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TRANSLATION HYPERSURFACES WITH CONSTANT CURVATURE IN 4-DIMENSIONAL ISOTROPIC SPACE

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Abstract. In the present study, we deal with translation hypersurfaces in the 4-dimensional isotropic space $\mathbb{I}^4$ generated by translating planar curves. Due to absolute figure of $\mathbb{I}^4$ there are four different types of such hypersurfaces. We classify these translation hypersufaces in $\mathbb{I}^4$ with constant Gauss-Kronecker and mean curvature.

1. Introduction

Dillen et al. [8] introduced a *translation hypersurface* M^{n-1} in a $n-$dimensional Euclidean space $\mathbb{R}^n$ as the graph of the form

$$y_n = f_1(y_1) + ... + f_{n-1}(y_{n-1}), \tag{1.1}$$

where $(y_1, ..., y_n)$ denote orthogonal coordinates in $\mathbb{R}^n$ and $f_1, ..., f_n$ smooth functions of single variable. The authors in [8] proved that if M^{n-1} is minimal, it is either a hyperplane or $M^{n-1} = M^2 \times \mathbb{R}^{n-3}$, where M^2 is the *Scherk's minimal surface* (see [34]) given in explicit form

$$y_3 = \frac{1}{c} \ln \left| \frac{\cos(cy_2)}{\cos(cy_1)} \right|, \ c \in \mathbb{R}, \ c \neq 0.$$

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In many different ambient spaces, it was tried to generalize the Scherk's result as defining the translation (hyper)surfaces, see [7, 9, 13, 14, 18, 22, 23, 24, 38, 39, 41]. In addition, Seo [35] extended the above result to the translation hypersurfaces with arbitrary constant Gauss-Kronecker and mean curvature.

Munteanu et al. [28] brought forward a different perspective by generalizing the usual notion of translation surface and called it *translation graph.* More precisely, a *translation graph* in $\mathbb{R}^{p+q}$ is given in explicit form

$$y_{p+q}\left(y_1, y_2, \ldots, y_{p+q-1}\right) = f_1\left(y_1, \ldots, y_p\right) + f_2\left(y_{p+1}, \ldots, y_{p+q-1}\right),$$

for smooth functions $f_1 : \mathbb{R}^p \to \mathbb{R}$ and $f_2 : \mathbb{R}^{q-1} \to \mathbb{R}$. They provided certain minimality results on the translation graphs. In addition, Lima et al. [17] proved that a translation graph in $\mathbb{R}^{p+q}$ has vanishing Gauss-Kronecker curvature if it has nonzero constant Gauss-Kronecker or mean curvature.

Moruz and Munteanu [27] dealt with the minimal graphs of the form

$$y_4\left(y_1, y_2, y_3\right) = f_1\left(y_1\right) + f_2\left(y_2, y_3\right),$$

which can be expressed as the sum of a curve in y_1y_4-plane and a surface in $y_2y_3y_4-$space.

Notice that the graph of the form (1.1) is formed by translating $n-1$ curves (called *generating curves*) lying in mutually perpendicular 2-planes. This bring with two restrictions on the translation hypersurfaces: one is that generating curves are planar and the second that the planes including the generating curves are mutually perpendicular. As the restrictions are removed, the different kinds of the translation hypersurfaces arise. For example; in the particular case $n = 3$, Liu and Yu [19] introduced the notion of *affine translation surface,* i.e., the translation surface that the generating curves lie in non-perpendicular planes. They obtained minimal affine translation surfaces, so called *affine Scherk surfaces.* Furthermore, arbitrary constant mean curvature and Weingarten affine translation surfaces were presented in [15, 20].

In this study, we are interested in the counterparts of translation hypersurfaces in isotropic geometry, i.e., a particular Cayley-Klein geometry (for details, see [16, 29, 40]). In 3-dimensional isotropic space $\mathbb{I}^3$, if the generating curves are chosen to lie in mutually perpendicular planes, then three types of translation surfaces exist due to the absolute figure. Let M^2 denote a translation surface in $\mathbb{I}^3$, then we have

Type 1. both generating curves lie in isotropic planes; that is, M^2 is a graph of the form

$$x_3(x_1, x_2) = f(x_1) + g(x_2),$$

where (x_1, x_2, x_3) denote the isotropically orthogonal coordinates in $\mathbb{I}^3$.

Type 2. One generating curve lies in non-isotropic plane and other in isotropic plane; that is, M^2 is a graph of the form

$$x_2(x_1, x_3) = f(x_1) + g(x_3).$$

Type 3. Both generating curves lie in non-isotropic planes; that is, M^2 is a graph of the form

$$x_1(x_2, x_3) = \frac{1}{2}\left(f(x_2 + x_3 - \pi/2) + g(\pi/2 - x_2 + x_3)\right).$$

As well as the non-isotropic planes, Strubecker [36] obtained the minimal translation surfaces in $\mathbb{I}^3$, so called *isotropic Scherk's surfaces of type 1,2,3.* These surfaces are respectively given as follows: for $c \in \mathbb{R},\ c \neq 0$, $x_3 = c\left(x_1^2 - x_2^2\right)$ $c \in \mathbb{R}$, $c \neq 0$ (type 1),

$$x_2 = \frac{1}{c}\ln\left|\frac{cx_3}{\cos cx_1}\right| \text{ (type 2) and } x_1 = \frac{1}{2c}\ln\left|\frac{\cos c(x_2 + x_3 - \pi/2)}{\cos c(\pi/2 - x_2 + x_3)}\right| \text{ (type 3).}$$

Recently, these results were generalized by Milin-Sipus [25] to the translation surfaces in $\mathbb{I}^3$ with arbitrary constant Gaussian and mean curvature. The situation that the generating curves in $\mathbb{I}^3$ are non-planar extends the above categorization and the results. For example, see [1, 4].

In $\mathbb{I}^4$, there are four types of translation hypersurfaces whose the generating curves lie in mutually perpendicular $k-$planes $(k = 2, 3)$, see Section 3. In more general case, i.e. in arbitrary dimensional isotropic spaces, the translation hypersurfaces of type 1 were studied in [3]. The present study concerns other three types of translation hypersurfaces in $\mathbb{I}^4$ with constant Gauss-Kronecker and mean curvature.

Due to the absolute figure of $\mathbb{I}^n$ $n \geq 3$, for a smooth real-valued function f the graph hypersurfaces associated with the form $x_n = f(x_1, ..., x_{n-1})$ differ from other hypersurfaces. For example; the Gauss-Kronecker and mean curvature for such a graph hypersurface in $\mathbb{I}^n$ correspond to determinant and the trace of the Hessian of f, respectively. The formulas of these curvatures were provided by Chen et al. [6]. As far as we know, this is first study formulating such fundamental curvatures for a generic hypersurface in $\mathbb{I}^n$.

2. Preliminaries

Some differential geometric approaches on curves and hypersurfaces in isotropic geometry can be found in [2, 5, 10, 12, 21, 26, 11, 30, 31, 32, 33].

Let $\mathbb{P}^n$ denote the $n-$dimensional real projective space, ω a hyperplane in $\mathbb{P}^n$ and $\mathbb{I}^n = \mathbb{P}^n \backslash \omega$ the obtained affine space. We call $\mathbb{I}^n$ *$n-$dimensional isotropic space* if ω contains a hypersphere $\mathbb{S}$ with null radius. Then the pair $\{\omega, \mathbb{S}\}$ is called *absolute figure* of $\mathbb{I}^n$ and parametrized in homogeneous coordinates by

$$\omega : u_0 = 0,\ \mathbb{S} : u_0 = u_1^2 + ... + u_{n-1}^2 = 0.$$

The vertex of $\mathbb{S}$ is $F\,(0 : 0 : ... : 1)$ called *absolute point*. Here, by a *vertex* we mean the intersection of all maximal generators of a quadric. For more details, see [37].

Denote affine coordinates $x_1 = \frac{u_1}{u_0}, ..., x_n = \frac{u_n}{u_0}$, $u_0 \neq 0$. Then the group of motions of $\mathbb{I}^n$ which preserves the absolute figure is given in terms of affine coordinates by

$$\begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix},$$

where A is an orthonogal $(n-1, n-1)-$matrix, B a real $(1, n-1)-$matrix.

Let $p = (p_1, ..., p_n)$, $q = (q_1, ..., q_n)$ be two points in $\mathbb{I}^n$. The *isotropic distance* between p and q is defined by

$$d_i\,(p, q) = \sqrt{\sum_{i=1}^{n-1} (p_i - q_i)^2}.$$

If $d_i = 0$, then the so-called *range* between p and q is defined as $d_i^r = |p_n - q_n|$.

A line is said to be *isotropic* if its point at infinity is absolute. Other lines are *non-isotropic*. We call a $k-$plane *isotropic* (*non-isotropic*) if it contains (does not) an isotropic line. In the affine model of $\mathbb{I}^n$, the isotropic lines and the isotropic $k-$planes are parallel to x_n-axis. For example; the following

$$a_1 x_1 + ... + a_n x_n = b,\ a_i, b \in \mathbb{R},$$

determines an isotropic (non-isotropic) hyperplane if $a_n = 0\ (\neq 0)$.

Note that the hyperplane $x_n = 0$, so-called *basic hyperplane*, is non-isotropic and therefore the Euclidean metric is used in it.

As distinct from the Euclidean case, the orthogonality in $\mathbb{I}^n$ does not bring with the perpendicularity. Obviously, two non-isotropic lines are orthogonal if their projections onto the basic hyperplane are perpendicular up to the Euclidean metric. Nevertheless, an isotropic line is orthogonal to some non-isotropic line. As a consequence, each non-isotropic hyperplane is orthogonal to the isotropic one. In addition, two isotropic hyperplanes are orthogonal if their projections onto the basic hyperplane are perpendicular.

We call a curve *isotropic* (*non-isotropic*) $k-$*planar* if it lies in an isotropic (non-isotropic) $k-$plane.

2.1. **Curvature theory of hypersurfaces.** This part of isotropic geometry is similar to the Euclidean case.

Let M^{n-1}, $n \geq 3$, be a hypersurface in $\mathbb{I}^n$ whose the tangent hyperplane at each point is non-isotropic. Such a hypersurface is said to be *admissible.* Then the coefficients g_{ij} of the first fundamental form are calculated by the induced metric from $\mathbb{I}^n$. The normal vector field U of M^{n-1} is completely isotropic, i.e. $(0, 0, ..., 1)$.

For the second fundamental form, let us consider a curve r on M^{n-1} with isotropic arclength s and the tangent vector $t(s) = r'(s) = \frac{dr}{ds}$. Denote S the projection of $r''(s) = \frac{d^2r}{ds^2}$ onto the tangent hyperplane of M^{n-1}. Then, the following decomposition occurs:

$$r''(s) = \kappa_g S + \kappa_n U,$$

where κ_g and κ_n are *geodesic* and *normal curvatures* of r, respectively. Hence, it follows $\kappa_g = \|r''(s)\|_i$, where $\|\cdot\|_i$ indicates the isotropic norm. In addition, by a direct computation, we have

$$\kappa_n = \frac{1}{\sqrt{\det g_{ij}}} \sum_{i,j=1}^{n-1} \det\left(r_{x_1}, ..., r_{x_{n-1}}, r_{x_i x_j}\right) \frac{dx_i}{ds}\frac{dx_j}{ds}, \tag{2.1}$$

where $r_{x_i} = \frac{\partial r}{\partial x_i}$ and $r_{x_i x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j}$, $1 \leq i, j \leq n-1$. If we put

$$h_{ij} = \frac{\det\left(r_{x_1}, ..., r_{x_{n-1}}, r_{x_i x_j}\right)}{\sqrt{\det g_{ij}}}$$

into (2.1) then one can be rewritten in the matrix form as

$$\kappa_n = \tilde{t}^T \cdot [h_{ij}] \cdot \tilde{t}, \ \tilde{t} = \left(\frac{dx_1}{ds}, ..., \frac{dx_{n-1}}{ds}\right)^T, \tag{2.2}$$

where "$\cdot$" denotes the matrix multiplication. If r is a curve with arbitrary parameter, then (2.2) turns to

$$\kappa_n = \frac{\tilde{t}^T \cdot [h_{ij}] \cdot \tilde{t}}{\tilde{t}^T \cdot [g_{ij}] \cdot \tilde{t}}.$$

The extreme values of κ_n, which we call *principal curvatures,* correspond to the eigenvalues of the matrix $[h_{ij}] \cdot [g_{ij}]^{-1}$. Let us denote the principal curvatures $\kappa_1, ..., \kappa_{n-1}$ and put $[a_{ij}] =$

$[h_{ij}] \cdot [g_{ij}]^{-1}$. Therefore, the characteristic equation of $[a_{ij}]$ follows

$$\det\left([a_{ij}] - \lambda I_{n-1}\right) = \lambda^{n-1} - tr\,[a_{ij}]\,\lambda^{n-2} + ... + (-1)^{n-1}\det[a_{ij}] = 0,$$

which provides the fundamental curvatures, called *isotropic Gauss-Kronecker curvature* (or *relative curvature*) and *isotropic mean curvature.* We shortly call them *Gauss-Kronecker* (K) and *mean curvature* (H). Obviously, one obtains

$$K = \kappa_1 ... \kappa_{n-1} = \det[a_{ij}] \text{ or } K = \frac{\det[h_{ij}]}{\det[g_{ij}]}$$

and

$$(n-1)\,H = \kappa_1 + ... + \kappa_{n-1} = tr\,[a_{ij}]\,,$$

where tr denotes the trace of a matrix.

A hypersurface is said to be flat (minimal) if K (H) is identically zero.

Notice that the isotropic counterpart for the notion of *shape operator* in the Euclidean sense of a hypersurface is indeed a zero map. The matrix $[a_{ij}]$ however plays the role of the matrix corresponding shape operator in $\mathbb{I}^n$.

3. Categorization of translation hypersurfaces

Let M^3 be a translation hypersurface in $\mathbb{I}^4$ generated by translating three curves lying in mutually perpendicular $k-$planes, $k = 2, 3$. Denote the generating curves α, β, γ. Up to the absolute figure of $\mathbb{I}^4$ there are four types of such hypersurfaces given as follows:

Type 1. Three of α, β, γ are isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u, v, w, f(u) + g(v) + h(w))\,,$$

where α, β and γ lie in x_1x_4-plane, x_2x_4-plane and x_3x_4-plane, respectively.

Type 2. α is non-isotropic 2-planar and β, γ isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u + v, w, f(u)\,, g(v) + h(w))\,,$$

where α, β and γ lie in x_1x_3-plane, x_1x_4-plane and x_2x_4-plane, respectively. Admissibility implies that f is a non-constant function.

Type 3. α, β are non-isotropic 2-planar and γ isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u + v + w, f(u)\,, g(v)\,, h(w))\,,$$

where α, β and γ lie in x_1x_2-plane, x_1x_3-plane and x_1x_4-plane, respectively. Admissibility implies that neither f nor g is a constant function.

Type 4. Three of α, β, γ are non-isotropic hyperplanar. The curves α, β, γ and the hyperplanes $P_\alpha, P_\beta, P_\gamma$ containing them can be choosen as

$$\alpha(u) = (f(u), u, u, u+\pi), \; P_\alpha : -2x_2 + x_3 + x_4 = \pi;$$
$$\beta(v) = \left(g(v), v, v, -v + \tfrac{\pi}{3}\right), \; P_\beta : 2x_2 + x_3 + 3x_4 = \pi;$$
$$\gamma(w) = \left(h(w), 6w, -w, w - \tfrac{\pi}{2}\right), \; P_\gamma : x_2 + 4x_3 - 2x_4 = \pi.$$

Then M^3 is parameterized by

$$r(u,v,w) = \left(f(u) + g(v) + h(w), u + v + 6w, u + v - w, u - v + w + \frac{5\pi}{6}\right),$$

where $\frac{df}{du} - \frac{dg}{dv} \neq 0$ because admissibility.

A translation hypersurface of above one type is no equivalent to that of other type due to the absolute figure of $\mathbb{I}^4$.

We hereinafter denote the derivatives of f, g, h with respect to the given variable by a prime and so.

4. Translation hypersurfaces of type 2

For a translation hypersurface of type 2, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 1 + (f')^2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [g_{ij}]^{-1} = \frac{1}{(f')^2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 + (f')^2 & 0 \\ 0 & 0 & (f')^2 \end{pmatrix}$$

and

$$[h_{ij}] = \begin{pmatrix} -\frac{f''g'}{f'} & 0 & 0 \\ 0 & -g'' & 0 \\ 0 & 0 & -h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature follows respectively

$$K = \frac{g'f''g''h''}{(f')^3} \tag{4.1}$$

and

$$3H = \frac{f''g'}{(f')^3} + g''\frac{1 + (f')^2}{(f')^2} + h''. \tag{4.2}$$

Theorem 4.1. *Let M^3 be a flat translation hypersurface of type 2 in $\mathbb{I}^4$. Then it is a cylindrical hypersurface with non-isotropic rulings. Furthermore if M^3 has nonzero constant*

Gauss-Kronecker curvature then, up to suitable constants and translations of u, v, w, the following holds

$$f(u) = \lambda u^{\frac{1}{2}}, \; g(v) = \mu v^{\frac{3}{2}}, \; h(w) = \xi w^2,$$

where $\lambda, \mu, \xi \in \mathbb{R}$ and $\lambda\mu\xi \neq 0$.

Proof. The (4.1) follows that K vanishes if at least one of f, g, h is a linear function with respect to the given variable, that is, at least one of the generating curves turns to a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1 u + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Hence, one can be parameterized by

$$r(u, v, w) = u(1, 0, c_1, 0) + (v, w, c_2, g(v) + h(w)),$$

which means that M^3 is congruent a cylindrical hypersurface with non-isotropic rulings. Now, let assume that the Gauss-Kronecker curvature is a nonzero constant K_0. So, the (4.1) leads to

$$\frac{f''}{(f')^3} = c_3, \; g'g'' = c_4, \; h'' = c_5, \tag{4.3}$$

for $c_3, c_4, c_5 \in \mathbb{R}$ and $K_0 = c_3c_4c_5 \neq 0$. After solving (4.3), we obtain

$$f(u) = \pm\frac{1}{c_3}\sqrt{-2c_3u + c_6} + c_7, \quad g(v) = \pm\frac{1}{3c_4}(2c_4v + c_8)^{\frac{3}{2}} + c_9$$

and

$$h(w) = \frac{c_5}{2}w^2 + c_{10}w + c_{11},$$

where $c_6, ..., c_{11} \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$one may assume $c_7 = c_9 = c_{11} = 0$ and up to a translation on u, v, wchoose $c_6 = c_8 = c_{10} = 0$. Besides putting $\lambda = \frac{\pm\sqrt{-2c_3}}{c_3}$, $\mu = \frac{\pm(2c_4)^{\frac{3}{2}}}{3c_4}$ and $\xi = \frac{c_5}{2}$ completes the proof.

Theorem 4.2. *Let M^3 be a minimal translation hypersurface of type 2 in $\mathbb{I}^4$. Then M^3 is either a non-isotropic hyperplane or, up to suitable constants and translations of u, v, w, one of the following cases occurs:*

(i) $f = f(u)$, $f' \neq 0$, $g(v) = \lambda$, $h(w) = \mu w$;

(ii) $f(u) = \lambda u^{\frac{1}{2}}$, $g(v) = \mu v$, $h(w) = \frac{\mu}{\lambda^2}w^2$, $\lambda\mu \neq 0$;

(iii) $f(u) = \lambda u$, $g(v) = \mu v^2$, $h(w) = -\frac{1+\lambda^2}{\lambda^2}\mu w^2$, $\lambda\mu \neq 0$;

(iv) *$M^3 = S^2 \times \mathbb{R}$, where S^2 is the isotropic Scherk's surface of type 2 in $\mathbb{I}^3$,*

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. The (4.2) leads to

$$\frac{f''g'}{(f')^3}+\frac{1+(f')^2}{(f')^2}g''+h''=0. \tag{4.4}$$

The partial derivative of (4.4) with respect to w implies $h''=h_0$, $h_0\in\mathbb{R}$. If $g'=0$, we get $h(w)=c_1w+c_2$. Putting $c_1=\mu$ and applying a translation on w implies that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Afterwards we assume $g'\neq 0$. Then the partial derivative of (4.4) with respect to v yields

$$\frac{f''}{f'}g''+\left[1+(f')^2\right]g'''=0. \tag{4.5}$$

If g'' and g''' are linearly independent then the contradiction $1+(f')^2=0$ is obtained. Hence we have either $g''=0$ or $g'''=kg''$, $g''\neq 0$ and $k\in\mathbb{R}$.

(1) $g''=0$. (4.4) can be rewritten by putting $g'=g_0\neq 0$ as

$$\frac{f''}{(f')^3}g_0+h_0=0. \tag{4.6}$$

Being $f''=0=h_0$ is a solution to (4.6), which leads M^3 to be a non-isotropic hyperplane. If $f''h_0\neq 0$, (4.6) turns to

$$\frac{f''}{(f')^3}=-\frac{h_0}{g_0}. \tag{4.7}$$

By solving (4.7), we derive

$$f(u)=\pm\frac{g_0}{h_0}\sqrt{2\frac{h_0}{g_0}u+c_3}+c_4,\quad g(v)=g_0v+c_5$$

and

$$h(w)=\frac{h_0}{2}w^2+c_6w+c_7.$$

Up to congruency of $\mathbb{I}^4$ one may assume $c_4=c_7=0$ and up to a translation on u,v,w choose $c_3=c_5=c_6=0$. After putting $\lambda=\pm\frac{g_0}{h_0}\sqrt{2\frac{h_0}{g_0}}$ and $\mu=g_0$ we obtain that M^3 is congruent to the hypersurface given in the case (ii) of the theorem.

(2) $g'''=kg''$, $g''\neq 0$. (4.5) leads to

$$f''=-kf'\left[1+(f')^2\right]. \tag{4.8}$$

Being $f''=0=k$ is a solution for (4.8). Therefore we write

$$f(u)=c_8u+c_9,\ g(v)=\frac{c_{10}}{2}v^2+c_{11}v+c_{12},$$

where $c_8, ..., c_{12} \in \mathbb{R}$, $c_8 c_{10} \neq 0$. Considering it into (4.4) concludes $h'' = -\frac{1+c_8^2}{c_8^2} c_{11}$ or

$$h(w) = -\frac{1+c_8^2}{c_8^2} c_{10} w^2 + c_{13} w + c_{14},$$

for $c_{13}, c_{14} \in \mathbb{R}$. As in the previous cases, up to suitable constants and translations, we obtain that M^3 is congruent to the hypersurface given in the case (iii) of the theorem. Assuming $k \neq 0$ in (4.8) yields $f'' \neq 0$. Also we have $g'' = kg' + c_{15}$, $l \in \mathbb{R}$ by integrating $g''' = kg''$. Hence substituting (4.8) into (4.4) gives

$$c_{15} \frac{1 + (f')^2}{(f')^2} + h_0 = 0. \tag{4.9}$$

The admissibility implies that f is a non-constant function and thus we conclude from (4.9) that $c_{15} = h_0 = 0$, i.e. $h(w) = c_{16} w + c_{17}$ for $c_{16}, c_{17} \in \mathbb{R}$. Because (4.8) and being $g'' = kg'$, we write

$$\frac{f''}{f' \left[1 + (f')^2\right]} = -k = -\frac{g''}{g'}. \tag{4.10}$$

After solving (4.10), we obtain

$$f(u) = \pm \frac{1}{k} \arccos\left(c_{18} e^{-ku}\right), \ g(v) = -\frac{c_{19}}{k} e^{kv}, \tag{4.11}$$

for $c_{18}, c_{19} \in \mathbb{R}$, $c_{18} c_{19} \neq 0$. By a change of parameter in (4.11) M^3 can be parameterized as

$$r(\tilde{u}, \tilde{v}, w) = \left(\frac{1}{k} \ln\left|\frac{k\tilde{v}}{\cos k\tilde{u}}\right|, 0, \tilde{u}, \tilde{v}\right) + w(0, 1, 0, c_{16})$$

and is congruent to $S^2 \times \mathbb{R}$, where S^2 is the isotropic Scherk's surface of type 2 in $\mathbb{I}^3$. This completes the proof.

Theorem 4.2. immediately implies the following corollary

Corollary 4.1. *Let M^3 be a translation hypersurface of type 2 in $\mathbb{I}^4$. Then, $H = 0$ implies $K = 0$.*

Theorem 4.3. *Let M^3 be a translation hypersurface of type 2 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u, v, w, one of the following cases occurs:*

(i) $f = f(u)$, $f' \neq 0$, $g(v) = \lambda$, $h(w) = \frac{3H_0}{2} w^2$;

(ii) $f(u) = \lambda u$, $g(v) = \mu v$, $h(w) = \frac{3H_0}{2} w^2$, $\lambda\mu \neq 0$;

(iii) $f(u) = \lambda u^{\frac{1}{2}}$, $g(v) = \mu v$, $h(w) = \xi w^2$, $\lambda\mu \neq 0$, $3H_0 = \frac{-2\mu}{\lambda^2} + 2\xi \neq 0$;

(iv) $f(u) = \lambda u,\ g(v) = \mu v^2,\ h(w) = \xi w^2,\ \lambda\mu \neq 0,\ 3H_0 = \frac{2\mu(1+\lambda^2)}{\lambda^2} + 2\xi \neq 0$;

(v) $M^3 = S^2 \times P$, *where* S^2 *is the isotropic Scherk's surface of type 2 in* $\mathbb{I}^3$ *and* P *is a parabolic circle in* $\mathbb{I}^2$ *with isotropic curvature* $3H_0$,

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. Reconsidering (4.2) leads to $h'' = h_0$, $h_0 \in \mathbb{R}$ and therefore we get

$$3H_0 = \frac{f''g'}{(f')^3} + g''\frac{1+(f')^2}{(f')^2} + h_0. \tag{4.12}$$

To solve (4.12), we distinguish two cases:

(1) $g' = g_0$, $g_0 \in \mathbb{R}$. In particular; if $g_0 = 0$, then we conclude $h_0 = 3H_0$ and

$$h(w) = \frac{3}{2}H_0 w^2 + c_1 w + c_2,\ c_1, c_2 \in \mathbb{R},$$

which implies that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Nevertheless; if $g_0 \neq 0$ then, by (4.12) we get

$$\frac{3H_0 - h_0}{g_0} = \frac{f''}{(f')^3}. \tag{4.13}$$

If $3H_0 = h_0$ in (4.13), we immediately obtain the proof the case (ii) of the theorem. Otherwise, after solving (4.13) we obtain

$$f(u) = \pm\frac{g_0}{3H_0 - h_0}\sqrt{\frac{-6H_0 + 2h_0}{g_0}u + c_3} + c_4,$$

where $3H_0 \neq h_0$ and $c_3, c_4 \in \mathbb{R}$. Hence, after suitable translations and constants, we obtain that M^3 is congruent to the hypersurface given in the case (iii) of the theorem.

(2) $g'' \neq 0$. We consider two cases:

(a) $f' = f_0 \neq 0,\ f_0 \in \mathbb{R}$. (4.12) leads to

$$3H_0 = \frac{1+f_0^2}{f_0^2}g'' + h_0,$$

which implies the proof of the case (iv) of the theorem up to constants and suitable translations.

(b) $f'' \neq 0$. (4.12) implies $h_0 = 3H_0$ and

$$\frac{f''}{(f')^3} = c_5\frac{1+(f')^2}{(f')^2}, \quad g'' = -c_5 g', \tag{4.14}$$

where $c_5 \in \mathbb{R}$, $c_5 \neq 0$. After solving (4.14), we obtain

$$f(u) = \pm\frac{1}{\lambda}\arccos\left(c_6 e^{c_5 u}\right),\ g(v) = -\frac{c_7}{\lambda}e^{-c_5 v} \tag{4.15}$$

for $c_6, c_7 \in \mathbb{R}$, $c_6 c_7 \neq 0$. By a change of parameter in (4.15), M^3 can be written as

$$r(\tilde{u}, \tilde{v}, w) = \left(\frac{1}{\lambda} \ln \left| \frac{\cos \lambda \tilde{u}}{\lambda \tilde{v}} \right|, 0, \tilde{u}, \tilde{v} \right) + \left(0, w, 0, \frac{3}{2} H_0 w^2 \right),$$

which completes the proof of the theorem.

5. Translation hypersurfaces of type 3

For a translation hypersurface of type 3, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 1 + (f')^2 & 1 & 1 \\ 1 & 1 + (g')^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad [g_{ij}]^{-1} = \begin{pmatrix} \frac{1}{(f')^2} & 0 & \frac{-1}{(f')^2} \\ 0 & \frac{1}{(g')^2} & \frac{-1}{(g')^2} \\ \frac{-1}{(f')^2} & \frac{-1}{(g')^2} & 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \end{pmatrix}$$

and

$$[h_{ij}] = \begin{pmatrix} \frac{f'' h'}{f'} & 0 & 0 \\ 0 & \frac{g'' h'}{g'} & 0 \\ 0 & 0 & h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature are respectively

$$K = \frac{(h')^2 f'' g'' h''}{(f' g')^3} \tag{5.1}$$

and

$$3H = h' \left[\frac{f''}{(f')^3} + \frac{g''}{(g')^3} \right] + h'' \left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right]. \tag{5.2}$$

Theorem 5.1. *Let M^3 be a flat translation hypersurface of type 3 in $\mathbb{I}^4$. Then it is a cylindrical hypersurface with non-isotropic rulings. Furthermore if M^3 has nonzero constant Gauss-Kronecker curvature then, up to suitable constants and translations of u, v, w, the following holds*

$$f(u) = \lambda u^{\frac{1}{2}}, \; g(v) = \mu v^{\frac{1}{2}}, \; h(w) = \xi w^{\frac{4}{3}},$$

where $\lambda, \mu, \xi \in \mathbb{R}$ and $\lambda \mu \xi \neq 0$.

Proof. The (5.1) follows that K vanishes if at least one of f, g, h is a linear function with respect to the given variable; that is, at least one of the generating curves turns to be a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1 u + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Hence, M^3 can be parameterized by

$$r(u, v, w) = u(1, c_1, 0, 0) + (v + w, c_2, g(v) + h(w)),$$

which means that it is congruent to a cylindrical hypersurface with non-isotropic rulings. Now, let us assume that K is a nonzero constant. So, (5.1) leads to

$$\frac{f''}{(f')^3} = c_3, \ \frac{g''}{(g')^3} = c_4, \ (h')^2 h'' = c_5, \tag{5.3}$$

for $c_3, c_4, c_5 \in \mathbb{R}$ and $c_3 c_4 c_5 \neq 0$. After solving (5.3), we obtain

$$f(u) = \pm \frac{1}{c_3} \sqrt{-2c_3 u + c_6} + c_7, \ g(v) = \pm \frac{1}{c_4} \sqrt{-2c_4 v + c_8} + c_9$$

and

$$h(w) = \frac{1}{4c_5} (3c_5 w + c_{10})^{\frac{4}{3}} + c_{11},$$

where $c_6, ..., c_{11} \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$one may assume $c_7 = c_9 = c_{11} = 0$ and up to a translation on u, v, wwe choose $c_6 = c_8 = c_{10} = 0$. Eventually, putting $\lambda = \frac{\pm\sqrt{-2c_3}}{c_3}$, $\mu = \frac{\pm\sqrt{-2c_4}}{c_4}$ and $\xi = \frac{(3c_5)^{\frac{4}{3}}}{4c_5}$ completes the proof.

Theorem 5.2. *Let M^3 be a minimal translation hypersurface of type 3 in $\mathbb{I}^4$. Then, M^3 is either a non-isotropic hyperplane or, up to suitable constants and translations of u, v, w, one of the following cases occurs:*

(i) $f = f(u), \ f' \neq 0, \ g = g(v), \ g' \neq 0, \ h(w) = \lambda$;
(ii) $f(u) = \lambda (-u)^{\frac{1}{2}}, \ g(v) = \lambda v^{\frac{1}{2}}, \ h(w) = \mu w, \ \lambda\mu \neq 0$;
(iii) $M^3 = S^2 \times \mathbb{R}$, *where S^2 is the isotropic Scherk' s surface of type 2 in $\mathbb{I}^3$;*
(iv) $f(u) = \eta \ln \left| \frac{1+\sqrt{1+\kappa e^{\lambda u}}}{1-\sqrt{1+\kappa e^{\lambda u}}} \right|$ *or* $f(u) = \kappa e^{\lambda u}$, $g(v) = \mu \ln \left| \frac{1+\sqrt{1+\xi e^{\varpi v}}}{1-\sqrt{1+\xi e^{\varpi v}}} \right|$ *or* $g(v) = \xi e^{\varpi v}$, $h(w) = \rho e^{\tau w}$, *where $\eta, \kappa, \lambda, \mu, \xi, \varpi, \rho, \tau$ are nonzero constants.*

Proof. Due to $H = 0$, (5.2) reduces to

$$h' \left[\frac{f''}{(f')^3} + \frac{g''}{(g')^3} \right] + h'' \left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right] = 0. \tag{5.4}$$

It immediately follows from (5.4) that h' and h'' can not be linearly independent. In this manner we have either $h' = 0$ or $h'' = c_1 h'$, $h' \neq 0$ and $c_1 \in \mathbb{R}$. Being $h' = 0$ implies that M^3 is congruent to the hypersurface given in the first case of the theorem. Now we assume that $h'' = c_1 h'$ and $h' \neq 0$. To solve (5.4) there are two cases:

(1) $c_1 = 0$, i.e. $h(w) = c_2 w + c_3$, $c_2, c_3 \in \mathbb{R}$, $c_2 \neq 0$. Hence (5.4) reduces to

$$\frac{f''}{(f')^3} = c_4 = \frac{-g''}{(g')^3}, \ c_4 \in \mathbb{R}. \tag{5.5}$$

If $c_4 = 0$, then M^3 turns to be a non-isotropic hyperplane. Otherwise, i.e. $c_4 \neq 0$, solving (5.5) leads to

$$f(u) = \pm \frac{1}{c_4} \sqrt{-2c_4 u + c_5} + c_6, \; g(v) = \pm \frac{1}{c_4} \sqrt{2c_4 v + c_7} + c_8,$$

where $c_6, ..., c_8 \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$ one may assume $c_6 = c_8 = 0$. We may aslo choose $c_3 = c_5 = c_7 = 0$ up to a translation on u, v, w. By putting $\lambda = \frac{\pm\sqrt{2c_4}}{c_4}$ and $\mu = c_3$, we obtain that M^3 is congruent to the hypersurface given in the case (ii) of the theorem.

(2) $c_1 \neq 0$. Then (5.4) yields

$$\frac{f''}{(f')^3} + \frac{c_1}{(f')^2} + \frac{g''}{(g')^3} + \frac{c_1}{(g')^2} = -c_1, \tag{5.6}$$

where the roles of f and g are symmetric and thus it is enough to discuss the situation on f. We have two cases:

(a) $f' = f_0 \in \mathbb{R}$. (5.6) implies

$$\frac{f_0^2 g''}{g' \left[\left(1 + f_0^2\right) (g')^2 + f_0^2 \right]} = -c_1. \tag{5.7}$$

After solving (5.7), we obtain

$$g(v) = \pm \frac{f_0}{\sqrt{1 + f_0^2}} \arccos\left(c_9 \left[1 + f_0^2 \right] e^{-c_1 v} \right), \; c_9 \in \mathbb{R}, \; c_9 \neq 0.$$

On the other hand, since $h'' = c_1 h'$ we get $h(w) = c_{10} e^{c_1 w}$, $c_{10} \in \mathbb{R}$, $c_{10} \neq 0$. By a change of parameter and up to suitable constants and translations we derive that M^3 is congruent to the hypersurface given in the case (iii) of the theorem.

(b) $f'' \neq 0$. By symmetry, we have $g'' \neq 0$. Thereby, (5.6) implies

$$\frac{f''}{(f')^3} + \frac{c_1}{(f')^2} = c_{11}, \tag{5.8}$$

and

$$\frac{g''}{(g')^3} + \frac{c_1}{(g')^2} = c_{12}, \tag{5.9}$$

where $c_{11}, c_{12} \in \mathbb{R}$ and $c_{12} = -c_1 - c_{11}$. From (5.8), we have

$$f'(u) = \pm \left(\frac{c_{11}}{c_1} + \frac{c_{13}}{c_1} e^{2\mu u} \right)^{\frac{-1}{2}}, \; c_{13} \in \mathbb{R}, \; c_{13} \neq 0. \tag{5.10}$$

If $c_{11}=0$ in (5.10), then we can derive $f(u)=\mp\left(\frac{c_{13}}{c_1}\right)^{\frac{-1}{2}}e^{-c_1u}$. Otherwise, we get

$$f(u)=-\frac{1}{\sqrt{c_1c_{11}}}\tanh^{-1}\left(\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}\right)=-\frac{1}{2\sqrt{c_1c_4}}\ln\left|\frac{1+\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}}{1-\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}}\right|.$$

Same solutions are also satisfied to (5.9). Up to suitable constants we complete the proof.

Theorem 5.3. *Let M^3 be a translation hypersurface of type 3 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u,v,w, one of the following cases occurs:*

(i) $f(u)=\lambda u,\ g(v)=\left(\frac{-2\mu}{3H_0}v\right)^{\frac{1}{2}},\ h(w)=\mu w,\ \lambda\mu\neq 0;$

(ii) $f(u)=\lambda u^{\frac{1}{2}},\ g(v)=\mu v^{\frac{1}{2}},\ h(w)=\xi w,\ \lambda\mu\xi\neq 0;$

(iii) $f(u)=\lambda u,\ g(v)=\mu v,\ h(w)=\frac{3H_0(\lambda\mu)^2}{2\left[(\lambda\mu)^2+\lambda^2+\mu^2\right]}w^2,\ \lambda\mu\neq 0,$

where $\lambda,\mu,\xi\in\mathbb{R}$,

Proof. Due to $H_0\neq 0$, h cannot be constant in (5.2). We have to distinguish several cases to solve (5.2):

(1) $h'=h_0\in\mathbb{R}$, $h_0\neq 0$. Then we write $h(w)=h_0w+c_1$, $c_1\in\mathbb{R}$. (5.2) reduces to

$$\frac{3H}{h_0}=\frac{f''}{(f')^3}+\frac{g''}{(g')^3}, \tag{5.11}$$

where the roles of f and g are symmetric and so the situation on g is only considered.

(a) $g''=0$, $g(v)=c_2v+c_3$, $c_2,c_3\in\mathbb{R}$, $c_2\neq 0$. Then (5.11) reduces to

$$\frac{f''}{(f')^3}=\frac{3H}{h_0} \tag{5.12}$$

and solving (5.12) gives

$$f(u)=\pm\frac{1}{\frac{3H_0}{h_0}}\sqrt{-\frac{6H_0}{h_0}u+c_4}+c_5,\ \ c_4,c_5\in\mathbb{R}.$$

Up to congruency of $\mathbb{I}^4$ one may assume $c_5=0$ and up to a translation on u, v and w choose $c_1=c_3=c_4=0$. Furthermore by putting $\lambda=\frac{\pm h_0\sqrt{-\frac{6H_0}{h_0}}}{3H_0}$, $c_2=\mu$ and $h_0=\xi$ we conclude that M^3 is congruent to the hypersurface given in the case (i) of the theorem.

(b) $g''\neq 0$. Hence (5.11) implies

$$\frac{f''}{(f')^3}=\frac{3H}{h_0}-c_6 \text{ and } \frac{g''}{(g')^3}=c_6,\ \ c_6\in\mathbb{R},\ \ c_6\neq 0. \tag{5.13}$$

Solving (5.13) gives

$$f(u) = \pm \frac{1}{\frac{3H_0}{h_0} - c_6} \sqrt{-2\left(\frac{3H_0}{h_0} - c_6\right) u + c_7} + c_8$$

and

$$g(v) = \pm \frac{1}{c_6} \sqrt{-2c_6 v + c_9} + c_{10},$$

for $c_7, ..., c_{10} \in \mathbb{R}$. As in previous case, after applying suitable translations and choosing constants, the case (ii) of the theorem is proved.

(2) $h'' \neq 0$. If $f'' = 0 = g''$, then (5.2) leads to

$$h(w) = c_{11} w^2 + c_{12} w + c_{13},$$

where $c_{11}, c_{12}, c_{13} \in \mathbb{R}$. Up suitable translations and constants this implies the proof of the case (iii) of the theorem. If $f''g'' \neq 0$, dividing (5.2) with h' and taking its partial derivative with respect to w, we deduce

$$-3H_0 \frac{h''}{(h')^2} = \left(\frac{h''}{h'}\right)' \left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}\right]. \tag{5.14}$$

Both-hand side must be nonzero in (5.14) and thus we can rewrite it as follows:

$$-3H_0 \frac{h''}{(h')^2} \left[\left(\frac{h''}{h'}\right)'\right]^{-1} = 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}. \tag{5.15}$$

This is a contradiction due to the fact that the right-hand side of (5.15) cannot be a constant. This completes the proof.

Theorem 5.3. immediately implies

Corollary 5.1. *Let M^3 be a translation hypersurface of type 3 in $\mathbb{I}^4$. Then, $H = const. \neq 0$ implies $K = 0$.*

6. Translation hypersurfaces of type 4

For a translation hypersurface of type 4, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 2 + (f')^2 & 2 + f'g' & 5 + f'h' \\ 2 + f'g' & 2 + (g')^2 & 5 + g'h' \\ 5 + f'h' & 5 + g'h' & 37 + (h')^2 \end{pmatrix},$$

$$[g_{ij}]^{-1} = \frac{1}{49(f' - g')^2} \times$$

$$\times\begin{pmatrix} 37(g')^2+2(h')^2-10g'h'+49 & 5f'h'+5g'h'-37f'g'-2(h')^2-49 & 5f'g'+2g'h'-2f'h'-5(g')^2 \\ 5f'h'+5g'h'-37f'g'-2(h')^2-49 & 37(f')^2+2(h')^2-10f'h'+49 & 5f'g'+2f'h'-2g'h'-5(f')^2 \\ 5f'g'+2g'h'-2f'h'-5(g')^2 & 5f'g'+2f'h'-2g'h'-5(f')^2 & 2(f')^2+2(g')^2-4f'g' \end{pmatrix}$$

and

$$[h_{ij}] = \frac{2}{f'-g'}\begin{pmatrix} f'' & 0 & 0 \\ 0 & g'' & 0 \\ 0 & 0 & h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature are respectively

$$K = \frac{8f''g''h''}{49(f'-g')^5} \tag{6.1}$$

and

$$\begin{aligned} 3H = \tfrac{2}{49(f'-g')^3}\Big\{\Big[37(g')^2+2(h')^2-10g'h'+49\Big]f''+ \\ +\Big[37(f')^2+2(h')^2-10f'h'+49\Big]g''+2h''(f'-g')^2\Big\}. \end{aligned} \tag{6.2}$$

The roles of f and g are symmetric in (6.2) and, while solving it, the situations depending on f are only considered.

Theorem 6.1. *If a translation hypersurface of type 4 in $\mathbb{I}^4$ has nonzero constant Gauss-Kronecker curvature K_0, then it is a cylindrical hypersurface with non-isotropic rulings, namely $K_0 = 0$.*

Proof. Assume that $K = K_0 \neq 0$, it then follows from (6.1) that $f''g''h'' \neq 0$. Hence (6.1) reduces to

$$\frac{49K_0}{8h_0} = \frac{f''g''}{(f'-g')^5}, \tag{6.3}$$

where $h'' = h_0 \neq 0$, $h_0 \in \mathbb{R}$. The partial derivative of (6.3) with respect to u yields

$$f'''(f'-g') - 5(f'')^2 = 0. \tag{6.4}$$

The fact that the coefficient of the term g' in (6.4) must be zero leads to the contradiction $f'' = 0$. Therefore K vanishes and at least one of f, g, h is a linear function with respect to the given variable; that is, at least one of the generating curves turns to be a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1u + c_2$, $c_1, c_2 \in \mathbb{R}$. Hence, one can be parameterized by

$$\begin{aligned} r(u,v,w) &= u(c_1,1,1,1)+ \\ &\left(c_2+g(v)+h(w), v+6w, v-w, -v+w+\tfrac{5\pi}{6}\right). \end{aligned}$$

which means that it is congruent to a cylindrical hypersurface with non-isotropic rulings.

Theorem 6.2. *Let M^3 be a minimal translation hypersurface of type 4 in $\mathbb{I}^4$. Then M^3 is either a non-isotropic hyperplane or, up to suitable constants and translations of u, v, w, one of the following cases holds:*

(i) $f(u) = \lambda u$, $g(v) = \lambda v - \frac{1}{\mu}\ln|\mu v|$, $h(w) = \frac{5\lambda}{2}w + \frac{1}{\mu}\ln\left|\cos\frac{7\sqrt{2+\lambda^2}\mu}{2}w\right|$, $\mu \neq 0$;

(ii) $f(u) = \lambda u - \frac{1}{\mu}\ln|\cos\xi w|$, $g(v) = \lambda v + \frac{1}{\mu}\ln|\cos\xi w|$, $h(w) = \frac{37\lambda}{5}w$, $\mu\xi \neq 0$,

where $\lambda, \mu, \xi \in \mathbb{R}$,

Proof. The (6.2) follows

$$\begin{aligned} 0 &= \left[37(g')^2 + 2(h')^2 - 10g'h' + 49\right] f'' \\ &+ \left[37(f')^2 + 2(h')^2 - 10f'h' + 49\right] g'' + 2h''(f' - g')^2 . \end{aligned} \tag{6.5}$$

We have two cases to solve (6.5):

(1) $f' = f_0$, $f_0 \in \mathbb{R}$. (6.5) can be rewritten as

$$\frac{g''}{(f_0 - g')^2} + \frac{2h''}{2(h')^2 - 10f_0h' + 37f_0^2 + 49} = 0. \tag{6.6}$$

The situation that $g'' = h'' = 0$, $g' \neq f_0$, leads M^3 to be a non-isotropic hyperplane. If $g''h'' \neq 0$, (6.6) implies

$$\frac{g''}{(f_0 - g')^2} = c_1 = \frac{-2h''}{2(h')^2 - 10f_0h' + 37f_0^2 + 49}, \tag{6.7}$$

where $c_1 \in \mathbb{R}$, $c_1 \neq 0$. After solving (6.7), we conclude

$$g(v) = f_0v - \frac{1}{c_1}\ln|c_1v + c_2| + c_3$$

and

$$h(w) = \frac{5f_0}{2}w + \frac{1}{c_1}\ln\left|\cos\left(-\frac{7c_1\sqrt{2 + f_0^2}}{2}w + c_4\right)\right| + c_5,$$

where $c_2, ..., c_5 \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$ one may assume $c_3 = c_5 = 0$ and up to a translation on v and w, choose $c_2 = c_4 = 0$. Furthermore by putting $\lambda = f_0$ and $\mu = c_1$, we obtain that M^3 is congruent to the hypersurface given in the case (i) of the theorem.

(2) $f'' \neq 0$. By symmetry, we deduce $g'' \neq 0$. We have two cases:

(a) $h' = h_0$, $h_0 \in \mathbb{R}$. (6.5) can be rewritten as

$$\frac{f''}{37(f')^2 - 10h_0f' + 49 + 2h_0^2} = c_6 = \frac{-g''}{37(g')^2 - 10h_0g' + 49 + 2h_0^2}, \tag{6.8}$$

for $c_6 \in \mathbb{R}$, $c_6 \neq 0$. Solving (6.8), we conclude

$$f(u) = \frac{-1}{37c_6}\ln|\cos(c_6ku + c_7)| + \frac{5h_0}{37}u + c_8$$

and

$$g(v) = \frac{1}{37c_6}\ln|\cos(-k\lambda v + c_9)| + \frac{5h_0}{37}v + c_{10},$$

where $c_7, ..., c_{10} \in \mathbb{R}$ and $k = \sqrt{1813 + 12h_0^2}$. As in previous case; after applying suitable translations and choosing constants, we complete the proof of the case (ii) of the theorem

(b) $h'' \neq 0$. Taking partial derivative of (6.5) with respect to u, v, w and then dividing $f''g''h''$ yields

$$5\frac{f'''}{f''} + 5\frac{g'''}{g''} + 2\frac{h'''}{h''} = 0,$$

which leads to

$$f''' = c_{11}f'', \ g''' = c_{12}g'', \ h''' = c_{13}h'', \tag{6.9}$$

for $c_{11}, c_{12}, c_{13} \in \mathbb{R}$ with $5c_{11} + 5c_{12} + 2c_{13} = 0$. Integrating (6.9) gives

$$f'' = c_{11}f' + c_{14}, \ g'' = c_{12}g' + c_{15}, \ h'' = c_{13}h' + c_{16},$$

for $c_{14}, c_{15}, c_{16} \in \mathbb{R}$. On the other hand, taking partial derivative of (6.5) with respect to w and dividing h'' leads to

$$(4h' - 10g')\, f'' + (4h' - 10f')\, g'' + 2h''' \, (f' - g')^2 = 0. \tag{6.10}$$

If we substitute (6.9) into (6.10), then

$$(4h' - 10g')\,(c_{11}f' + c_{14}) + (4h' - 10f')\,(c_{12}g' + c_{15}) + 2c_{13}\,(f' - g')^2 = 0,$$

which is a polynomial equation on f' or g'. This immediately gives $c_{13} = 0$, i.e.

$$(4h' - 10g')\,(c_{11}f' + c_{14}) + (4h' - 10f')\,(c_{12}g' + c_{15}) = 0. \tag{6.11}$$

Taking partial derivative of (6.11) with respect to w and dividing $4h''$ implies

$$c_{11}f' + c_{12}g' + c_{14} + c_{15} = 0,$$

which means $c_{11} = c_{12} = 0$, i.e. $f''' = g''' = h''' = 0$. Considering it into (6.5) and then taking parital derivatives its with respect to u and v yield

$$-4f''g''h'' = 0,$$

which gives a contradiction.

Theorem 6.3. *Let M^3 be a translation hypersurface of type 4 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u, v, w, one of the following cases holds:*

(i) $f(u) = \lambda u, \ g(v) = \mu v, \ h(w) = \frac{147H_0}{8}(\lambda - \mu)\, w^2, \ \lambda \neq \mu;$
(ii) $f(u) = \lambda u, \ g(v) = \lambda v + \mu v^{\frac{1}{2}}, \ h(w) = \xi w, \ \mu \neq 0,$

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. We have several cases to solve (6.2):

(1) $f' = f_0 \in \mathbb{R}$. (6.2) then reduces to

$$c_1 (f_0 - g')^3 = \left[2 (h')^2 - 10 f_0 h' + 37 f_0^2 + 49\right] g'' + 2h'' (f_0 - g')^2, \tag{6.12}$$

where $c_1 = \frac{147H_0}{2} \neq 0$. If $g' = g_0 \in \mathbb{R}$ and $f_0 \neq g_0$ in (6.12), then we immediately have

$$h (w) = \frac{147H_0}{8} (f_0 - g_0) w^2 + c_2 w + c_3, \ c_2, c_3 \in \mathbb{R}.$$

If we put $\lambda = f_0$, $\mu = g_0$ and apply suitable translations on u, v, w, then we prove that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Next we assume $g'' \neq 0$ and consider the following cases:

(a) $h' = h_0 \in \mathbb{R}$. (6.12) follows

$$\frac{g''}{(f_0 - g')^3} = c_4, \tag{6.13}$$

for $c_4 = \frac{c_1}{37f_0^2 + 2h_0^2 - 10h_0 f_0 + 49}$. Solving (6.13) leads to

$$g (v) = f_0 v \pm \frac{1}{c_4} (2c_4 v + c_5)^{\frac{1}{2}} + c_6, \ c_5, c_6 \in \mathbb{R}.$$

As in previous case; after applying suitable translations and choosing constants, we prove the case (ii) of the theorem.

(b) $h'' \neq 0$. The partial derivative of (6.12) with respect to w gives

$$\frac{g''}{(f_0 - g')^2} + \frac{h'''}{h'' (2h' - 5f_0)} = 0, \tag{6.14}$$

where $h''' \neq 0$ due to $g'' \neq 0$. (6.14) implies

$$\frac{g''}{(f_0 - g')^2} = c_7 = -\frac{h'''}{h'' (2h' - 5f_0)}, \tag{6.15}$$

where $c_7 \in \mathbb{R}$, $c_7 \neq 0$. Considering (6.15) into (6.12) leads to

$$\lambda (f_0 - g') = \left[2 (h')^2 - 10h' f_0 + 37f_0^2 + 49\right] c_7 + 2h''. \tag{6.16}$$

The contradiction $g'' = 0$ is obtained by taking partial derivative of (6.16) with respect to v.

(2) $f'' \neq 0$. The symmetry follows $g'' \neq 0$. We have two cases:

(a) $h'' = 0$, $h' = h_0$. (6.2) can be rewritten as

$$\begin{aligned} &\frac{147H_0 (f' - g')^3}{2} = \\ &\left[37 (g')^2 + 2h_0^2 - 10g' h_0 + 49\right] f'' + \left[37 (f')^2 + 2h_0^2 - 10f' h_0 + 49\right] g''. \end{aligned} \tag{6.17}$$

Twice partial derivatives of (6.17) with respect to u and v give

$$\left(\frac{f'''}{f'}\right)' g'' + \left(\frac{g'''}{g''}\right)' f'' = 0,$$

which yields

$$\left(\frac{f'''}{f''}\right)' = c_8 f'', \quad \left(\frac{g'''}{g''}\right)' = -c_8 g'', \ c_8 \in \mathbb{R}. \tag{6.18}$$

Integrating (6.18) leads to

$$\frac{f'''}{f''} = c_8 f' + c_9, \quad \frac{g'''}{g''} = -c_8 g' + c_{10}, \; c_9, c_{10} \in \mathbb{R}.$$

Now taking partial derivative of (6.17) with respect to u and dividing f'' gives

$$\frac{441H_0\left(f'-g'\right)^2}{2} = \left[37\left(g'\right)^2 + 2h_0^2 - 10g'h_0 + 49\right]\left(c_8 f' + c_{10}\right) + \left[74f' + 10h_0\right]g''.$$

The last equation is a polynomial equation on f'; however, the leading coefficient is $\frac{441H_0}{2}$ which cannot be zero. This is a contradiction.

(b) $h'' \neq 0$. Let us put $\Phi = \frac{147H_0}{2}\left(f'-g'\right)^3$. Then considering (6.2) and the equation $\frac{\Phi_{uvw}}{f''g''h''} = 0$ gives

$$5\frac{f'''}{f''} + 5\frac{g'''}{g''} + 2\frac{h'''}{h''} = 0,$$

or

$$f''' = c_{11} f'', \; g''' = c_{12} g'', \; h''' = c_{13} h'', \tag{6.19}$$

for $c_{11}, c_{12}, c_{13} \in \mathbb{R}$ with $5c_{11} + 5c_{12} + 2c_{13} = 0$. Integrating (6.19) gives

$$f'' = c_1 f' + c_{14}, \; g'' = c_2 g' + c_{15}, \; h'' = c_3 h' + c_{16}. \tag{6.20}$$

Plugging (6.20) into (6.2) gives a polynomial equation on f'; however, the leading coefficient is $\frac{147H_0}{2}$ which cannot be zero. This completes the proof.

Theorem 6.3. immediately implies

Corollary 6.1. *Let M^3 be a translation hypersurface of type 4 in $\mathbb{I}^4$. Then, $H = const. \neq 0$ implies $K = 0$.*

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