K W, U) + 2\Psi U(\theta) + U(\theta)\mu(K) \\
&= 4\Psi U(\theta) + 3\mu(K)U(\theta) + 2\theta\bar{g}(\bar{\nabla}_K W, U).
\end{aligned} \tag{3.26}$$

By equating Eq.(3.25) and Eq.(3.26), we obtain  $2\theta U(\Psi) = \Psi U(\theta) + 2\theta d\mu(K, U)$ , which gives item (iii).

Now, let  $U, V \in \mathfrak{X}(\bar{\Sigma})$ , we have

$$\begin{aligned} \overline{R}_{UV}K &= \overline{\nabla}_U \overline{\nabla}_V K - \overline{\nabla}_V \overline{\nabla}_U K - \overline{\nabla}_{[U,V]}K \\ &= \overline{\nabla}_U(\Psi V + \mu(V)K) - \overline{\nabla}_V(\Psi U + \mu(U)K) - \Psi[U, V] - \mu([U, V])K \\ &= U(\Psi)V + \Psi \overline{\nabla}_U V + U(\mu(V))K + \mu(V)\overline{\nabla}_U K - V(\Psi)U - \Psi \overline{\nabla}_V U - V(\mu(U))K \\ &\quad - \mu(U)\overline{\nabla}_V K - \Psi[U, V] - \mu([U, V])K \\ &= U(\Psi)V + U(\mu(V))K + \mu(V)(\Psi U + \mu(U)K) - V(\Psi)U - V(\mu(U))K \\ &\quad - \mu(U)(\Psi V + \mu(V)K) - \mu([U, V])K \\ &= (U(\Psi) - \Psi\mu(U))V - (V(\Psi) - \Psi\mu(V))U - (d\mu(U, V))K, \end{aligned}$$

which gives item (iv).

If  $\{e_1, e_2, \dots, e_{n+2}\}$  an orthonormal frame basis of  $(\bar{\Sigma}, \bar{g})$ , then using Eq.(3.10), we have

$$\begin{aligned} \overline{Ric}(U, K) &= \sum_{i=1}^{n+2} \epsilon_i \bar{g}(\overline{R}(e_i, U)K, e_i) \\ &= \sum_{i=1}^{n+2} \epsilon_i \bar{g}((e_i(\Psi) - \Psi\mu(e_i))U - (U(\Psi) - \Psi\mu(U))e_i - (d\mu(e_i, U))K, e_i) \\ &= \sum_{i=1}^{n+2} \epsilon_i \bar{g}((\bar{g}(\overline{\nabla}\Psi, e_i)U - \Psi\bar{g}(W, e_i)U - U(\Psi)e_i + \Psi\mu(U)e_i - (d\mu(e_i, U))K, e_i) \\ &= \sum_{i=1}^{n+2} \epsilon_i (\bar{g}(\overline{\nabla}\Psi, e_i)\bar{g}(U, e_i) - \Psi\bar{g}(W, e_i)\bar{g}(U, e_i) - U(\Psi)\bar{g}(e_i, e_i) + \Psi\mu(U)\bar{g}(e_i, e_i) \\ &\quad - d\mu(e_i, U)\bar{g}(K, e_i) \\ &= \bar{g}(\overline{\nabla}\Psi, U) - \Psi\bar{g}(W, U) - (n+2)U(\Psi) + (n+2)\Psi\mu(U) - d\mu\left(\sum_{i=1}^{n+2} \epsilon_i \bar{g}(K, e_i)e_i, U\right) \\ &= \bar{g}(\overline{\nabla}\Psi, U) - \Psi\mu(U) - (n+2)\bar{g}(\overline{\nabla}\Psi, U) + (n+2)\Psi\mu(U) - d\mu(K, U) \\ &= -(n+1)(\bar{g}(\overline{\nabla}\Psi, U) - \Psi\mu(U)) - d\mu(K, U), \end{aligned}$$

which gives item (iv) and completes the proof of the Lemma. □

**Corollary 3.5.** *From item (iv) of Lemma (3.2), for a torse-forming vector field  $K$  on an  $(n+2)$  dimensional semi-Riemannian manifold, we have :*

$$\overline{Ric}(K, K) = (n+1)(\Psi\mu(K) - \bar{g}(\overline{\nabla}\Psi, K)). \tag{3.27}$$

**Corollary 3.6.** *Let  $K \in \Gamma(\bar{\Sigma})$  be a torse-forming vector field and  $U \in \Gamma(\bar{\Sigma})$  such that  $\bar{g}(U, K) = \bar{g}(U, W) = 0$ . Then  $U \cdot \theta = 0$ .*

*Proof.* The proof follows from Eq. (3.24). □

The subsequent lemma serves as a generalization of [5, Lemma 5. p. 8], where it was assumed that  $K$  is a closed conformal vector field.

**Proposition 3.3.** *Let  $K \in \mathfrak{X}(\bar{\Sigma})$  be a torse-forming vector field on a Lorentzian manifold  $(\bar{\Sigma}, \bar{g})$ .*

- (i) If  $K$  is never null and it has constant length with  $\bar{g}(K, K) \neq 0$ , then  $\gamma$  is closed and  $\Psi^2 = \theta\mu(W)$ , where  $W$  is the dual vector of the generating form  $\mu$  and  $\theta = \bar{g}(K, K)$ .
- (ii) If  $K$  is never null torqued and has constant length, then either  $K$  is parallel or  $W$  is lightlike.
- (iii) If  $K$  is torqued, then either  $K$  is a null recurrent or  $W$  is lightlike.

*Proof.* We recall  $\theta = \bar{g}(K, K)$ . If  $\theta$  is constant then from item (i) of Lemma 3.2 it worth noting that  $\Psi K = -\theta W$ . Thus,  $d\gamma = 0$  as  $K$  and  $W$  are linearly related. Also setting  $U = K$  in Eq.(3.23) together with the fact that  $\theta$  is constant, we have  $0 = K(\theta) = 2\Psi\theta + 2\theta\mu(K)$ , which implies  $\Psi = -\mu(K)$ , that is,  $\Psi^2 = \theta\mu(W)$  by applying  $\mu$  to the first item of Lemma 3.2. Which proof item (i). Next, if  $K$  is torqued, i.e.,  $\mu(K) = 0$ , then from item (i), we have  $\theta\mu(W) = 0$  and  $\Psi = 0$ . From the later equation, we have for any  $U \in \mathfrak{X}(\bar{\Sigma})$   $0 = U\bar{g}(K, K) = 2\bar{g}(\bar{\nabla}_U K, K) = 2\bar{g}(\mu(U)K, K) = 2\mu(U)\bar{g}(K, K)$ , which implies  $\mu(U) = 0$  as  $\bar{g}(K, K) \neq 0$ , which gives the proof of item (ii). (iii) If  $K$  is torqued, i.e,  $\mu(K) = 0$ , then it is worth noting that  $0 = K \cdot \mu(K) = K \cdot \bar{g}(K, W) = \bar{g}(K, K)\bar{g}(W, W) = \theta\mu(W)$ . That is either,  $K$  is null or  $W$  is null. From Remark 3.1, the claims follows.  $\square$

We remember the following from [9],[15].

**Theorem 3.3.** [9] *Every torqued vector field  $K$  associated with a twisted product manifold  $I \times_f F$  is of the form  $K = \Psi f \partial_t$ , where  $t$  is an arc-length parameter of  $I$ ,  $\Psi$  is a nonzero function on  $F$ , and  $f$  is the twisting function.*

We have the following classification of torqued vector fields on Einstein manifolds, which arises as a consequence of Theorem 3.3.

**Theorem 3.4.** [9] *Every torqued vector field  $K$  on an Einstein manifolds is of the form*

$$K = \Phi Z, \tag{3.28}$$

where  $Z$  is a concircular vector field and  $\Phi$  is a function satisfying  $Z(\Phi) = 0$ .

Conversely, every vector field of the form (3.28) is a torqued vector field on an Einstein manifold.

To end up this section, we assume that  $K$  is a torqued vector field tangent to the screen distribution and we have the following Lemmas :

**Lemma 3.3.** *Let  $K$  be a torse-forming vector field on  $(\bar{\Sigma}, \bar{g})$ . Assume that  $K$  is a torqued vector field. For any  $U \in \mathfrak{X}(\bar{\Sigma})$  with  $U \perp K$ , we have:*

$$\Psi\mu(U) - U(\Psi) = d\mu(U, K) \quad \text{and} \quad \bar{Ric}(U, K) = (n+2)(\Psi\mu(U) - U(\Psi)). \tag{3.29}$$

*Proof.* From item (iv) of Lemma 3.2 together with the fact that  $K$  is torqued and  $\theta = \bar{g}(K, K)$ , we have

$$\begin{aligned} \bar{R}_{UK}K &= (U(\Psi) - \Psi\mu(U))K - (K(\Psi) - \Psi\mu(K))U + (d\mu(U, K))K \\ &= (U(\Psi) - \Psi\mu(U))K - K(\Psi)U + (d\mu(U, K))K. \end{aligned}$$

This leads to  $0 = \bar{g}(\bar{R}_{UK}K, K) = \theta(U(\Psi) - \Psi\mu(U)) + \theta d\mu(U, K)$ , which implies  $d\mu(U, K) = -U(\Psi) + \Psi\mu(U)$ , as  $\theta \neq 0$ . Replace this just in the second point of item (iv) of Lemma 3.2 give the second equality.  $\square$

From the previous Lemma and item (iii) of Lemma 3.2, for any  $U \in \mathfrak{X}(\bar{\Sigma})$  with  $U \perp K$ , it holds that  $\Psi$  and  $\theta$  are related by

$$2U(\Psi) - \Psi\mu(U) = \Psi U(\log(\sqrt{\theta})). \tag{3.30}$$

We have the following Lemma.

**Lemma 3.4.** *From the above Lemma and since  $\xi \perp K$  and  $N \perp K$  with  $\theta = \bar{g}(K, K)$ , we get :*

$$\begin{aligned} \xi(\theta) &= 2\theta\mu(\xi), & \text{and} & & N(\theta) &= 2\theta\mu(N). \\ \xi(\Psi) &= \mu(\xi), & \text{and} & & N(\Psi) &= \mu(N). \end{aligned}$$

**Corollary 3.7.** *If the dual vector field  $W$  of the generating form  $\mu$  belongs to  $\mathcal{S}(E)$  then :*

$$\xi(\theta) = N(\theta) = 0, \quad \text{and} \quad \xi(\Psi) = N(\Psi) = 0.$$

The main result of this section is the following Theorems.

**Theorem 3.5.** *Let  $(\Sigma, E)$  be a screen quasi-conformal normalized null hypersurface of a Lorentzian manifold  $(\bar{\Sigma}, \bar{g})$ , furnished with a nonzero torse-forming vector field  $K$  tangent to  $\mathcal{S}(E)$  with conformal scalar  $\Psi$  and generating vector field  $W$ . If  $W$  is tangent to  $span(\xi) \oplus span(N)$ , then  $\Sigma$  is locally a triple product manifold  $c_\xi \times c_K \times \Sigma'$ , where  $c_\xi$  is a null curve tangent to  $T\Sigma^\perp$ ,  $c_K$  is a curve of the line bundle  $\langle K \rangle$  and  $\Sigma'$  is an  $(n - 1)$ -dimensional leaf of the orthogonal complement of  $K_S$  in  $\mathcal{S}(\zeta)$ .*

*Proof.* Since  $\Sigma$  is screen quasi-constant then  $\mathcal{S}(E)$  is integrable and from [25],  $\Sigma$  is locally a product manifold  $c_\xi \times \overset{\circ}{\Sigma}$ , where  $c_\xi$  is a null curve tangent to  $span(\xi)$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ . Also, as  $K$  belongs to  $\mathcal{S}(E)$ , Corollary 3.1 gives  $\nabla_X^* K = \Psi PX + \mu(X)K$ ,  $\Psi = C(\xi, K)$ ,  $B(X, K) = 0$ . But,  $\Psi = C(\xi, K) \stackrel{(2.9)}{=} \kappa B(\xi, K) + \sigma g(\xi, K) = 0$ , and  $\mu(X) = 0$  as  $W \in span(\xi) \oplus span(N)$ . That is,  $K$  is parallel on  $\mathcal{S}(\zeta)$ . Moreover, it worth noting that  $g(K, \overset{\star}{\nabla}_X Y) = 0, \forall X, Y \in T \overset{\circ}{\Sigma}$  and orthogonal to  $K$ . Thus, if  $L$  is the complementary distribution of the line bundle  $\langle K \rangle$  spanned by  $K$  in  $\overset{\circ}{\Sigma}$ , then  $L$  is parallel and thus, an integrable distribution over  $\mathcal{S}(E)$ . Therefore, by de-Ram’s decomposition Theorem in [12], each leaf  $\overset{\circ}{\Sigma}$  of  $\mathcal{S}(E)$  is locally a product manifold  $c_K \times \Sigma'$ , where  $c_K$  is a curve of the line bundle  $\langle K \rangle$  spanned by  $K$  and  $\Sigma'$  is an  $(n - 1)$ -dimensional leaf of  $L$ . □

**Theorem 3.6.** *Let  $(\Sigma, E)$  be a screen integrable normalized null hypersurface of a Lorentzian manifold  $(\bar{\Sigma}, \bar{g})$ , admitting a nonzero torqued vector field  $K$  tangent to  $\mathcal{S}(\zeta)$ . Then  $\Sigma$  is locally a triple product manifold  $c_\xi \times I \times_f \Sigma'$ , where  $c_\xi$  is a null curve tangent to  $T\Sigma^\perp$ ,  $f$  is a non zero smooth function on an open set  $I$  with arc length  $t$ .  $\partial_t = e_1$  is a unit vector field satisfying  $K = \sqrt{\theta}e_1$ , and  $\Sigma'$  is a semi-Riemannian  $(n - 1)$ -dimensional manifold, where  $\theta = \bar{g}(K, K)$ .*

*Proof.* As  $\mathcal{S}(\zeta)$  is integrable,  $C$  is symmetric, hence,  $\Sigma$  is a product manifold  $c_\xi \times \overset{\circ}{\Sigma}$ , where  $c_\xi$  is a null curve tangent to  $T\Sigma^\perp$  and  $\overset{\circ}{\Sigma}$  is a leaf of  $\mathcal{S}(E)$ . Since  $K$  belongs to  $\mathcal{S}(E)$ , Corollary 3.1 gives

$$\nabla_X^* K = \Psi PX + \mu(X)K, \quad \Psi = C(\xi, K), \quad B(X, K) = 0.$$

Now let us extend  $e_1$  to a quasi-orthonormal basis  $\{e_1\}^\perp \cup \{e_1\} = \{e_1, e_2, \dots, e_n\}$  of  $\overset{\circ}{\Sigma}$ . Thus, from Eq.(3.30), we have  $2e_i(\Psi) - \Psi\mu(e_i) = \Psi e_i(\log(\sqrt{\theta}))$ . On the other hands we have :

$$\begin{aligned}\nabla_{e_i}^* K &= \Psi e_i + \mu(e_i)\sqrt{\theta}e_1, \\ \nabla_{e_i}^* K &= e_i(\sqrt{\theta})e_1 + \sqrt{\theta}\nabla_{e_i}^* e_1,\end{aligned}$$

which give

$$e_i(\log(\theta)) = 2\mu(e_i), \text{ and } \nabla_{e_i}^* e_1 = \frac{\Psi}{\sqrt{\theta}}e_i, \text{ with } i \neq 1$$

and since  $K$  is torqued, we get  $0 = \mu(K) = \sqrt{\theta}\mu(e_1)$ , which implies  $\mu(e_1) = 0$ .

Hence, by assuming  $\Psi \neq 0$ , we have :

$$\begin{aligned}2e_i(\Psi) &= \Psi\mu(e_i) + \Psi e_i(\log(\sqrt{\theta})) \\ &= \Psi e_i(\log(\sqrt{\theta})) + \Psi e_i(\log(\sqrt{\theta})) \\ &= 2\Psi e_i(\log(\sqrt{\theta})) \\ \frac{e_i(\Psi)}{\Psi} &= e_i(\log(\sqrt{\theta})) \implies e_i(\log(\Psi) - \log(\sqrt{\theta})) = 0.\end{aligned}$$

This implies that,  $\{e_1\}^\perp$  is an integrable distribution since  $\nabla_{e_i}^* \in \{e_1\}^\perp$  whose leaves are totally umbilical hypersurface of  $\overset{\circ}{\Sigma}$  with umbilicity factor  $-\frac{\Psi}{\sqrt{\theta}}$ . There for from [27, Proposition 35, p. 206], the leaf  $\overset{\circ}{\Sigma}$  is a semi-Riemannian warped product  $I \times_f \Sigma'$ , where  $f$  is a function on the open interval  $I$  and  $\Sigma'$  is a Riemannian  $(n-1)$ -dimensional manifold, and  $\theta$  is a positive function on  $I \times \Sigma'$ .

Since  $e_1 = \partial_t$ , we may set  $\sqrt{\theta} = \Psi f(t)$  and admit that  $\Psi$  is define only on  $\Sigma'$ .

Thus it is comes that  $\Sigma$  is locally in the form  $c_\xi \times I \times_{f(t)} \Sigma'$ .  $\square$

#### 4. COMPACT NULL HYPERSURFACES AND TORSE-FORMING VECTOR FIELD

The question here is to know if it is always possible to select  $E$  such that it is torse forming, along with prescribed geometric properties for the null hypersurface.

**Theorem 4.1.** *Let  $(\Sigma, E)$  be a simply connected compact normalized null hypersurface of Lorentzian manifold  $(\bar{\Sigma}, \bar{g})$  furnished with a torse-forming vector field  $E$ . Then the generating vector field  $W$  can not be a rigged for  $\Sigma$ .*

*Proof.* Assume there exists a compact null hypersurface in  $\bar{\Sigma}$  with a torse-forming vector field  $E$  and a rigged generating vector field  $W$ . Then  $\alpha = \gamma$  and  $\mu(U) = 0$  for all  $U \in \Gamma(T\Sigma)$ . Substituting this in Eq.(3.21), we have that  $d\gamma(U, V) = d\alpha(U, V) = 0$ ,  $\forall U, V \in \Gamma(T\Sigma)$ , which implies that  $\omega = i^*\alpha$  is a closed 1-form on  $\Sigma$  and since  $\Sigma$  is simply connected, there exists  $f \in C^\infty(\Sigma)$  such that  $\omega = df$ . It follows that  $\tilde{\nabla}f = W$ . However,  $\Sigma$  being compact, the function  $f$  has a critical point  $x \in \Sigma$ , where  $W(x) = 0$  and this contradicts the fact  $\tilde{g}(\tilde{\nabla}f, \tilde{\nabla}f) = 1$ .  $\square$

B.Y.Chen and K.Yano[11], introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifolds  $(\bar{\Sigma}, \bar{g})$  endowed with the curvature tensor  $\bar{R}$

satisfying the following equation:

$$\begin{aligned} \bar{g}(\bar{R}(U, V)Z, T) &= s\{\bar{g}(V, Z)\bar{g}(U, T) - \bar{g}(U, Z)\bar{g}(V, T)\} \\ &+ t\{\bar{g}(U, T)\theta(V)\theta(Z) - \bar{g}(U, Z)\theta(V)\theta(T) \\ &+ \bar{g}(V, Z)\theta(U)\theta(T) - \bar{g}(V, T)\theta(U)\theta(Z)\}, \end{aligned} \tag{4.31}$$

for any vector fields  $U, V, Z, T \in \Gamma(T\bar{\Sigma})$ , where  $s$  and  $t$  are smooth functions and

$$\theta(U) = \bar{g}(U, \zeta),$$

is  $\bar{g}$ -dual to a non vanishing smooth vector field  $\zeta$  called the curvature vector field of  $\bar{\Sigma}$ . It is well known that if the curvature tensor  $\bar{R}$  is of the form  $(1, 1)$ , then  $\bar{\Sigma}$  is conformally flat. If  $t = 0$ , then  $\bar{\Sigma}$  is a space of constant curvature.

Here, we suppose that the curvature vector field  $\zeta$  never belongs to the tangent space of the null hypersurface  $\Sigma$ . In this case  $\zeta$  can be taken as a rigging for  $\Sigma$ . This leads to the following definition.

**Definition 4.1.** [19] *A null hypersurface  $\Sigma$  of of Lorentzian manifolds of quasi-constant curvature such that the curvature vector field  $\zeta$  is a rigging for  $\Sigma$  is said to be  $\zeta$ -rigging null hypersurface.*

It is known that a totally umbilic null hypersurface with a closed normalization locally decomposes as a twisted product, [16, Theorem 5.3]. Here, we demonstrate that if it has a torse-forming rigging within an ambient of quasi-constant curvature, the local twisted product structure of the Riemann structure  $(\Sigma, \tilde{g})$  is actually a warped product.

**Theorem 4.2.** *Let  $(\bar{\Sigma}^{n+2}, \bar{g})$  be a simply connected Lorentzian manifold of quasi-constant curvature (with  $n \geq 2$ ) and  $\Sigma$  a  $\zeta$ -rigging totally umbilic null hypersurface such that  $\zeta$  is a torse forming vector field with the generating form  $W$  as a rigged for  $\Sigma$ . Then given  $x \in \Sigma$ ,  $(\Sigma, \tilde{g})$  is locally isometric to a warped product  $(\mathbb{R} \times \overset{\circ}{\Sigma}, dr^2 + f^2g_0)$ , where  $\overset{\circ}{\Sigma}$  is the leaf of  $\mathcal{S}(\zeta)$  through  $x$ , and  $g_0$  is a conformal metric to  $g_{|\overset{\circ}{\Sigma}}$ . Moreover, the decomposition is global whenever  $\Sigma$  is simply connected and the rigged vector field  $\xi$  complete, .*

*Proof.* Utilizing [16, Theorem 5.3] as mentioned earlier, our sole focus will be to demonstrate the warped decomposition of  $(\Sigma, \tilde{g})$ .

In [16, Theorem 4.8], it is demonstrated that for  $X, Y \in T\Sigma$ , the following is true,

$$R(X, Y)\xi - \tilde{R}(X, Y)\xi = \bar{g}(\bar{R}(X, Y)\xi, N)\xi - \tau(X)\overset{\star}{A}_\xi(Y) + \tau(Y)\overset{\star}{A}_\xi(X).$$

As the generating vector field  $W$  is rigged for  $\Sigma$ , then from Eq.(3.20), the 1-form  $\tau$  vanishes along the null hypersurfaces. Using the Gauss-Codazzi equation we have  $R(X, Y)\xi = \bar{R}(X, Y)\xi$ . Finally  $\bar{R}(X, Y)\xi = t(\alpha(Y)X - \alpha(X)Y)$  as  $(\bar{\Sigma}, \bar{g})$  has quasi-constant curvature (see (4.31)) and the above equality becomes  $\tilde{R}(X, Y)\xi = t(\alpha(Y)X - \alpha(X)Y)$ ,  $\forall X, Y \in \Gamma(T\Sigma)$ . Then,  $\widetilde{Ric}(U, \xi) = 0$  for all  $\mathcal{S}(\zeta)$ -valued vector field  $U$ . The conclusion is derived from the condition of mixed Ricci flat, as stated in [17, Theorem 1]. □

**Theorem 4.3.** *Let  $(\bar{\Sigma}, \bar{g})$  be a Lorentzian manifold of quasi-constant curvature with  $t_{|M}$  non-negative and  $\Sigma$  a  $\zeta$ -rigging simply connected compact null hypersurface of  $\bar{\Sigma}$ . If  $\zeta$  is a*

torse-forming vector field with conformal scalar  $\Psi$  and generating form  $\mu$ , then  $\Sigma$  is totally geodesic in  $\bar{\Sigma}$  and  $\mathcal{S}(\zeta)$  is totally umbilic in  $\bar{\Sigma}$  with umbilic factor  $\sigma$ .

*Proof.* Since  $\zeta$  is torse forming, from above Theorem 3.2,  $\zeta$  is screen quasi-conformal with  $A_N(U) = -\frac{1}{2}\bar{g}(\zeta, \zeta) \overset{*}{A}_\xi(U) - \Psi PU$  i.e  $\kappa = -\frac{1}{2}\bar{g}(\zeta, \zeta)$  and  $\sigma = -\Psi$ . From (2.6), we have

$$\bar{g}(\bar{R}(U, V)\xi, N) = C(V, \overset{*}{A}_\xi X) - C(U, \overset{*}{A}_\xi V) - 2d\tau(U, V), \tag{4.32}$$

for all  $U, V \in \Gamma(T\Sigma)$ . Given that  $\bar{\Sigma}$  possesses quasi-constant curvature, the left side of (4.32) vanishes. In addition, since  $\Sigma$  is screen quasi-conformal, we obtain

$$C(V, \overset{*}{A}_\xi U) - C(U, \overset{*}{A}_\xi V) = \bar{g}(A_N V, \overset{*}{A}_\xi U) - \bar{g}(A_N U, \overset{*}{A}_\xi V) = 0,$$

implying that  $d\tau = 0$ , i.e.,  $\tau$  is closed. Given that  $\Sigma$  is simply connected, we can find a function  $f$  defined on  $\Sigma$  such that  $\tau(U) = df(U)$  holds for every vector field  $U \in \Gamma(T\Sigma)$ . By introducing a new rigging defined as  $\hat{\zeta} = \gamma\zeta$ , the associated rigged vector field is expressed as  $\hat{\xi} = \gamma^{-1}\xi$ , leading to the relation  $\tau(U) = \hat{\tau}(U) + X(\log(\gamma^{-1}))$ . Hence, by setting  $\gamma = \exp(f)^{-1}$  in this expression, we conclude that  $\hat{\tau}(U) = 0$  for all  $U \in \Gamma(T\Sigma)$ .

Let  $\hat{\mathbf{H}}_\xi$  represent the mean curvature function and  $\overset{*}{A}_{\hat{\zeta}}$  denote the screen shape operator associated with  $\hat{\zeta}$ . It is a well-established fact that

$$\overline{Ric}(\hat{\xi}) = \hat{\xi}(n\hat{\mathbf{H}}_\xi) + n\hat{\tau}(\hat{\xi})\hat{\mathbf{H}}_\xi - \|\overset{*}{A}_{\hat{\zeta}}\|^2.$$

But  $\overline{Ric}(\hat{\xi}) = nt_{|\Sigma}$  and  $\tau(\hat{\xi}) = 0$ , it follows that  $\hat{\xi}(\hat{\mathbf{H}}_\xi) - |\overset{*}{A}_{\hat{\zeta}}|^2 - nt_{|\Sigma} = 0$ . From the inequality  $|\overset{*}{A}_{\hat{\zeta}}|^2 \geq \frac{1}{n}\hat{\mathbf{H}}_\xi^2$ , we get  $\hat{\xi}(\hat{\mathbf{H}}_\xi) - \frac{1}{n}\hat{\mathbf{H}}_\xi^2 - nt_{|\Sigma} \geq 0$ , and since  $\hat{\xi}$  is complete ( $\Sigma$  being compact), we obtain that  $\hat{\mathbf{H}}_\xi = 0$ . From the equation  $\hat{\xi}(\hat{\mathbf{H}}_\xi) - |\overset{*}{A}_{\hat{\zeta}}|^2 - nt_{|\Sigma} = 0$ , it can be deduced that  $|\overset{*}{A}_{\hat{\zeta}}|^2 = -nt_{|\Sigma}$ , resulting in  $\overset{*}{A}_{\hat{\zeta}} = 0$  and  $t_{|\Sigma} = 0$  on  $\Sigma$  given that  $t_{|\Sigma}$  is non-negative. Since  $\Sigma$  is totally geodesic and considering that  $\zeta$  possesses a quasi-conformal screen distribution, we can derive from Eq.(2.9) that  $\mathcal{S}(\zeta)$  is totally umbilic with the umbilic factor  $\sigma$ .  $\square$

**Theorem 4.4.** *Let  $(\bar{\Sigma}^4, \bar{g})$  be a 4-dimensional Lorentzian manifold of quasi-constant curvature with  $t_{|\Sigma}$  non-negative and  $\Sigma$  a compact  $\zeta$ -rigging null hypersurface. If  $\Sigma$  has finite fundamental group then the normalization  $\zeta$  can not be torse-forming.*

*Proof.* Let  $\Sigma$  be defined as above. Assuming there exists a torse-forming normalization  $\zeta$ , and given that  $\Sigma$  has a finite fundamental group, the first De Rham cohomology group  $H^1(\Sigma, \mathbb{R})$  must be trivial. From Theorem 4.3, it can be deduced that  $\Sigma$  is totally geodesic. Additionally, since  $\zeta$  is established as torse-forming,  $\mathcal{S}(\zeta)$  is integrable and leads to a foliation on  $\Sigma$ . We demonstrate the leaves of  $\mathcal{S}(\zeta)$  are totally geodesic in  $(\Sigma, \tilde{g})$ . To support this, let's recall (from [16], Proposition 3.7) that for  $U$  and  $V$  in  $\mathcal{S}(\zeta)$ ,

$$\tilde{\nabla}_U V = \overset{*}{\nabla}_U V - \tilde{g}(\tilde{\nabla}_U \xi, V)\xi,$$

but we also have

$$\tilde{g}(\tilde{\nabla}_U \xi, V) + \tilde{g}(\tilde{\nabla}_V \xi, U) = L_\xi \tilde{g}(U, V) = -2B(U, V).$$

Given that  $\mathcal{S}(\zeta)$  is integrable, we conclude that  $\tilde{g}(\tilde{\nabla}_U \xi, V) = \tilde{g}(\tilde{\nabla}_V \xi, U)$ . This leads to the result that  $\tilde{g}(\tilde{\nabla}_U \xi, V) = -B(U, V)$ , from which we derive that

$$\tilde{\nabla}_U V = \overset{\star}{\nabla}_U V + B(U, V)\xi.$$

In other words, the second fundamental form of each leaf in  $\mathcal{S}(\zeta)$  within the manifold  $(M, \tilde{g})$  is  $B$ , leading to the conclusion that all of these leaves are totally geodesic in  $(\Sigma, \tilde{g})$ , since  $\Sigma$  is totally geodesic in the  $(\bar{\Sigma}^4, \bar{g})$ . Consequently, there exists a totally geodesic foliation of codimension one on the compact 3-manifold  $\Sigma$ , which implies that  $\Sigma$  must possess an infinite fundamental group (refer to [18]), resulting in a contradiction.  $\square$

### 5. $K_{\mathcal{S}}$ AS A RICCI SOLITON POTENTIAL ON LEAVES OF $\mathcal{S}(E)$

Consider a normalized null hypersurface  $(\Sigma, E)$  within a  $(n+2)$ -dimensional semi-Riemannian manifold  $(\bar{\Sigma}, \bar{g})$  that possesses a torse-forming vector field  $K$ . In this context, we examine the screen distribution  $\mathcal{S}(E)$ , which is integrable on  $\Sigma$ , ensuring that the structure  $(\overset{\circ}{\Sigma}, \overset{\circ}{g}, K_{\mathcal{S}}, \Lambda)$  forms a Ricci soliton. Here,  $\overset{\circ}{\Sigma}$  denotes a leaf of  $\mathcal{S}(E)$ ,  $\Lambda$  is constant, and  $\overset{\circ}{g} = g|_{\mathcal{S}(E)}$ . In this scenario, we have :

$$L_{K_{\mathcal{S}}} \overset{\circ}{g} + 2 \overset{\circ}{Ric} = 2\Lambda \overset{\circ}{g}, \tag{5.33}$$

where  $L_{K_{\mathcal{S}}}$  is the Lie derivative operator,  $\overset{\circ}{Ric}$  is the Ricci tensor of  $\overset{\circ}{\Sigma}$  and  $\Lambda$  is a real constant.

**Proposition 5.1.**  $(\overset{\circ}{\Sigma}, \overset{\circ}{g}, K_{\mathcal{S}}, \Lambda)$  is a Ricci soliton if and only if :

$$\overset{\circ}{Ric}(U, V) = (\Lambda - \Psi) \overset{\circ}{g}(U, V) - aB(U, V) - bC(U, V) - \frac{1}{2}\{\gamma(U)\mu(V) + \gamma(V)\mu(U)\}, \tag{5.34}$$

for any  $U, V \in \overset{\circ}{\Sigma}$ .

*Proof.* As  $\overset{\star}{\nabla}$  is a  $\overset{\circ}{g}$ -metric connection, for any  $U, V \in \overset{\circ}{\Sigma}$ , from Eq.(3.12) we have :

$$\begin{aligned} L_{K_{\mathcal{S}}} \overset{\circ}{g}(U, V) &= \overset{\circ}{g}(\nabla_U^{\star} K_{\mathcal{S}}, V) + \overset{\circ}{g}(U, \nabla_V^{\star} K_{\mathcal{S}}) \\ &= \overset{\circ}{g}(a \overset{\star}{A}_{\xi}(U) + bA_N(U) + \Psi U + \mu(U)K_{\mathcal{S}}, V) \\ &+ \overset{\circ}{g}(U, a \overset{\star}{A}_{\xi}(V) + bA_N(V) + \Psi V + \mu(V)K_{\mathcal{S}}) \\ &= 2\Psi \overset{\circ}{g}(U, V) + 2aB(U, V) + 2bC(U, V) + \gamma(U)\mu(V) + \gamma(V)\mu(U). \end{aligned}$$

From this together with Eq.(5.33), the claim follows.  $\square$

If  $aB + bC = \eta g$ , for some smooth function  $\eta$ , then  $\Sigma$  is screen quasi-conformal and Eq.(5.34) becomes :

$$\overset{\circ}{Ric}(U, V) = (\Lambda - \Psi - \eta) \overset{\circ}{g}(U, V) - \frac{1}{2}\{\gamma(U)\mu(Y) + \gamma(V)\mu(U)\}.$$

This leads to the following Corollary

**Corollary 5.1.** Under the previous mentioned hypothesis, the scalar curvature  $\overset{\circ}{Sc}$  of  $\overset{\circ}{\Sigma}$  is :

$$\overset{\circ}{Sc} = n(\Lambda - \Psi - \eta) - \overset{\circ}{g}(K_{\mathcal{S}}, W_{\mathcal{S}}). \tag{5.35}$$

Additionally, if the dual vector  $W$  of the torse-forming vector field belongs to  $(span(\xi) \oplus span(N))$ , then  $\mu(X) = 0$  for any  $X \in \overset{\circ}{\Sigma}$ . From this, Eq.(5.35) becomes  $\overset{\circ}{S}c = n(\Lambda - \Psi - \sigma)$ .

**Theorem 5.1.** *Let  $K$  be a torse-forming vector field on a Lorentzian manifold  $(\overset{\circ}{\Sigma}^{n+1}, \overset{\circ}{g})$  where  $n \geq 4$ , and let  $(\Sigma, E)$  be a normalized null hypersurface of  $\overset{\circ}{\Sigma}$ . Suppose that  $(\overset{\circ}{M}, \overset{\circ}{g}, K_{\mathcal{S}}, \Lambda)$  is a Ricci soliton on  $\Sigma$  with  $aB + bC = \eta \overset{\circ}{g}$ , where  $\eta$  is smooth function, and the dual vector field  $W$  of the generating form  $\mu$  is a rigging. Then the following are true :*

- (i)  $\Psi + \eta$  is constant function, and given by  $\Psi + \sigma = \Lambda$  ;
- (ii)  $\Sigma = c_{\xi} \times I \times_{f(t)} \Sigma'$ , where  $c_{\xi}$  is a null curve tangent to  $T\Sigma^{\perp}$ ,  $I$  is an open interval with arc length  $t$ ,  $f$  is smooth function on  $\Sigma'$ , and  $\Sigma'$  is an Einstein  $(n - 2)$ -dimensional manifold,

*Proof.* Since the screen distribution  $\mathcal{S}(E)$  is integrable (given that  $a \overset{\star}{A}_{\xi} + bA_N = \eta P$ ), Theorem 3.2 implies that  $\Sigma$  is locally structured as a product manifold, specifically  $c_{\xi} \times \Sigma'$ , where  $c_{\xi}$  represents a null curve tangent to  $T\Sigma^{\perp}$  and  $\overset{\circ}{\Sigma}$  is a Riemannian leaf of  $\mathcal{S}(E)$ . Furthermore, by utilizing Eq.(5.34) alongside the fact that  $W$  serves as a rigging for  $\Sigma$  and  $aB + bC = \eta \overset{\circ}{g}$ , we get  $\overset{\circ}{Ric}(U, V) = (\Lambda - \Psi - \eta) \overset{\star}{g}(U, V)$ , for all  $U, V \in \mathcal{S}(E)$ . Which means that  $\overset{\circ}{M}$  is Einstein manifold. Since  $dim\Sigma \geq 4$ , it follows that  $dim \overset{\circ}{\Sigma} \geq 3$ , thus  $\overset{\circ}{\Sigma}$  has constant curvature ([10]). Thus  $\Lambda - \Psi - \eta$  is constant, hence  $\Psi + \eta$  is constant. Moreover, by Eq.(3.12) with the fact that  $W$  is a rigging for  $\Sigma$ , we have  $\nabla_U^* K_{\mathcal{S}} = (\Psi + \eta)U$ . Now it is worth noting that

$$\begin{aligned} \overset{\circ}{R}(U, K_{\mathcal{S}})K_{\mathcal{S}} &= \nabla_U^* \nabla_{K_{\mathcal{S}}}^* K_{\mathcal{S}} - \nabla_{K_{\mathcal{S}}}^* \nabla_U^* K_{\mathcal{S}} - \nabla_{[U, K_{\mathcal{S}}]}^* K_{\mathcal{S}} \\ &= \nabla_U^* (\Psi + \sigma)K_{\mathcal{S}} - \nabla_{K_{\mathcal{S}}}^* (\Psi + \sigma)U - (\Psi + \sigma)[U, K_{\mathcal{S}}] \\ &= U(\Psi + \eta)K_{\mathcal{S}} - K_{\mathcal{S}}(\Psi + \eta)U \\ &= 0. \end{aligned}$$

This leads to

$$\overset{\circ}{Ric}(K_{\mathcal{S}}, K_{\mathcal{S}}) = \sum_{i=1}^{n-1} \overset{\circ}{g}(\overset{\circ}{R}(e_i, K_{\mathcal{S}})K_{\mathcal{S}}, e_i) = 0.$$

Using this with Eq.(5.34), we get  $(\Lambda - \Psi - \eta) \overset{\circ}{g}(X, Y) = 0$ , which gives  $\Lambda = \Psi + \eta$  since  $\overset{\circ}{g}$  is Riemannian. To show item (ii), we see that  $K_{\mathcal{S}}$  is closed conformal vector field on  $\overset{\circ}{\Sigma}$ , and by Theorem 3.6  $\overset{\circ}{\Sigma}$  is a warped product manifold  $I \times_{f(t)} \Sigma'$ , where  $f(t)$  is a positive smooth function. □

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