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A GENERALIZED TOPOLOGICAL CONVERGENCE OF FUNCTION'S SEQUENCES CONFINED BY WEIGHTS

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Abstract. In this article, an extended method based on weight functions for pointwise and uniform statistical convergence for function sequences in topological settings is presented. Several fundamental theorems are proven and new definitions are suggested in order to rigorously describe these sorts of generalized convergence. Illustrations and counterexamples that highlight the differences and advantages of the suggested approaches provide further support for the theoretical development. Furthermore, this study has also examined the topological implications of these new types of convergence with the presence of Dini's theorem.

Keywords: Sequence of function, Weighted statistical convergence, Weighted statistical limit of sequence of function, Weight function.

MSC (2020): 40A30, 54A20.

1. Introduction

The traditional concept of convergence for sequences frequently turns out to be inappropriate for some findings. To overcome this restriction, H. Fast proposed the concept of statistical convergence in 1951 [15] with the advancement of A. Zygmund's summability technique [30], providing a more adaptable substitute that allows for a limited number of terms differ from the limit. Independently, J.A. Schoenberg [24] explored the same, connecting statistical convergence with summability theory. The notion of natural density (also known as asymptotic density) for a set $Q \subseteq \mathbb{N}$, has been defined as,

$$\delta(Q) = \lim_{n\to\infty} \frac{|\{l \leq n : l \in Q\}|}{n}$$

if the limit exists.

A significant breakthrough was made in 2008 when G. Di Maio and L. Kočinac [21] expanded the idea to topological spaces, allowing for the analysis of convergence behavior beyond purely metric or numerical contexts. As evidenced by further research [4, 16, 18, 19],

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this cleared the path for more generalizations and applications, expanding the theory's use across a different branch of subjects.

Within the realm of function spaces, statistical convergence gained prominence with the work of A. Gökhan and M. Güngür [17] in 2002. They introduced pointwise statistical convergence and statistical Cauchy conditions for real-valued function sequences, demonstrating their equivalence under suitable conditions. Their results established a solid foundation for extending statistical convergence to function-level analysis. Later, Balcerzak, Dems, and Komisarski [3] contributed further by considering statistical and ideal convergence for function sequences in metric contexts.

Explorations into more generalized normed structures followed, with Sarabadan and Talebi [23], and subsequently Yegül and Dündar [27], examining statistical convergence within 2-normed spaces. These efforts revealed rich structural behavior of function sequences under alternative topological constraints and highlighted the versatility of statistical convergence as an analytical tool. Numerous work on statistical convergence can be found in [5, 6, 7, 8, 9, 11, 12, 14, 25, 28, 29].

In an effort to refine these convergence methods further, the concept of weighted statistical convergence emerged. By replacing the standard counting function with a suitable weight function g, a generalized weighted density was introduced in 2020 by Adem et. al. [1] that defined as,

$$\delta_g(Q) = \lim_{n\to\infty} \frac{|\{l \leq n : l \in Q\}|}{g(n)}.$$

A generalized work has been paved way in the field of Topology by Das et. al. in 2024 [10] using the weight function g that offers some important applications in the field of weighted statistical convergence in topological space.

In parallel to these developments, a classical result such as Dini's theorem which asserts the uniform convergence of monotonic, pointwise-converging sequences of continuous functions on compact domains have attracted renewed interest. Various researchers have investigated how this theorem might be extended under statistical convergence schemes on metric spaces and normed spaces.

In view of these developments, this current study presents a framework for examining uniform statistical convergence and pointwise statistical convergence of function sequences in topological spaces by the weights. Lastly, the work provides novel versions of Dini's theorem in the weighted context.

2. Preliminaries

Before delving into the new variant of statistical convergence, some prerequisite definitions, terms and notations are added for the reader's convenience. For common symbols, nomenclature and concepts we follow [13].

Definition 2.1. [1] *Let $g : \mathbb{N} \longrightarrow [0, \infty)$ be a function is called weight function if* $\lim_{n\to\infty} g(n) = \infty$ *and* $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$.

Definition 2.2. [2] *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions where $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ for all $n \in \mathbb{N}$. Then for an arbitrary $a \in X$,*

$$\diamondsuit^{-}_{f(a)}(f_n(a)) = \{n \in \mathbb{N} : \{f_n(a)\} \cap V = \emptyset \textit{ for at least one neighborhood } V \textit{ of } f(a)\}.$$

Definition 2.3. [10] *Let g be a weight function. A sequence $\{x_n : n \in \mathbb{N}\}$ is said to be g-statistically convergent to x_0 in a topological space (X, τ), if for every neighborhood U of x_0, $\delta_g(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.*

Theorem 2.1. [22] *A pointwise convergent sequence $\{f_n\}$ of functions is also uniformly convergent f on A if the following conditions are satisfied:*

i) *A is compact set in a metric space (X, d);*
ii) *f and each f_n is continuous on A for $n \in \mathbb{N}$;*
iii) *$\{f_n\}$ is decreasing: $0 \leq f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$ and all $x \in A$.*

3. On s_g-limit for sequence of function

Based on the operator approach of statistical convergence for sequence of function by the author of [2], this paper aims to propose a new variant of statistical convergence restricted with a weight function to control the rate of convergence under the topological settings.

Throughout the paper, a mapping $g : \mathbb{N} \to \mathbb{R}^+$ is considered as the weight function defined as $g(n) = \log(1 + n)$ and $g(n) = n^p$ for all $n \in \mathbb{N}$ and $0 < p \leq 1$ with the existence of $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$. These weight functions are used for the development of examples.

Definition 3.1. *A sequence of function $(\varphi_m(x))_{m\in\mathbb{N}}$ where $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ is termed as pointwise weighted statistically convergent (or shortly pointwise s_g-convergent) to $\varphi(x)$, if $\delta_g(\diamondsuit^{-}_{\varphi(x)}(\varphi_m(x))) = 0$ for all $x \in X$. Mathematically indicated as Pt-$s_g - \lim_{m\to\infty} \varphi_m = \varphi$ or $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$.*

Definition 3.2. *A sequence of function $(\varphi_m(x))_{m\in\mathbb{N}}$ where $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ is termed as uniform weighted statistically convergent (or shortly uniform s_g-convergent) to $\varphi(x)$, if $\delta_g(\bigcup_{x\in X} \diamondsuit^{-}_{\varphi(x)}(\varphi_m(x))) = 0$. Mathematically indicated as U-$s_g - \lim_{m\to\infty} \varphi_m = \varphi$ or $\varphi_m \overset{U-s_g-\lim}{\longrightarrow} \varphi$.*

Theorem 3.1. *Every uniform s_g-convergent sequence of function is pointwise s_g-convergent.*

Proof. Let $(\varphi_m)_{m\in\mathbb{N}}$ be uniform s_g-convergent sequence of function that converges to φ, i.e., $\delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$. Now, it is obvious that $\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\} \subseteq \bigcup_{x\in X}\{m \in \mathbb{N} : \varphi(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}$ that gives $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) \leq \delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$. As a result, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$ for all $x \in X$. Thus, $\delta_g(\diamondsuit^{-}_{\varphi(x)}\varphi_m(x)) = 0$ for all $x \in X$. Hence $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$. □

Example 3.1. *Any sequence of function that is pointwise s_g-convergent does not imply the sequence of function to be uniform s_g-convergent.*

Let us consider a function $\varphi_m, \varphi : (X,\tau) \longrightarrow (Y,\sigma)$ for all $m \in \mathbb{N}$ such that $Y = \{p_1, p_2\}$ and $\sigma = \{\emptyset, \{p_1\}, Y\}$ defined for all $x \in X$ and let the sequence of function $(\varphi_m(x))$ is defined by

$$\varphi_m(x) = \begin{cases} p_2, & \text{if } m = l^l : l \in \mathbb{N} \\ p_1, & \text{otherwise} \end{cases}$$

Here the neighborhoods of $\varphi(x) = p_1$ are $W_1 = \{p_1\}$ and $W_2 = \{Y\}$. For the neighborhood W_1 of p_1, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_1 = \emptyset$ for the neighborhood $\{p_1\}$ of $p_1\}) = \delta_g(\{1, 4, 27, 256, \cdots\})$ $= 0$ and for the neighborhood W_2 of p_1, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_2 = \emptyset$ for the neighborhood Y of $p_1\}) = \delta_g(\emptyset) = 0$ for all $x \in X$.
Thus, for every $x \in X$, $\varphi_m(x) \overset{Pt-s_g-\lim}{\longrightarrow} \varphi(x)$.
But, $\delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) \neq 0$.
Thus, $\varphi_m(x) \overset{U-s_g-\lim}{\not\longrightarrow} \varphi(x)$.

Example 3.2. *Limit of a pointwise s_g-convergent sequence of function may have more than one limit point.*
Let us consider a function $\varphi_m, \varphi : (X,\tau) \longrightarrow (Y,\sigma)$ such that $Y = \{a_1, a_2, a_3\}$ and $\sigma = \{\emptyset, Y, \{a_1, a_2\}, \{a_3\}\}$. Then for all $m \in \mathbb{N}$ the sequence of function (φ_m) is defined as,

$$\varphi_m = \begin{cases} a_1, & \text{if } m = l^l, l \in \mathbb{N} \\ a_2, & \text{otherwise} \end{cases}$$

Here the open neighborhoods of a_2 are $W_1 = \{a_1, a_2\}$ and $W_2 = Y$. For the neighborhood W_1 of a_2, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_1 = \emptyset$ for the neighborhood $\{a_1, a_2\}$ of $a_2\}) = \delta_g(\{1, 4, 27, \cdots\}) = 0$ and for the neighborhood W_2 of a_2, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_2 = \emptyset$ for the neighborhood Y of $a_2\}) = \delta_g(\emptyset) = 0$ for all $x \in X$.
Thus, for every neighborhood W of a_2, $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} a_2$. But the neighborhoods of a_2 are same as the neighborhoods of a_1. So, for every neighborhood of a_1, $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} a_1$.
Thus, limit of a pointwise s_g-convergent sequence of function has more than one limit point.

Theorem 3.2. *Every pointwise weighted statistically convergent sequence of function does not have more than one limit point in a Hausdorff co-domain space.*

Proof. Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence of function that converges pointwise weighted statistically to two limit points $\varphi(b)$ and $\varpi(b)$ with $\varphi(b) \neq \varpi(b)$ for any specified $b \in X$.
But (Y,σ) is a Hausdorff space so, there exist two open neighborhoods $W_1, W_2 \subseteq \sigma$ of the limit points $\varphi(b)$ and $\varpi(b)$ such that $\varphi(b) \in W_1$, $\varpi(b) \in W_2$ and also $W_1 \cap W_2 = \emptyset$ that implies $W_2 \subseteq W_1^{\complement}$.
Since $W_2 \subseteq W_1^{\complement}$ so, for a specified $b \in X$,
$\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\} \subseteq \{m \in \mathbb{N} : \varphi_m(b) \cap W_1^{\complement} \neq \emptyset$ for at least one neighborhood W_1 of $\varphi(b)\}$.
i.e., $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) \leq \delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_1^{\complement} \neq \emptyset$ for at least one neighborhood W_1 of $\varphi(b)\}) = 0$.
i.e, $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) = 0$. But $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 = \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) \neq 0$ which contradicts the fact that $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi(b)$.

As a result, $\varphi(b) = \varpi(b)$ for a specified $b \in X$.
Hence, limit of a pointwise s_g-convergent sequence of function is always unique in Hausdorff co-domain space. $\square$

Theorem 3.3. *If $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ and $\varpi_m, \varpi : (Y, \sigma) \longrightarrow (Z, \eta)$ are functions such that $\varpi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi$, then $\varpi_m \circ \varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi \circ \varphi$.*

Proof. As, $\varpi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi$ so, $\delta_g(\Diamond^-_{\varpi(y)}\varpi_m(y)) = 0$ for all $y \in Y$.
Let $b \in X$ be arbitrary. Therefore, $\varphi(b) \in Y$ that results $\delta_g(\Diamond^-_{\varpi(\varphi(b))}\varpi_m(\varphi_m(b))) = 0$.
Thus, $\delta_g(\Diamond^-_{\varpi\circ\varphi(b)}\varpi_m \circ \varphi_m(b)) = 0$ for all $x \in X$. Hence $\varpi_m \circ \varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi \circ \varphi$. $\square$

From the article [10], it is clear that every s_g-convergent sequence is an s-convergent. So, it can be possible to show that every pointwise s_g-convergent sequence of function is pointwise s-convergent. Also every uniform s_g-convergent sequence of function is uniform s-convergent. For this applications, the following diagram is depicted below.

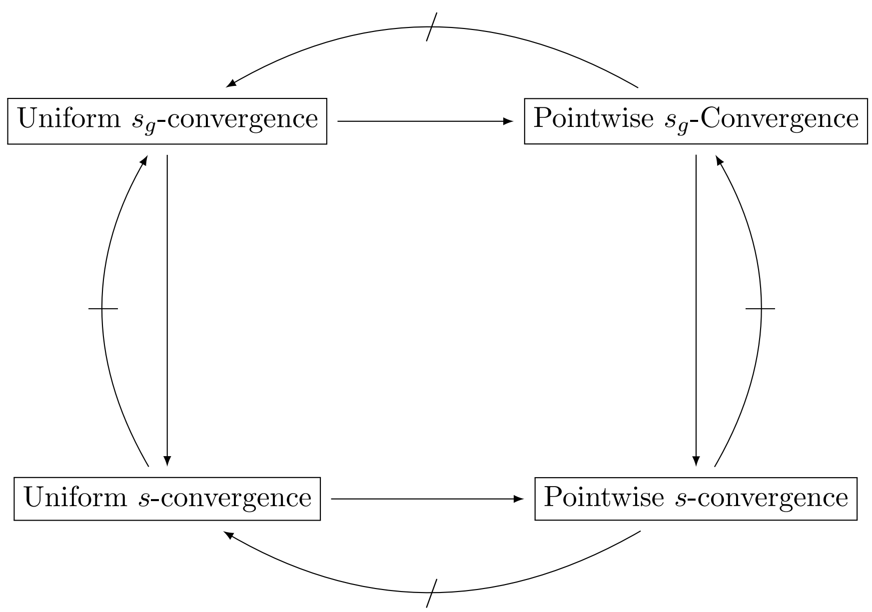

Figure 1. Relationship chart

Theorem 3.4. *A pointwise weighted statistical convergent sequence of function (φ_n) that converges to φ is also uniform weighted statistically convergent to φ, if the following conditions holds,*

i) *X is countable and (X, τ) is sequential compact.*
ii) *φ and every φ_n is continuous on X for all $n \in \mathbb{N}$.*
iii) *φ_n is monotonic for all $n \in \mathbb{N}$.*

Proof. Let $\varphi_n \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$ and φ_n is a monotone increasing sequence. Therefore, $\delta_g(\Diamond^-_{\varphi(x)}\varphi_n(x)) = 0$ for all $x \in X$.
Let $\varpi_n(x) = \varphi_n(x) - \varphi(x)$ for all $x \in X$ therefore, $\varpi_n \overset{Pt-s_g-\lim}{\longrightarrow} 0$ as φ_n is increasing. Since

φ and φ_n are continuous, ϖ_n is also continuous for all $n \in \mathbb{N}$. Therefore, $\delta_g(\diamondsuit_0^- \varpi_n(x)) = 0$ for all $x \in X$.
If possible, let us assume $\delta_g(\bigcup_{x_m \in X} \diamondsuit_0^- \varpi_n(x_m)) \neq 0$. Then by sequential compactness $\{x_m\}$ has a convergent sub sequence $x_{m_k} \longrightarrow x_0 \in X$. Also, ϖ_n is continuous so, by continuity $\varpi_n(x_{m_k}) \longrightarrow \varpi_n(x_0)$. But, $\varpi_n(x_0) \overset{Pt-s_g-\lim}{\longrightarrow} 0$. So, $\varpi_n(x_{m_k}) \overset{U-s_g-\lim}{\longrightarrow} 0$ for all $k \in \mathbb{N}$ which contradicts the fact that $\delta_g(\bigcup_{x_m \in X} \diamondsuit_0^- \varpi_n(x_m)) \neq 0$.
Hence $\varphi_n \overset{U-s_g-\lim}{\longrightarrow} \varphi$. □

4. Conclusion

In this work, we developed two new variants of statistical convergence (pointwise and uniform) for function's sequences that controlled with the weight function g to witness the pace of convergence under topological space and also established the relation among them. We showed that the pointwise weighted statistically convergent sequence of functions admits multiple limit points in a topological space while in a Hausdorff codomain space, the limit becomes unique. It was further shown that the composition of two different pointwise weighted statistical convergent sequence of function also result a pointwise statistical convergent. Lastly, a theorem has been developed to generalize the Dini's theorem with the weight function, that provides a broader framework for convergence of function's sequences.

This work may be further extended by the regulation of new parameter α that will effect the rate of convergence under topological settings.

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