



CHARACTERIZATION OF MATRIX CLASSES ASSOCIATED WITH p -ABSOLUTELY SUMMABLE SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. In this paper, we have characterized the bi-complex matrix transformations on $\ell_p(\mathbb{C}_2)$ space. We provide the necessary and sufficient requirements for an infinite bi-complex matrix to belong to $B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$. Additionally, we study when an infinite bi-complex matrix transforms $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$ and provide a counterexample for the converse statement. We also provide some non trivial examples to support our claims.

Keywords: Bi-complex numbers, $\mathbb{C}_2$ -linear operators, Characterization of matrix classes, p -Absolutely summable sequences, Sequence spaces.

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1. INTRODUCTION

The concept of bi-complex numbers are four-dimensional vectors that can be recognized similarly to complex numbers, which are two-dimensional vectors. Bi-complex numbers were initially developed using pairs of complex numbers. Why were bi-complex numbers used? What are their advantages and disadvantages? In order to answer those questions, the notion of bi-complex numbers was introduced by Segre [14]. The work of Price [12] contains a detailed review of the bi-complex space. In recent years, many authors [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15] have studied bi-complex numbers. Their theory finds applications in various fields of mathematics and physics, such as electromagnetic theory, fluid dynamics, and special relativity. Now, it has become a subject of research in both physics and mathematics, attracting significant interest from researchers because of its numerous applications in various domains of mathematical sciences and related disciplines.

Throughout the article, we denote the set of all bi-complex sequences by $\omega(\mathbb{C}_2)$.

Segre [14] defined the bi-complex number as follows:

$$z = w_1 + jw_2 = (a + ib) + j(c + id) = a + ib + jc + ijd,$$

where $w_1, w_2 \in \mathbb{C}(i)$; $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = -1, ij = ji$.

Here we consider $\mathbb{C}(i)$ as the set of all complex numbers with imaginary units i .

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The set of bi-complex numbers $\mathbb{C}_2$ is defined by

$$\mathbb{C}_2 = \{z = w_1 + jw_2 : w_1, w_2 \in \mathbb{C}(i)\}.$$

The four idempotent elements in $\mathbb{C}_2$ are $0, 1, e_1$ and e_2 , where $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$.

Every bi-complex number $z = w_1 + jw_2$ can be uniquely expressed as complex combination of e_1 and e_2 as follows;

$$z = \alpha_1 e_1 + \alpha_2 e_2,$$

where $\alpha_1 = w_1 - iw_2$ and $\alpha_2 = w_1 + iw_2$.

If we consider the two bi-complex numbers $z = \alpha'_1 e_1 + \alpha'_2 e_2$ and $w = \alpha''_1 e_1 + \alpha''_2 e_2$, then we have the following:

$$\begin{aligned} z \pm w &= (\alpha'_1 \pm \alpha''_1) e_1 + (\alpha'_2 \pm \alpha''_2) e_2, \\ zw &= (\alpha'_1 \alpha''_1) e_1 + (\alpha'_2 \alpha''_2) e_2. \end{aligned}$$

A function $\|\cdot\|_{\mathbb{C}_2} : \mathbb{C}_2 \rightarrow \mathbb{R}^+ \cup \{0\}$ such that

$$\begin{aligned} \|z\|_{\mathbb{C}_2} &= \sqrt{|w_1|^2 + |w_2|^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{|\alpha_1|^2 + |\alpha_2|^2} \end{aligned}$$

is a real valued norm on $\mathbb{C}_2$, where $z = w_1 + jw_2 = \alpha_1 e_1 + \alpha_2 e_2$.

This norm satisfies the following properties:

$$\|zw\|_{\mathbb{C}_2} \leq \sqrt{2} \|z\|_{\mathbb{C}_2} \|w\|_{\mathbb{C}_2} \text{ and } \|z \pm w\|_{\mathbb{C}_2} \leq \|z\|_{\mathbb{C}_2} + \|w\|_{\mathbb{C}_2}.$$

With the above norm, $\mathbb{C}_2$ forms a modified Banach space.

Let $(X(\mathbb{C}_2), \|\cdot\|_{X(\mathbb{C}_2)})$ be any bi-complex sequence space, then $X(\mathbb{C}_2)$ can be written as $X(\mathbb{C}_2) = X_1(\mathbb{C})e_1 + X_2(\mathbb{C})e_2$, where $X_1(\mathbb{C}) = \{(\xi_{k,1}) : \xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in X(\mathbb{C}_2)\}$ and $X_2(\mathbb{C}) = \{(\xi_{k,2}) : \xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in X(\mathbb{C}_2)\}$ and the norm $\|\cdot\|_{X(\mathbb{C}_2)}$ is given by

$$\|\xi\|_{X(\mathbb{C}_2)} = \frac{1}{\sqrt{2}} \sqrt{\left(\|\xi_{k,1}\|_{X_1(\mathbb{C})}^2 + \|\xi_{k,2}\|_{X_2(\mathbb{C})}^2\right)},$$

where $\|\cdot\|_{X_1(\mathbb{C})}$ and $\|\cdot\|_{X_2(\mathbb{C})}$ are the norms over the spaces $X_1(\mathbb{C})$ and $X_2(\mathbb{C})$, respectively.

Again, if $(Y(\mathbb{C}), \|\cdot\|_{Y(\mathbb{C})})$ is any given complex sequence space, we can always define a bi-complex sequence space $(Y(\mathbb{C}_2), \|\cdot\|_{Y(\mathbb{C}_2)})$, where

$Y(\mathbb{C}_2) = \left\{ \xi = (\xi_{k,1}e_1 + \xi_{k,2}e_2) : (\xi_{k,1}), (\xi_{k,2}) \in Y(\mathbb{C}) \right\}$ and the norm is given by

$$\|\xi\|_{Y(\mathbb{C}_2)} = \frac{1}{\sqrt{2}} \sqrt{\left(\|\xi_{k,1}\|_{Y(\mathbb{C})}^2 + \|\xi_{k,2}\|_{Y(\mathbb{C})}^2\right)},$$

Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite matrix of bi-complex numbers $\hat{a}_{nk}$, where

$$\hat{a}_{nk} = a_{nk}^\dagger e_1 + a_{nk}^\ddagger e_2, \quad a_{nk}^\dagger, a_{nk}^\ddagger \in \mathbb{C}(i), \text{ for all } n, k \in \mathbb{N}.$$

Then the matrix $\hat{A}$ can be uniquely written as $\hat{A} = A^\dagger e_1 + A^\ddagger e_2$, where $A^\dagger = [a_{nk}^\dagger]$ and $A^\ddagger = [a_{nk}^\ddagger]$ are infinite matrix of complex numbers.

Now, for any sequence $\xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in \omega(\mathbb{C}_2)$, we obtain the sequence $\hat{A}\xi$, the

$\widehat{A}$ -transform of ξ , in the following way;

$$\begin{aligned}
 (\widehat{A}\xi)_n &= \sum_{k=1}^{\infty} \widehat{a}_{nk} \xi_k \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2) (\xi_{k,1} e_1 + \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1} e_1 + a_{nk}^{\ddagger} \xi_{k,2} e_2) \\
 &= e_1 \sum_{k=1}^{\infty} a_{nk}^{\dagger} \xi_{k,1} + e_2 \sum_{k=1}^{\infty} a_{nk}^{\ddagger} \xi_{k,2}.
 \end{aligned} \tag{1.1}$$

Therefore,

$$(\widehat{A}\xi)_n = (A^{\dagger} \xi_{k,1})_n e_1 + (A^{\ddagger} \xi_{k,2})_n e_2$$

provided both the series on the right hand side of (1.1) converge for each $n \in \mathbb{N}$. In this way, we transform the $\mathbb{C}_2$ -sequence ξ into the $\mathbb{C}_2$ -sequence $\widehat{A}\xi$.

Let $\omega_1(\mathbb{C}_2)$ and $\omega_2(\mathbb{C}_2)$ be any two bi-complex sequence spaces. For any sequence $\xi \in \omega_1(\mathbb{C}_2)$, if $\widehat{A}\xi$ exist and is in $\omega_2(\mathbb{C}_2)$, then we say that $\widehat{A}$ defines a $\mathbb{C}_2$ -matrix mapping from $\omega_1(\mathbb{C}_2)$ into $\omega_2(\mathbb{C}_2)$. The class of all $\mathbb{C}_2$ -matrices $\widehat{A}$ which transform every sequence $\xi \in \omega_1(\mathbb{C}_2)$ into $\omega_2(\mathbb{C}_2)$ is denoted by $(\omega_1(\mathbb{C}_2), \omega_2(\mathbb{C}_2))$. In additionally, it can easily be verified that this transformation is linear. For more details one may refer to [5].

Definition 1.1. [4]

$$\begin{aligned}
 \ell_{\infty}(\mathbb{C}_2) &= \left\{ \xi = (\xi_k) \in \omega(\mathbb{C}_2) : \sup_{k \in \mathbb{N}} \|\xi_k\|_{\mathbb{C}_2} < \infty \right\}, \\
 \ell_p(\mathbb{C}_2) &= \left\{ \xi = (\xi_k) \in \omega(\mathbb{C}_2) : \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2}^p < \infty \right\}, \quad 0 < p < \infty.
 \end{aligned}$$

Lemma 1.1. [4] $\ell_{\infty}(\mathbb{C}_2)$ is a Banach space with the norm $\|\cdot\|_{\ell_{\infty}(\mathbb{C}_2)}$ defined by

$$\|\xi\|_{\ell_{\infty}(\mathbb{C}_2)} = \sup_{k \in \mathbb{N}} \|\xi_k\|_{\mathbb{C}_2}, \text{ for all } \xi = (\xi_k) \in \ell_{\infty}(\mathbb{C}_2).$$

Lemma 1.2. [4] For $1 \leq p < \infty$, $\ell_p(\mathbb{C}_2)$ is a Banach space with the norm $\|\cdot\|_{\ell_p(\mathbb{C}_2)}$ defined by

$$\|\xi\|_{\ell_p(\mathbb{C}_2)} = \left(\sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}}, \text{ for all } \xi = (\xi_k) \in \ell_p(\mathbb{C}_2).$$

Lemma 1.3. [4] (Bi-complex Hölder's Inequality) Let p and q be two real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\xi_k, \eta_k \in \mathbb{C}_2$, for $k \in \{1, 2, 3, \dots, n\}$. Then

$$\sum_{k=1}^n \|\xi_k \cdot \eta_k\|_{\mathbb{C}_2} \leq \sqrt{2} \left(\sum_{k=1}^n \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n \|\eta_k\|_{\mathbb{C}_2}^q \right)^{\frac{1}{q}}.$$

Lemma 1.4. [4] (Bi-complex Minkowski's Inequality) Let p be a real numbers with $1 < p < \infty$ such that $\xi_k, \eta_k \in \mathbb{C}_2$, for $k \in \{1, 2, 3, \dots, n\}$. Then

$$\left(\sum_{k=1}^n \|\xi_k + \eta_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n \|\eta_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}}.$$

Theorem 1.1. [10] *Let $A = [a_{nk}]$ be an infinite matrix satisfying the condition*

$$\sup_k \sum_{n=1}^{\infty} |a_{nk}|^2 < \infty.$$

Then $A \in B(\ell_2, \ell_2)$ and $\|A\| = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^2$.

Theorem 1.2. *Let $A = [a_{nk}]$ be an infinite matrix such that $A \in B(\ell_2, \ell_2)$. Then, the bounded linear transformation A assigns a matrix $A = [a_{nk}]$ such that $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$, for every $x = (x_k) \in \ell_2$. In addition, the following condition is satisfied:*

$$\sum_{k=1}^{\infty} |a_{nk}|^2 < \infty, \text{ for all } n \in \mathbb{N}.$$

Theorem 1.3. [10] *Let $A = [a_{nk}]$ be an infinite matrix. If $\sum_{k=0}^{\infty} |a_{nk}| < \infty$, for all $n \in \mathbb{N}$ and*

$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$. Then $\sum_{k=1}^{\infty} |a_{nk}|$ is uniformly convergent in $n \in \mathbb{N}$.

Theorem 1.4. *Let $A = [a_{nk}]$ be an infinite matrix. The necessary and sufficient condition for the matrix $A \in B(\ell_1, \ell_p)$, $1 \leq p < \infty$ is $\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty$.*

Theorem 1.5. *Let $A = [a_{nk}]$ be an infinite matrix satisfying the condition $\sup_{n,k} |a_{nk}| < \infty$, then $A \in B(\ell_1, \ell_{\infty})$.*

2. MAIN RESULTS

In this section, we establish the main results of this article.

Theorem 2.1. *Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix where $\hat{a}_{nk} = a_{nk}^{\dagger}e_1 + a_{nk}^{\ddagger}e_2$, for all $n, k \in \mathbb{N}$. If $\hat{A} = A^{\dagger}e_1 + A^{\ddagger}e_2 \in (X(\mathbb{C}_2), Y(\mathbb{C}_2))$, where $X(\mathbb{C}_2) = X_1(\mathbb{C})e_1 + X_2(\mathbb{C})e_2$ and $Y(\mathbb{C}_2) = Y_1(\mathbb{C})e_1 + Y_2(\mathbb{C})e_2$. Then $A^{\dagger} \in (X_1(\mathbb{C}), Y_1(\mathbb{C}))$ and $A^{\ddagger} \in (X_2(\mathbb{C}), Y_2(\mathbb{C}))$.*

Proof. Given that $\hat{A} = A^{\dagger}e_1 + A^{\ddagger}e_2 \in (X(\mathbb{C}_2), Y(\mathbb{C}_2))$. We have, for any sequence $\xi = (\xi_k) \in X(\mathbb{C}_2)$, where $\xi_k = \xi_{k,1}e_1 + \xi_{k,2}e_2$, for all $k \in \mathbb{N}$; $(\xi_{k,1}) \in X_1(\mathbb{C})$ and $(\xi_{k,2}) \in X_2(\mathbb{C})$, which implies that $\hat{A}\xi \in Y(\mathbb{C}_2) = Y_1(\mathbb{C})e_1 + Y_2(\mathbb{C})e_2$.

Then,

$$\begin{aligned}
 (\widehat{A}\xi)_n &= \sum_{k=1}^{\infty} \widehat{a}_{nk} \xi_k \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2) (\xi_{k,1} e_1 + \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1} e_1 + a_{nk}^{\ddagger} \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1}) e_1 + \sum_{k=1}^{\infty} (a_{nk}^{\ddagger} \xi_{k,2}) e_2.
 \end{aligned} \tag{2.2}$$

Since, both the series of the right hand side of (2.2) are convergent, So we have

$$(\widehat{A}\xi)_n = (A^{\dagger}\xi)_n e_1 + (A^{\ddagger}\xi)_n e_2,$$

where $A^{\dagger}\xi = \left(\sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1}) \right) \in Y_1(\mathbb{C})$ and $A^{\ddagger}\xi = \left(\sum_{k=1}^{\infty} (a_{nk}^{\ddagger} \xi_{k,2}) \right) \in Y_2(\mathbb{C})$.

So, $A^{\dagger}$ transforms the sequence $(\xi_{k,1}) \in X_1(\mathbb{C})$ to the space $Y_1(\mathbb{C})$ and $A^{\ddagger}$ transforms the sequence $(\xi_{k,2}) \in X_2(\mathbb{C})$ to the space $Y_2(\mathbb{C})$. Hence, $A^{\dagger} \in (X_1(\mathbb{C}), Y_1(\mathbb{C}))$ and $A^{\ddagger} \in (X_2(\mathbb{C}), Y_2(\mathbb{C}))$. $\square$

Theorem 2.2. Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix satisfying the condition $\|M\|_{\mathbb{C}_2} = \sup_k \sum_{n=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty$. Then $\widehat{A} \in B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ and $\|\widehat{A}\| = \|M\|_{\mathbb{C}_2}$.

Proof. We know that

$$\begin{aligned}
 \|M\|_{\mathbb{C}_2} &= \sup_k \sum_{n=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty \\
 &= \sup_k \sum_{n=1}^{\infty} \|a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2\|_{\mathbb{C}_2}^2 < \infty \\
 &= \sup_k \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \sqrt{|a_{nk}^{\dagger}|^2 + |a_{nk}^{\ddagger}|^2} \right)^2 < \infty \\
 &= \sup_k \frac{1}{2} \sum_{n=1}^{\infty} \left(|a_{nk}^{\dagger}|^2 + |a_{nk}^{\ddagger}|^2 \right) < \infty.
 \end{aligned}$$

This implies,

$$M_1 = \sup_k \sum_{n=1}^{\infty} |a_{nk}^{\dagger}|^2 < \infty \text{ and } M_2 = \sup_k \sum_{n=1}^{\infty} |a_{nk}^{\ddagger}|^2 < \infty.$$

Then by Theorem 1.1, we have $A^{\dagger}, A^{\ddagger} \in B(\ell_2, \ell_2)$ and $\|A^{\dagger}\| = M_1, \|A^{\ddagger}\| = M_2$. Hence, $\widehat{A} = A^{\dagger} e_1 + A^{\ddagger} e_2 \in (\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ and it is clear that $\widehat{A}$ is $\mathbb{C}_2$ -linear.

Now, we have to prove that $\widehat{A}$ is a bounded linear operator on $\ell_2(\mathbb{C}_2)$.

As $A^{\dagger}, A^{\ddagger} \in B(\ell_2, \ell_2)$, we can write

$$\|A^{\dagger}\xi_1\|_{\ell_2} \leq k_1 \|\xi_1\|_{\ell_2} \text{ and } \|A^{\ddagger}\xi_2\|_{\ell_2} \leq k_2 \|\xi_2\|_{\ell_2},$$

where $\xi_1 = (\xi_{k,1})$ and $\xi_2 = (\xi_{k,2}) \in \ell_2$.

We have,

$$\begin{aligned} \|\hat{A}\xi\|_{\ell_2(\mathbb{C}_2)} &= \frac{1}{\sqrt{2}} \sqrt{\|A^\dagger \xi_1\|_{\ell_2}^2 + \|A^\ddagger \xi_2\|_{\ell_2}^2} \\ &\leq \frac{1}{\sqrt{2}} \sqrt{k_1^2 \|\xi_1\|_{\ell_2}^2 + k_2^2 \|\xi_2\|_{\ell_2}^2} \\ &\leq \frac{k}{\sqrt{2}} \sqrt{\|\xi_1\|_{\ell_2}^2 + \|\xi_2\|_{\ell_2}^2}, \text{ where } k = \max\{k_1, k_2\}. \end{aligned}$$

Thus, $\|\hat{A}\xi\|_{\ell_2(\mathbb{C}_2)} \leq k\|\xi\|_{\ell_2(\mathbb{C}_2)}$.

On the other hand, if we take

$$M = M_1 e_1 + M_2 e_2,$$

we get

$$\begin{aligned} \|\hat{A}\| &= \frac{1}{\sqrt{2}} \sqrt{\|A^\dagger\|^2 + \|A^\ddagger\|^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{M_1^2 + M_2^2} = \|M\|_{\mathbb{C}_2}. \end{aligned}$$

□

Note: The necessity of the stated condition cannot be relaxed. To illustrate this, we consider the following counterexample. The counterexample below shows that if the condition is relaxed, the conclusion no longer holds. In particular, the matrix $\hat{A} = (\hat{a}_{nk})$ does not belong to $B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ because it fails to satisfy the required condition.

Example 2.1. Consider the matrix $\hat{A} = (\hat{a}_{nk})$, where

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \end{bmatrix}.$$

Here, for the first column of the matrix $\hat{A}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\| \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 \right\|_{\mathbb{C}_2}^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{2} \left(\left| \frac{1}{\sqrt{n}} \right|^2 + \left| \frac{1}{\sqrt{n}} \right|^2 \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

So, this matrix does not satisfy the given condition.

Now, $e_1 = (1, 0, 0, \dots) \in \ell_2(\mathbb{C}_2)$. Then

$$\widehat{A}e_1 = \begin{bmatrix} e_1 + e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_1 + e_2 \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 \\ \vdots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 \\ \vdots \end{bmatrix}.$$

But $\left(e_1 + e_2, \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2, \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2, \dots \right) \notin \ell_2(\mathbb{C}_2)$.

Theorem 2.3. Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix such that $\widehat{A} \in B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$. Then the bounded linear transformation $\widehat{A}$ assigns a matrix $\widehat{A} = [\widehat{a}_{nk}]$ such that

$$(\widehat{A}\xi)_n = \sum_{k=1}^{\infty} \widehat{a}_{nk} \xi_k, \text{ for every } \xi = (\xi_k) \in \ell_2(\mathbb{C}_2).$$

In addition, the following condition is satisfied

$$\sum_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2}^2 < \infty, \text{ for all } n \in \mathbb{N}.$$

Proof. Consider the bi-complex sequence (e'_k) defined as $e'_k = (0, 0, \dots, e_1 + e_2, 0, 0, \dots) = e_k^\dagger e_1 + e_k^\ddagger e_2$, is a basis for $\ell_2(\mathbb{C}_2)$, where $e_k^\dagger = e_k^\ddagger = (0, 0, \dots, 1, 0, 0, \dots)$, $(e_k^\dagger), (e_k^\ddagger)$ is a basis of ℓ_2 . Therefore, for any sequence $\xi = (\xi_k) \in \ell_2(\mathbb{C}_2)$ such that

$$\xi = \sum_{k=1}^{\infty} \xi_k e'_k = \left(\sum_{k=1}^{\infty} \xi_{k,1} e_k^\dagger \right) e_1 + \left(\sum_{k=1}^{\infty} \xi_{k,2} e_k^\ddagger \right) e_2.$$

Since, $\widehat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation, we have

$$\widehat{A}\xi = \sum_{k=1}^{\infty} \xi_k \widehat{A}e'_k.$$

On the other hand, since, $e'_k \in \ell_2(\mathbb{C}_2)$, for every $k \in \mathbb{N}$. By hypothesis

$$\widehat{A}e'_k = (\widehat{a}_{1k}, \widehat{a}_{2k}, \dots, \widehat{a}_{nk}, \dots) \in \ell_2(\mathbb{C}_2).$$

This implies,

$$\widehat{A}\xi = \sum_{k=1}^{\infty} \xi_k \widehat{a}_{nk}.$$

For the next part, since

$$\|\widehat{a}_{nk}\|_{\mathbb{C}_2} = \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2}.$$

So

$$\begin{aligned} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 &= \frac{1}{2} \left(|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2 \right) \\ &\leq |a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2. \end{aligned}$$

Taking summation on both sides, we get

$$\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 \leq \sum_{k=1}^{\infty} |a_{nk}^\dagger|^2 + \sum_{k=1}^{\infty} |a_{nk}^\ddagger|^2.$$

Then by Theorem 1.2, we can write

$$\sum_{k=1}^{\infty} |a_{nk}^\dagger|^2 < \infty \text{ and } \sum_{k=1}^{\infty} |a_{nk}^\ddagger|^2 < \infty.$$

Hence, $\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty$, for all $n \in \mathbb{N}$. This completes the proof. $\square$

Theorem 2.4. *Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix. If $\sum_{k=0}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \infty$, for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} = 0$. Then $\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}$ is uniformly convergent in $n \in \mathbb{N}$.*

Proof. Given that

$$\begin{aligned} & \sum_{k=0}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \infty, \text{ for each } n \in \mathbb{N}, \\ \Rightarrow & \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \infty, \text{ for each } n \in \mathbb{N} \\ \Rightarrow & \sum_{k=1}^{\infty} |a_{nk}^\dagger| < \infty \text{ and } \sum_{k=1}^{\infty} |a_{nk}^\ddagger| < \infty, \text{ for each } n \in \mathbb{N} \end{aligned}$$

where

$$\widehat{a}_{nk} = a_{nk}^\dagger e_1 + a_{nk}^\ddagger e_2.$$

Since $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} = 0$, then for all $\varepsilon > 0$ there exist $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } n > N(\varepsilon).$$

As, $|a_{nk}^\dagger| < \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \varepsilon$ and $|a_{nk}^\ddagger| < \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \varepsilon$.

Then, $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}^\dagger| = 0$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}^\ddagger| = 0$.

Then using the Theorem 1.3, there exist $M_1(\varepsilon)$ and $M_2(\varepsilon)$ such that

$$\sum_{k \geq M_1} |a_{nk}^\dagger| < \frac{\varepsilon}{2} \text{ and } \sum_{k \geq M_2} |a_{nk}^\ddagger| < \frac{\varepsilon}{2}, \text{ for all } n \in \mathbb{N}.$$

Take $M = \max\{M_1, M_2\}$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k \geq M} \|\hat{a}_{nk}\|_{\mathbb{C}_2} &= \frac{1}{\sqrt{2}} \sum_{k \geq M} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} \\ &\leq \frac{1}{\sqrt{2}} \sum_{k \geq M} |a_{nk}^\dagger| + \frac{1}{\sqrt{2}} \sum_{k \geq M} |a_{nk}^\ddagger| \\ &\leq \sum_{k \geq M} |a_{nk}^\dagger| + \sum_{k \geq M} |a_{nk}^\ddagger| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, the theorem is proved. $\square$

Theorem 2.5. *Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix. Then the necessary and sufficient condition for the matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$, $1 \leq p < \infty$, is that*

$$\sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty.$$

Proof. Necessity: Let us assume that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$. i.e., $\hat{A}$ is a $\mathbb{C}_2$ - bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_p(\mathbb{C}_2)$.

We have to prove that $\sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty$ or $\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty$, for all $k \in \mathbb{N}$.

Consider the bi-complex sequence (e'_k) defined as $e'_k = (0, 0, \dots, e_1 + e_2, 0, 0, \dots)$, for each $k \in \mathbb{N}$.

Then, $\hat{A}e'_k = (\hat{a}_{1k}, \hat{a}_{2k}, \hat{a}_{3k}, \dots, \hat{a}_{nk}, \dots)$, which is the k -th column of $\hat{A}$.

Now $\hat{A}e'_k \in \ell_p(\mathbb{C}_2)$, then $\|\hat{A}e'_k\|_{\ell_p(\mathbb{C}_2)} < \infty$. This implies that

$$\left(\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} < \infty, \text{ for all } k \in \mathbb{N} \Rightarrow \sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty.$$

Hence, the necessary condition is proved.

Sufficiency: Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix such that

$$M = \sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty, 1 \leq p < \infty. \quad (2.3)$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then

$$\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty.$$

Now,

$$\begin{aligned}
\|\hat{A}\xi\|_{\ell_p(\mathbb{C}_2)} &= \left(\sum_{n=1}^{\infty} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \|\hat{a}_{nk} \xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} (\sqrt{2})^p \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&= \sum_{k=1}^{\infty} \sqrt{2} \|\xi_k\|_{\mathbb{C}_2} \left(\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sqrt{2} M^{\frac{1}{p}} \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty.
\end{aligned}$$

This implies that $\hat{A}\xi = \left(\sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right)_n \in \ell_p(\mathbb{C}_2)$, i.e., $\hat{A}$ transforms any sequence in $\ell_1(\mathbb{C}_2)$ into the space $\ell_p(\mathbb{C}_2)$, for $1 \leq p < \infty$. Also, it is clear that $\hat{A}$ is a $\mathbb{C}_2$ -linear transformation. However, $\|\hat{A}\xi\|_{\ell_p(\mathbb{C}_2)} \leq \sqrt{2} M^{\frac{1}{p}} \|\xi\|_{\ell_1(\mathbb{C}_2)}$. This implies that $\hat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_p(\mathbb{C}_2)$. Hence, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$. $\square$

Note: The above condition can not be relaxed. For this we consider the following example.

Example 2.2. Consider the matrix $\hat{A} = (\hat{a}_{nk})$, where

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & e_1 + e_2 & \dots & e_1 + e_2 \dots \\ \frac{1}{2}e_1 + \frac{1}{2}e_2 & \frac{1}{2}e_1 + \frac{1}{2}e_2 & \dots & \frac{1}{2}e_1 + \frac{1}{2}e_2 \dots \\ \frac{1}{3}e_1 + \frac{1}{3}e_2 & \frac{1}{3}e_1 + \frac{1}{3}e_2 & \dots & \frac{1}{3}e_1 + \frac{1}{3}e_2 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{n}e_1 + \frac{1}{n}e_2 & \frac{1}{n}e_1 + \frac{1}{n}e_2 & \dots & \frac{1}{n}e_1 + \frac{1}{n}e_2 \dots \\ \vdots & \vdots & \dots & \vdots \ddots \end{bmatrix}.$$

For any column of the matrix $\hat{A}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\| \frac{1}{n}e_1 + \frac{1}{n}e_2 \right\|_{\mathbb{C}_2}^p \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \sqrt{\left| \frac{1}{n} \right|^2 + \left| \frac{1}{n} \right|^2} \right)^p \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^p \left(\frac{\sqrt{2}}{n} \right)^p \\
&= \sum_{n=1}^{\infty} \frac{1}{n^p}.
\end{aligned}$$

If $p = 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$. Thus the matrix does not satisfy the given condition. If we take $\xi = (e_1 + e_2, 0, 0, \dots) \in \ell_1(\mathbb{C}_2)$, then we have

$$\hat{A}\xi = \left(e_1 + e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2, \dots, \frac{1}{n}e_1 + \frac{1}{2}e_2, \dots \right), \text{ does not belong to } \ell_p(\mathbb{C}_2).$$

This implies that $\hat{A} \notin B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$.

However, if we take $1 < p < \infty$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.

Therefore, the matrix $\hat{A}$ satisfies the given condition, i.e., $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$.

Theorem 2.6. Let $A = [\hat{a}_{nk}]$ be an infinite bi-complex matrix satisfying the condition $\sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$, then $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_{\infty}(\mathbb{C}_2))$. The converse, however, is not true.

Proof. Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix such that

$$M = \sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty.$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then $\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

For all $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2} &\leq \sum_{k=1}^{\infty} \|\hat{a}_{nk} \xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} M \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} M \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} \\ &= \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)} < \infty. \end{aligned}$$

Therefore, $\sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2} \leq \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)} < \infty$ and hence, $\hat{A}\xi \in \ell_{\infty}(\mathbb{C}_2)$. Also, it is clear that $\hat{A}$ is a $\mathbb{C}_2$ -linear transformation.

On the other hand, $\|\hat{A}\xi\|_{\ell_{\infty}(\mathbb{C}_2)} \leq \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)}$, for all $\xi \in \ell_1(\mathbb{C}_2)$, this implies that $\hat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_{\infty}(\mathbb{C}_2)$. Hence, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_{\infty}(\mathbb{C}_2))$. This completes the proof. $\square$

For the converse part, consider the following example.

Example 2.3. Let $\hat{A} = [\hat{a}_{nk}]$ be a infinite matrix bi-complex matrix such that

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & 0 & 0 & \dots & 0\dots \\ 0 & 2e_1 + 2e_2 & 0 & \dots & 0\dots \\ 0 & 0 & 3e_1 + 3e_2 & \dots & 0\dots \\ \vdots & \vdots & \vdots & \dots & \vdots\dots \\ 0 & 0 & 0 & \dots & ne_1 + ne_2\dots \\ \vdots & \vdots & \vdots & \dots & \vdots\dots \end{bmatrix}.$$

Here,

$$\begin{aligned} \|a'_{nn}\|_{\mathbb{C}_2} &= \|ne_1 + ne_2\|_{\mathbb{C}_2} \\ &= \frac{1}{\sqrt{2}}\sqrt{n^2 + n^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{2n^2} \\ &= n. \end{aligned}$$

Then,

$$\sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} = \infty.$$

But we can show that, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

For any sequence $\xi = (\xi_n) = (\xi_{n,1}e_1 + \xi_{n,2}e_2) \in \ell_1(\mathbb{C}_2)$, we have

$$\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{n=1}^{\infty} \|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} < \infty.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} = 0.$$

This implies that, for some real number $c > 0$ and for $n \in \mathbb{N}$, we have

$$\|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} \leq \frac{c}{n^2}.$$

Thus, $\hat{A}\xi = \left(\sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right) = (a'_{nn}\xi_n)$. To show $\hat{A}\xi = (a'_{nn}\xi_n) \in \ell_\infty(\mathbb{C}_2)$, we have to show that

$$\sup_n \|a'_{nn}\xi_n\|_{\mathbb{C}_2} < \infty.$$

Now,

$$\begin{aligned} \|a'_{nn}\xi_n\|_{\mathbb{C}_2} &= \|(ne_1 + ne_2) \cdot (\xi_{n,1}e_1 + \xi_{n,2}e_2)\|_{\mathbb{C}_2} \\ &\leq \sqrt{2}\|ne_1 + ne_2\|_{\mathbb{C}_2}\|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot \frac{1}{\sqrt{2}}\sqrt{n^2 + n^2} \cdot \frac{c}{n^2} \\ &= \frac{\sqrt{2}c}{n}. \end{aligned}$$

Therefore, $\hat{A}\xi \in \ell_\infty(\mathbb{C}_2)$. It is also easy to show that $\hat{A}$ is $\mathbb{C}_2$ -linear and bounded transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. This implies that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Theorem 2.6 gives only the sufficient condition for an infinite bi-complex matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

In the following theorem, we establish the necessary and sufficient conditions for an infinite bi-complex matrix $\hat{A}$ such that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Theorem 2.7. *Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix. Then the necessary and sufficient condition for the matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$, is $\sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$.*

Proof. Necessity: Let $\hat{A} = [a_{nk}]$ be an infinite bi-complex matrix such that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then $M_1 = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

Thus, $\hat{A}\xi = \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \in \ell_\infty(\mathbb{C}_2)$. i.e., $\|\hat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} = \sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} < \infty$.

Now, for any $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} &\leq \sum_{k=1}^{\infty} \|\hat{a}_{nk}\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} M_1 \cdot \|\hat{a}_{nk}\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot M_1 \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

This implies that, $\sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$.

Sufficiency: Let $\hat{A} = [a_{nk}]$ be an infinite bi-complex matrix satisfying the condition

$$M_2 = \sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty.$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$ such that $\sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

To show $\hat{A}\xi = \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \in \ell_\infty(\mathbb{C}_2)$, we have to show that

$$\|\hat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} = \sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} < \infty.$$

Now, for any $n \in \mathbb{N}$

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot M_2 \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty. \end{aligned}$$

This implies that, $\sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} \leq \sqrt{2} \cdot M_2 \|\xi_k\|_{\ell_1(\mathbb{C}_2)} < \infty$. i.e., $\hat{A}\xi \in \ell_\infty(\mathbb{C}_2)$. So, $\hat{A}$ transform every sequence of $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. Also, it is easy to show that $\hat{A}$ is a $\mathbb{C}_2$ -linear

transformation.

On the other hand, since $\|\hat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} \leq \sqrt{2} \cdot M_2 \|\xi\|_{\ell_1(\mathbb{C}_2)}$, for all $\xi \in \ell_1(\mathbb{C}_2)$.

Therefore, $\hat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. Hence, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$. This completes the proof. $\square$

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