



## SOME IMPORTANT MAPS IN INTERVAL DIGITAL SIGNAL PROCESSING

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**Abstract.** In recent years there has been an increasing attention to intervals since an interval supplies necessary constraints on the uncertainties that arise from real world problems. Especially, in signal processing, when a signal value in a time  $t$  is completely unknown, intervals are used to process such signals, Mathematically, a signal is defined as a function from a subset of  $\mathbb{R}$  into  $\mathbb{C}$ . The notion of interval signal is a function from a subset of  $\mathbb{R}$  into  $\mathbb{I}_{\mathbb{R}}$  which is the set of all intervals. This paper presents some elemental maps for the mathematical functions of interval digital signal processing. Finally, the interval circular convolution map is introduced to provide a suitable way for processing of interval signals.

**Keywords:** Interval-valued map, interval signal, discrete-interval sequence, discrete-time systems, circular convolution operator

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### 1. INTRODUCTION

In recent years, interval analysis has been used in many application areas such as global optimization, economics, physics, control design, signal processing, robotics. The uncertainties that arise in real world problems have been taken into a suitable range thanks to the interval concept, and so the solution of the problem has been made more comfortable. In 1962 by R. Moore [7], the notion of interval has been widely handled to bound uncertainties arising from a system. In [7] is denoted a real interval  $x$  by  $x = [\underline{x}, \bar{x}]$  where  $\underline{x}$  and  $\bar{x}$  are the left and right endpoints of  $x$ , respectively. Further, the set of all real intervals is denoted by  $\mathbb{I}_{\mathbb{R}}$ . Algebraic structures on this set have studied in [1, 2]. Nowadays interval analysis tools are widely used by many authors in signal processing, [9, 3, 11].

Mathematically, a signal is a map an independent variable in any time. Sometimes, the value of a signal at any time  $t$  may not be determined exactly. We need the concept of interval-valued data and interval-based signal processing to resolve such uncertainties in process, [4, 10, 14]. In [11], a *discrete-interval sequence*  $X$  is a map from  $\mathbb{Z}$  into  $\mathbb{I}_{\mathbb{R}}$  and it is represented  $X = (X_n)_{n \in \mathbb{Z}}$ . Additionally, the concept of discrete-time interval signal and continuous-time interval signal are presented in [13, 5]. In these works, some new mathematical techniques

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are developed to obtain some approximate estimations on interval digital signal processing. Especially, in [11], some important tools for interval digital signal processing are presented. These tools are: casualty, stability, additivity and homogeneity. Further, the notion of interval convolution is advanced to represent uncertainty systems and signals.

This study provides convenience to the designers dealing with uncertainties digital signal processing thanks to new definitions and results. Firstly, linear interval systems, memoryless interval systems, casual interval systems and BIBO-stable interval systems are presented by using the interval sequence  $\mathbb{I}_{l_1}$  which denotes the set of all discrete-interval sequence  $X = (X_n)_{n \in \mathbb{Z}}$  such that

$$\sum_{n \in \mathbb{Z}} \|X_n\| < \infty.$$

Thus, many concepts in interval digital processing are based on mathematical foundations by using interval analysis and functional analysis techniques. Finally, the notion of interval circular convolution is denoted. This definition is interval version of circular convolution in classical signal processing and it is one of the most important operations of digital signal processing.

## 2. PRELIMINARIES

An interval  $x$  is the compact-convex subset of real numbers given by

$$x = [\underline{x}, \bar{x}] = \{t \in \mathbb{R} : \underline{x} \leq t \leq \bar{x}\}$$

If  $x$  contains a single real number then  $x$  is called degenerate interval and we write that  $\underline{x} = \bar{x}$ .

If  $x = [\underline{x}, \bar{x}]$  and  $y = [\underline{y}, \bar{y}]$  are intervals then

$$x + y = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \tag{2.1}$$

$$x - y = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]$$

where  $y = [-\bar{y}, -\underline{y}]$ ,

$$\alpha x = \alpha [\underline{x}, \bar{x}] = \begin{cases} [\alpha \underline{x}, \alpha \bar{x}], & \alpha \geq 0 \\ [\alpha \bar{x}, \alpha \underline{x}], & \alpha < 0 \end{cases} \tag{2.2}$$

and

$$x \cdot y = [\underline{x}, \bar{x}] \cdot [\underline{y}, \bar{y}] = [\min M, \max M] \tag{2.3}$$

where  $M = \{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}$ .

The details of interval arithmetic in this article can be founded in [6, 8].

The set of all intervals is denoted by  $\mathbb{I}_{\mathbb{R}}$  and it is not a linear space since some elements have not an additive inverse. For example, the interval  $[2, 3]$  has not an additive inverse, since

$$[2, 3] + (-1)[2, 3] = [-1, 1]$$

and the set  $[-1, 1]$  is different from the unit element  $\{0\}$ . The set  $\mathbb{I}_{\mathbb{R}}$  is a quasilinear space with the operations given by 2.1, 2.2 and the partial order relation "  $\preceq$  ",

$$x \preceq y \iff \underline{y} \leq \underline{x}, \bar{x} \leq \bar{y}. \tag{2.4}$$

The quasilinear spaces are investigated in [1]. If an interval  $x$  has an additive inverse then is called regular, otherwise it is called singular. The set of all regular elements of  $\mathbb{I}_{\mathbb{R}}$  is isometrically isomorphic to  $\mathbb{R}$ .

Further, the arithmetic interval has psedo-distributivity:

$$x(y + z) \preceq xy + xz$$

where  $x, y, z \in \mathbb{I}_{\mathbb{R}}$  and " $\preceq$ " is the partial order relation given by 2.4.

Now, let us give some basic notions with respect to the discrete interval sequences.

A discrete interval sequence is defined as follows:

$$X = (X_n)_{n \in \mathbb{Z}} = (\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots)$$

such that  $X_n \in \mathbb{I}_{\mathbb{R}}$  for  $n \in \mathbb{Z}$ , [13]. The sequence  $X$  is called as discrete-time interval signal in interval signal processing.

The set of all discrete interval sequences  $X = (X_n)_{n \in \mathbb{Z}}$  such that

$$\sum_{n \in \mathbb{Z}} \|X_n\|_{\mathbb{I}_{\mathbb{R}}}^2 < \infty$$

is denoted by  $\mathbb{I}_{l_2}$  and it is a Hilbert quasilinear space, [13]. Further, the norm on  $\mathbb{I}_{\mathbb{R}}$  is defined by

$$\|x\|_{\mathbb{I}_{\mathbb{R}}} = \|[x, \bar{x}]\|_{\mathbb{I}_{\mathbb{R}}} = \max\{|x|, |\bar{x}|\},$$

[1]. For example,  $\|[-3, 1]\| = \max\{|-3|, |1|\} = 3$ .

The set  $\mathbb{I}_{l_2}$  is in rapport with the set of all finite energy discrete-time interval signals. The addition, scalar multiplication operations and the relation are follows:

$$\begin{aligned} X + Y &= (\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots) + (\dots, Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, \dots) \\ &= (\dots, X_{-2} + Y_{-2}, X_{-1} + Y_{-1}, X_0 + Y_0, X_1 + Y_1, X_2 + Y_2, \dots), \end{aligned}$$

$$\begin{aligned} \lambda X &= \alpha(\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots) \\ &= (\dots, \alpha X_{-2}, \alpha X_{-1}, \alpha X_0, \alpha X_1, \alpha X_2, \dots) \end{aligned}$$

and the relation

$$(X_n)_{n \in \mathbb{Z}} \lesssim (Y_n)_{n \in \mathbb{Z}} \text{ iff } X_n \preceq Y_n \text{ for each } n \in \mathbb{Z}.$$

For example, if

$$X_n = \begin{cases} [-1, \cos(\frac{\pi}{2}n)] & , \text{ for } n \in \mathbb{Z}^+ \\ 0 & , \text{ otherwise} \end{cases}$$

then  $X = (X_n)_{n \in \mathbb{Z}} \notin \mathbb{I}_{l_2}$ . If

$$X_n = \begin{cases} [0, \frac{1}{n^2}] & , \text{ for } n \in \mathbb{Z}^+ \\ 0 & , \text{ otherwise} \end{cases}$$

then  $Y = (Y_n)_{n \in \mathbb{Z}} \in \mathbb{I}_{l_2}$ .

### 3. DISCRETE INTERVAL SYSTEMS AND SOME THEIR PROPERTIES

Discrete interval systems are important operators on the space of discrete interval sequences. Some of the most commonly used systems in applications are: Linear, memoryless,

casual, stable interval systems. These systems is both important in applications and amenable to analysis. In this section, these discrete interval systems are examined and the properties of some important maps between interval sequence spaces are investigated.

Mathematically, a discrete interval system  $F$  maps an input discrete interval sequence  $X$  into an output discrete interval sequence  $Y$  such that

$$Y = F(x).$$

Typically, the space of discrete interval sequences is  $\mathbb{I}_{l_2}$  or a subspace of  $\mathbb{I}_{l_2}$ .

**3.1. Linear Interval System.** A discrete interval system  $F$  is called linear if for any discrete interval sequences  $X$  and  $Y$ ,

$$F(X + Y) = F(X) + F(Y),$$

$$F(\lambda X) = \lambda F(X)$$

and

$$\text{if } X \lesssim Y \text{ then } F(X) \lesssim F(Y)$$

where  $\lambda$  is a real constant.

The *Kronecker delta sequence* is defined in [12] as follows:

$$\delta_n = \begin{cases} 1 & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases} , n \in \mathbb{Z}.$$

According to this definition we define *an interval Kronecker delta sequence* as follows:

$$\Delta_n = [\delta_n, \delta_n].$$

In other words,

$$\Delta_n = \begin{cases} [1, 1] & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases} , n \in \mathbb{Z}.$$

For example, the product of the sequence  $\Delta_n$  with a constant sequence  $X = (\dots, [1, 2], [1, 2], [1, 2], \dots)$  is as follows:

$$\Delta_n.X = \begin{cases} [1, 2] & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases} , n \in \mathbb{Z}.$$

An interval sequence  $H_n$  is called the *impulse response* of linear interval system  $F$  if input  $\Delta_n$  produces  $H_n$ , i.e.;  $H_n = F(\Delta_n)$ .

**3.2. Memoryless Interval System.** A discrete interval system  $F$  is called a memoryless system if the output  $Y_n$  depends only on its inputs  $X_n$ .

**Definition 3.1.** *The projection via domain restriction operator  $1_U : \mathbb{I}_{l_2} \rightarrow \mathbb{I}_{l_2}$  is defined by*

$$Y = (Y_n)_{n \in \mathbb{Z}} = 1_U(X_n)_{n \in \mathbb{Z}}$$

with

$$Y_n = \begin{cases} X_n & , \text{ for } n \in U \\ 0 & , \text{ otherwise} \end{cases} ,$$

where  $U$  is a subset of  $\mathbb{Z}$ .

According to this definition,  $F$  is a memoryless system if at a time index  $t$

$$1_{\{t\}}X_n = 1_{\{t\}}X'_n$$

then

$$1_{\{t\}}Y_n = 1_{\{t\}}Y'_n$$

where  $X_n$  and  $X'_n$  are inputs,  $Y_n$  and  $Y'_n$  are outputs.

**3.3. Shift-Invariant Interval System.** A discrete interval system  $F$  is called a *shift-invariant system* when for any integer  $k$  and the inputs  $X$  and  $X'$ , if

$$X_n = X_{n-k}$$

then

$$F(X_n) = F(X'_{n-k}).$$

Namely, in a shift-invariant system, shifting in the time of the input has the same variation in the time of the output.

**3.4. Casual Interval System.** A discrete interval system  $F$  is called a *casual system* if the output of the system  $F$  at time  $n$  depends on the input only up to time  $n$ . Namely, in the casual interval system  $F$  if

$$1_{\{-\infty, \dots, k\}}X_n = 1_{\{-\infty, \dots, k\}}X'_n$$

then

$$1_{\{-\infty, \dots, k\}}Y_n = 1_{\{-\infty, \dots, k\}}Y'_n$$

where  $X_n$  and  $X'_n$  are inputs,  $Y_n$  and  $Y'_n$  are outputs,  $1_{\{-\infty, \dots, k\}}$  is domain restriction operator.

**3.5. BIBO Stable Interval System.** A discrete interval system  $F$  is called a *BIBO stable* if for each bounded interval sequence converts to bounded interval sequence.

If  $F$  is a BIBO stable interval system and

$$\sup_{n \in \mathbb{Z}} \|X_n\|_{\mathbb{R}} < \infty$$

then

$$\sup_{n \in \mathbb{Z}} \|Y_n\|_{\mathbb{R}} < \infty$$

where  $X_n = F(X_n)$ ,  $n \in \mathbb{Z}$ .

Now let us give an example of how these operators are used in applications.

Consider the discrete interval system  $F$  such that for the interval sequence  $X = (X_n)_{n \in \mathbb{Z}}$ ,

$$F(X_n) = \frac{1}{3}(X_{n-1} + X_n + X_{n+1}).$$

This system is called *averaging interval operator*.

Averaging interval operator is linear system:

$$\begin{aligned} F(X_n + \tilde{X}_n) &= \frac{1}{3}((X_{n-1} + \tilde{X}_{n-1}) + (X_n + \tilde{X}_n) + (X_{n+1} + \tilde{X}_{n+1})) \\ &= \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) + \frac{1}{3}(\tilde{X}_{n-1} + \tilde{X}_n + \tilde{X}_{n+1}) \\ &= F(X_n) + F(\tilde{X}_n) \end{aligned}$$

and

$$\begin{aligned} F(\lambda X_n) &= \frac{1}{3}(\lambda X_{n-1} + \lambda X_n + \lambda X_{n+1}) \\ &= \lambda\left(\frac{1}{3}(X_{n-1} + X_n + X_{n+1})\right) \\ &= \lambda F(X_n). \end{aligned}$$

Further, assume that  $X \lesssim \tilde{X}$ . Since  $X_n \subseteq \tilde{X}_n$  for each  $n \in \mathbb{Z}$ , we write that

$$\begin{aligned} F(X_n) &= \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) \\ &\lesssim \frac{1}{3}(\tilde{X}_{n-1} + \tilde{X}_n + \tilde{X}_{n+1}) \\ &= F(\tilde{X}_n). \end{aligned}$$

This system is BIBO stable:

Suppose that  $\sup_{n \in \mathbb{Z}} \|X_n\|_{\mathbb{I}_{\mathbb{R}}} < \infty$ . Then

$$\begin{aligned} \sup_{n \in \mathbb{Z}} \|F(X_n)\| &= \sup_{n \in \mathbb{Z}} \left\| \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) \right\| \\ &= \frac{1}{3} \sup_{n \in \mathbb{Z}} \|X_{n-1} + X_n + X_{n+1}\| \\ &\leq \frac{1}{3} \sup_{n \in \mathbb{Z}} \{\|X_{n-1}\| + \|X_n\| + \|X_{n+1}\|\} \\ &= \sup_{n \in \mathbb{Z}} \|X_n\| < \infty. \end{aligned}$$

But this discrete interval system is neither memoryless nor casual.

The matrix of the averaging interval operator is as follows:

For  $F(X_n) = Y_n, n \in \mathbb{Z}$

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ Y_{-1} \\ \boxed{Y_0} \\ Y_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \boxed{1} & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ X_{-1} \\ \boxed{X_0} \\ X_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Further, it is shown that this matrix is Toeplitz matrix.

As is known that the convolution operator has an important situation in digital signal processing. For this purpose, the interval convolution operator is defined in [11]. Now let us present the concept of interval circular convolution operator used to define the discrete-time Fourier transform.

In real life, many computations are done on finite impuls. For this purpose, the first method is to set  $X_i = [0, 0]$  for all  $i$  outside of  $\{0, 1, \dots, N - 1\}$ . Later; the second method is

as follows:

$$X = (\dots, X_{N-1}, X_0, X_1, \dots, X_{N-1}, X_0, X_1, \dots).$$

Namely, a finite-length interval sequence can be written as follows:

$$X_{(i+k)N} = X_i, k \in \mathbb{Z}.$$

Let  $F$  be a linear interval system and  $H_n$  be the impulse response of  $F$ . If  $H_n$  is an element of  $\mathbb{I}_{l_1}$ , then we obtain the interval convolution as usual,

$$\begin{aligned} Y_i &= (X * H)_i \\ &= \sum_{i \in \mathbb{Z}} X_i H_{i-k} \\ &= \sum_{i \in \mathbb{Z}} H_i X_{i-k}. \end{aligned}$$

Since  $X$  is  $N$ -periodic interval sequence, then  $Y$  is  $N$ -periodic:

$$\begin{aligned} Y_{i+N} &= \sum_{i \in \mathbb{Z}} H_i X_{i+N-k} \\ &= \sum_{i \in \mathbb{Z}} H_i X_{i-k} \\ &= Y_n. \end{aligned}$$

Let us write the impulse response  $H_n$  with period  $N$  as follows:

$$(H^N)_n = \sum_{i \in \mathbb{Z}} H_{n-iN}.$$

The last sum is convergent, since  $H \in \mathbb{I}_{l_1}$ .

Now we will write the interval convolution of a  $N$ - periodic interval input sequence  $X$  and a  $N$ - periodic interval impulse response  $H$ .

$$\begin{aligned} (H * X)_n &= \sum_{i \in \mathbb{Z}} H_i X_{n-i} \\ &= \sum_{k \in \mathbb{Z}} \sum_{i=kN}^{(k+1)N-1} H_i X_{n-i} \\ &= \sum_{k \in \mathbb{Z}} \sum_{l=0}^{N-1} H_{l+kN} X_{n-l-kN}. \end{aligned}$$

If we do the change of variable  $i = l$ .

$$\sum_{k \in \mathbb{Z}} \sum_{l=0}^{N-1} H_{l+kN} X_{n-l-kN} = \sum_{k \in \mathbb{Z}} \sum_{i=0}^{N-1} H_{i+kN} X_{n-i}.$$

Since  $H \in \mathbb{I}_{l_1}$  we can write that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_{i=0}^{N-1} H_{i+kN} X_{n-i} &= \sum_{i=0}^{N-1} \sum_{k \in \mathbb{Z}} H_{i+kN} X_{n-i} \\ &= \sum_{i=0}^{N-1} H_i^N X_{n-i} \\ &= \sum_{i=0}^{N-1} H_i^N X_{(n-i) \bmod N}. \end{aligned}$$

Thus, the interval circular convolution of  $N$ -periodic interval sequences  $H$  and  $X$  is defined as

$$\begin{aligned} (H \otimes X)_n &= \sum_{i=0}^{N-1} X_i H_{(n-i) \bmod N} \\ &= \sum_{i=0}^{N-1} X_{(n-i) \bmod N} H_i. \end{aligned}$$

#### 4. CONCLUSION

In this study, linear interval systems, memoryless interval systems, casual interval systems and BIBO-stable interval systems are presented by using the interval sequences. Further, the interval circular convolution is denoted by using interval analysis and functional analysis techniques. This study provides convenience to the designers dealing with uncertainties digital signal processing thanks to new definitions and results. Later, other important operators in the classical digital processing field can be adapted with the help of intervals to contribute to interval signal processing.

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