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CERTAIN GEOMETRIC PROPERTIES OF THE COTANGENT BUNDLE ENDOWED WITH THE HORIZONTALLY DEFORMED SASAKI METRIC

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Abstract. This study presents a new family of Riemannian metrics on the cotangent bundle of a Riemannian manifold, referred to as horizontally deformed Sasaki metric. These metrics extend the classical Sasaki metric by incorporating a horizontal deformation. The paper conducts a detailed examination of the geometric structure of the cotangent bundle endowed with the horizontally deformed Sasaki metric, focusing on the Levi-Civita connection and various curvature tensors. Additionally, the geometry of a naturally arising hypersurface within the cotangent bundle is investigated through the derivation of its induced Levi-Civita connection and associated curvature formulas. The findings broaden the existing class of Riemannian metrics defined on cotangent bundles and provide deeper insight into the geometric framework of Riemannian manifolds and their cotangent structures. These contributions enhance the understanding of invariant metric structures on cotangent bundles and offer a foundation for further research in differential geometry.
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1. Introduction

One of the earliest contributions in the literature that treats the cotangent bundle of a manifold as a Riemannian manifold is due to Patterson and Walker [10], who introduced a Riemannian metric on the cotangent bundle derived from an affine symmetric connection on the base manifold, now known as the Riemannian extension of the connection. This construction was subsequently generalized by Sekizawa [12], who developed a classification of natural transformations mapping affine connections on manifolds to metrics on their cotangent bundles. His work produced a two-parameter family of metrics termed natural Riemann extensions, which have since become the focus of extensive study.

This foundational work has inspired further research into alternative Riemannian metrics on cotangent bundles (see [2, 7, 9]), with the Sasaki metric [11] emerging as one of the most widely studied. In this context, Ağca introduced the notion of g-natural metrics on cotangent

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bundles, drawing from similar constructions previously developed for tangent bundles [1]. For related works, see [3, 4, 5, 8]. In addition, deformations of the Sasaki metric have been investigated in the setting of anti-para-Kähler and standard Kähler manifolds (see [6, 14, 15, 16, 18]).

In this work, we propose a new deformation, termed the horizontally deformed Sasaki metric, which acts on the horizontal distribution of the cotangent bundle. This construction broadens the class of invariant Riemannian metrics on cotangent bundles and sheds new light on the geometric interplay between a manifold and its cotangent bundle.

Motivated by fundamental questions concerning the influence of such deformations on curvature behavior, connections, and induced geometric structures, we undertake a systematic investigation of this metric. These inquiries bear relevance to both theoretical and applied aspects of differential geometry.

The structure of the paper is as follows:

Section 2 reviews fundamental results on the cotangent bundle.

Section 3 establishes the foundational properties of the horizontally deformed Sasaki metric and derives explicit formulas for the associated Levi-Civita connection.

Section 4 examines all types of curvature in the cotangent bundle under this metric, highlighting the geometric effects of the horizontal deformation in comparison with the classical Sasaki metric.

Section 5 explores the geometry of hypersurfaces within the cotangent bundle, deriving their induced Levi-Civita connections. These results generalize previous work and lay the groundwork for further study of submanifolds in the context of deformed Riemannian structures.

2. Basic information and preliminary results

Consider an m-dimensional Riemannian manifold (M^m, g), with its associated cotangent bundle T^*M and the canonical projection map $\pi : T^*M \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a corresponding local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$ on T^*M, where the index $\bar{i}$ is defined by $\bar{i} = i + 1$. Here p_i denote the components of a covector $p \in T^*_x M$, expressed relative to the canonical coframe $\{dx^i\}$ for each point $x \in U$.

Let $C^\infty(M)$ be the ring of smooth real-valued functions on M and $\Im^r_s(M)$ (respectively, $\Im^r_s(T^*M)$) denote the $C^\infty(M)$-module of smooth tensor fields of type (r, s) on the manifold M (respectively, on its cotangent bundle T^*M). The Levi-Civita connection associated with the Riemannian metric g is denoted by ∇, and its corresponding Christoffel symbols are written as Γ^k_{ij}.

The connection ∇ naturally defines a splitting of the tangent bundle of the cotangent bundle T^*M into a direct sum:

$$TT^*M = VT^*M \oplus HT^*M, \tag{2.1}$$

where VT^*M and HT^*M represent the vertical and horizontal distributions, respectively. These distributions are given explicitly by

$$\begin{aligned} V_{(x,p)}T^*M &= \ker(d_{(x,p)}\pi) = \{\omega_i\partial_{\bar{i}}|_{(x,p)} \mid \omega_i \in \mathbb{R}\}, \\ H_{(x,p)}T^*M &= \{X^i\partial_i|_{(x,p)} + X^i p_a\Gamma^a_{hi}\partial_{\bar{h}}|_{(x,p)} \mid X^i \in \mathbb{R}\} \end{aligned}$$

for each point $(x,p) \in T^*M$, V and H represent the projections onto the vertical and horizontal sub-bundles, induced by the connection ∇ respectively, and $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} = \frac{\partial}{\partial p_i}$ represent the coordinate vector fields.

The mapping

$$w \mapsto {}^V w = w_i\partial_{\bar{i}}|_{(x,p)}$$

defines an isomorphism between T^*_xM and $V_{(x,p)}T^*M$. Similarly, the mapping

$$v \mapsto {}^H v = v^i\partial_i|_{(x,p)} + v^i p_a\Gamma^a_{hi}\partial_{\bar{h}}|_{(x,p)}$$

establishes an isomorphism between the vector spaces T_xM and $H_{(x,p)}T^*M$. Consequently, any tangent vector $W \in T_{(x,p)}T^*M$ can be uniquely expressed as

$$W = {}^H v + {}^V w, \tag{2.2}$$

where $v \in T_xM$ and $w \in T^*_xM$ are uniquely determined.

Let $Z = Z^i\partial_i$ and $\omega = \omega_i dx^i$ be the local representations of a vector field $Z \in \Im^1_0(M)$ and a covector field $\omega \in \Im^0_1(M)$, respectively, in a local chart $(U, x^i)_{i=\overline{1,m}}$. The horizontal lifts of Z, denoted by ${}^HZ$ in $\Im^1_0(T^*M)$, as well as the vertical lift of ω, denoted by ${}^V\omega$ in $\Im^1_0(T^*M)$, are defined as follows:

$$\begin{aligned} {}^HZ &= Z^i\partial_i + p_h\Gamma^h_{ij}Z^j\partial_{\bar{i}}, && (2.3) \\ {}^V\omega &= \omega_i\partial_{\bar{i}}, && (2.4) \end{aligned}$$

In particular, the vertical distribution ${}^Vp$ on T^*M is defined by

$${}^Vp = p_i{}^V(dx^i) = p_i\partial_{\bar{i}},$$

which is also known as the canonical or Liouville vector field on T^*M.

The Lie bracket operation between vertical and horizontal vector fields on the cotangent bundle T^*M can be explicitly described by the formulas

$$\begin{cases} [{}^V\omega, {}^V\eta] = 0, \\ [{}^HZ, {}^V\eta] = {}^V(\nabla_Z\eta), \\ [{}^HZ, {}^HY] = {}^H[Z,Y] + {}^V(p\mathrm{R}(Z,Y)), \end{cases} \tag{2.5}$$

for each vector fields $Y, Z \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$[13], such that $p\mathrm{R}(Z,Y) = p_a\mathrm{R}^a_{ijk}Z^iY^jdx^k$, where R^a_{ijk} are local components of R on (M^m, g).

Given a Riemannian manifold (M^m, g) , the map

$$\begin{array}{cccc} \sharp : & \Im^0_1(M) & \to & \Im^1_0(M) \\ & \theta & \mapsto & \sharp\theta \end{array}$$

defined by

$$g(\sharp\theta, Z) = \theta(Z)$$

for each vector field $Z \in \Im^1_0(M)$ is $C^\infty(M)$-linear isomorphism.
Locally, for each $\theta = \theta_i dx^i \in \Im^0_1(M)$, we have

$$\sharp\theta = g^{ij}\theta_i\partial_j,$$

where (g^{ij}) is the inverse of the matrix (g_{ij}).

For each point $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by

$$g^{-1}(\omega, \eta) = g(\sharp\omega, \sharp\eta) = g^{ij}\omega_i\eta_j,$$

from this we write $\sharp\omega = g^{-1} \circ \omega$.

Lemma 2.1. *Given a Riemannian manifold (M^m, g). Then, the following results are satisfied:*

$$\begin{aligned}
\nabla_X \sharp\omega &= \sharp(\nabla_X\omega), && (2.6)\\
Xg^{-1}(\omega,\eta) &= g^{-1}(\nabla_X\omega,\eta) + g^{-1}(\omega,\nabla_X\eta), && (2.7)\\
\sharp(\omega \mathrm{R}(X,Y)) &= -\mathrm{R}(X,Y)\sharp\omega, && (2.8)\\
\omega(\nabla_X F) &= \nabla_X(\omega F) - (\nabla_X\omega)F, && (2.9)\\
\omega(\nabla_X \mathrm{R})(Y,Z) &= \nabla_X(\omega \mathrm{R}(Y,Z)) - (\nabla_X\omega)\mathrm{R}(Y,Z) - \omega \mathrm{R}(\nabla_X Y, Z) &&\\
&\quad -\omega \mathrm{R}(Y, \nabla_X Z), && (2.10)\\
\sharp(\omega(\nabla_X \mathrm{R})(Y,Z)) &= -(\nabla_X \mathrm{R})(Y,Z)\sharp\omega. && (2.11)
\end{aligned}$$

for each vector fields $X, Y, Z \in \Im^1_0(M)$, covector fields $\omega, \eta \in \Im^0_1(M)$ and $F \in \Im^1_1(M)$, where ∇ is the Levi-Civita connection of (M^m, g) and R is its curvature tensor.

3. The horizontally deformed Sasaki metric

Definition 3.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M, we introduce a new Riemannian metric g^f on T^*M defined as follows:*

$$\begin{aligned}
g^f({}^HZ, {}^HY)_{(x,p)} &= f(r)g_x(Z,Y), && (3.12)\\
g^f({}^HZ, {}^V\eta)_{(x,p)} &= 0 &&\\
g^f({}^V\omega, {}^V\eta)_{(x,p)} &= g_x^{-1}(\omega,\eta), && (3.13)
\end{aligned}$$

for each $(x,p) \in T^*M$, vector fields $Y, Z \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$, where $f : [0, +\infty[\to]0, +\infty[$ is a smooth real-valued function and $r = g^{-1}(p,p)$. We refer to this metric as the horizontally deformed Sasaki metric.
For $f = 1$, g^f coincides with the Sasaki metric introduced in [11].

Lemma 3.1. [17] *Let (M^m, g) be a Riemannian manifold, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth real-valued function. Then, the following results are satisfied:*

(1) ${}^HZ(f(r)) = 0$,
(2) ${}^V\omega(f(r)) = 2f'(r)g^{-1}(\omega, p)$,
(3) ${}^HZg(X,Y) = Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$,
(4) ${}^HZg^{-1}(\eta,\mu) = Zg^{-1}(\eta,\mu) = g^{-1}(\nabla_Z\eta,\mu) + g^{-1}(\eta,\nabla_Z\mu)$,
(5) ${}^HZg^{-1}(\eta,p) = g^{-1}(\nabla_Z\eta,p)$,

(6) ${}^{V}\omega g^{-1}(\eta, p) = g^{-1}(\eta, \omega)$,
(7) ${}^{V}\omega g(Y, Z) = 0$,
(8) ${}^{V}\omega g^{-1}(\eta, \mu) = 0$,

for each vector fields $X, Y, Z \in \Im^1_0(M)$ *and covector fields* $\omega, \eta, \mu \in \Im^0_1(M)$, *where* $r = g^{-1}(p, p)$.

Proof. Lemma 3.1 is a direct consequence of the expressions given in equations (2.3) and (2.4). □

Lemma 3.2. *Let* (M^m, g) *be a Riemannian manifold, and consider its cotangent bundle* T^*M *endowed with the horizontally deformed Sasaki metric* g^f. *Then, the following relations hold:*

$$\begin{aligned} (1)\ {}^{H}Z\left(g^f({}^{H}X, {}^{H}Y)\right) &= g^f({}^{H}(\nabla_Z X), {}^{H}Y) + g^f({}^{H}X, {}^{H}(\nabla_Z Y)), \\ (2)\ {}^{V}\mu\left(g^f({}^{H}X, {}^{H}Y)\right) &= 2f'(r)g(X, Y)g^{-1}(\mu, p), \\ (3)\ {}^{H}X\left(g^f({}^{V}\eta, {}^{V}\mu)\right) &= g^f({}^{V}(\nabla_X \eta), {}^{V}\mu) + g^f({}^{V}\eta, {}^{V}(\nabla_X \mu)), \end{aligned}$$

for each vector fields $X, Y, Z \in \Im^1_0(M)$ *and covector fields* $\eta, \mu \in \Im^0_1(M)$.

Proof. Lemma 3.2 is an immediate consequence of Lemma 3.1. □

We now define the Levi-Civita connection $\widetilde{\nabla}$ associated with the horizontally deformed Sasaki metric g^f on T^*M, derived through application of the Koszul formula

$$\begin{aligned} 2g^f(\widetilde{\nabla}_P Q, W) &= Pg^f(Q, W) + Qg^f(W, P) - Wg^f(P, Q) + g^f(W, [P, Q]) \\ &\quad + g^f(Q, [W, P]) - g^f(P, [Q, W]), \end{aligned} \tag{3.14}$$

for each vector fields $P, Q, W \in \Im^1_0(T^*M)$. The following theorem gives all the different forms of the Levi-Civita connection.

Theorem 3.1. *Let* (M^m, g) *be a Riemannian manifold, and consider its cotangent bundle* T^*M *endowed with the horizontally deformed Sasaki metric* g^f. *The associated Levi-Civita connection* $\widetilde{\nabla}$ *with respect to this metric satisfies the following properties:*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) - f'(r)g(X, Y){}^{V}p + \frac{1}{2}{}^{V}(p\mathrm{R}(X, Y)), \\ 2.\ \widetilde{\nabla}_{{}^{H}X}{}^{V}\eta &= {}^{V}(\nabla_X \eta) + \frac{f'(r)}{f(r)}g^{-1}(\eta, p){}^{H}X + \frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p, \sharp\eta)X), \\ 3.\ \widetilde{\nabla}_{{}^{V}\omega}{}^{H}Y &= \frac{f'(r)}{f(r)}g^{-1}(\omega, p){}^{H}Y + \frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p, \sharp\omega)Y), \\ 4.\ \widetilde{\nabla}_{{}^{V}\omega}{}^{V}\eta &= 0, \end{aligned}$$

for each vector fields $X, Y \in \Im^1_0(M)$ *and covector fields* $\omega, \eta \in \Im^0_1(M)$.

Proof. The proof relies on formula (2.5), together with Lemmas 2.1, 3.1, and 3.2, in addition to Koszul's formula (3.14). Through straightforward computations, we derive the following:

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{^HX}{}^HY, {}^HZ) &= {}^HX\big(g^f({}^HY, {}^HZ)\big) + {}^HY\big(g^f({}^HZ, {}^HX)\big) - {}^HZ\big(g^f({}^HX, {}^HY)\big) \\
&\quad + g^f({}^HZ, [{}^HX, {}^HY]) + g^f({}^HY, [{}^HZ, {}^HX]) - g^f({}^HX, [{}^HY, {}^HZ]) \\
&= g^f({}^H(\nabla_X Y), {}^HZ) + g^f({}^HY, {}^H(\nabla_X Z)) + g^f({}^H(\nabla_Y Z), {}^HX) \\
&\quad + g^f({}^HZ, {}^H(\nabla_Y X)) - g^f({}^H(\nabla_Z X), {}^HY) - g^f({}^HX, {}^H(\nabla_Z Y)) \\
&\quad + g^f({}^HZ, {}^H[X, Y]) + g^f({}^HY, {}^H[Z, X]) - g^f({}^HX, {}^H[Y, Z]) \\
&= 2g^f({}^H(\nabla_X Y), {}^HZ),
\end{aligned}$$

and

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{^HX}{}^HY, {}^V\mu) &= {}^HX\big(g^f({}^HY, {}^V\mu)\big) + {}^HY\big(g^f({}^V\mu, {}^HX)\big) - {}^V\mu\big(g^f({}^HX, {}^HY)\big) \\
&\quad + g^f({}^V\mu, [{}^HX, {}^HY]) + g^f({}^HY, [{}^V\mu, {}^HX]) - g^f({}^HX, [{}^HY, {}^V\mu]) \\
&= -2f'(r)g(X, Y)g^{-1}(\mu, p) - g^f({}^V\mu, {}^V(p\mathrm{R}(X, Y))) \\
&= -g^f\big(2f'(r)g(X, Y){}^Vp + {}^V(p\mathrm{R}(X, Y)), {}^V\mu\big),
\end{aligned}$$

from which we find the expression of $\widetilde{\nabla}_{^HX}{}^HY$.
Analogous computations to those presented above yield

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{^HX}{}^V\eta, {}^HZ) &= {}^HX\big(g^f({}^V\eta, {}^HZ)\big) + {}^V\eta\big(g^f({}^HZ, {}^HX)\big) - {}^HZ\big(g^f({}^HX, {}^V\eta)\big) \\
&\quad + g^f({}^HZ, [{}^HX, {}^V\eta]) + g^f({}^V\eta, [{}^HZ, {}^HX]) - g^f({}^HX, [{}^V\eta, {}^HZ]) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g^f({}^V\eta, {}^V(p\mathrm{R}(Z, X))) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g^{-1}(\eta, p\mathrm{R}(Z, X)) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g(\sharp\eta, \sharp(p\mathrm{R}(Z, X))) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) - g(\sharp\eta, \mathrm{R}(Z, X)\sharp p) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g(\mathrm{R}(\sharp p, \sharp\eta)X, Z) \\
&= \frac{2f'(r)}{f(r)}g^{-1}(\eta, p)g^f({}^HX, {}^HZ) + \frac{1}{f(r)}g^f({}^H(\mathrm{R}(\sharp p, \sharp\eta)X), {}^HZ).
\end{aligned}$$

Also it follows that

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{^HX}{}^V\eta, {}^V\mu) &= {}^HX\big(g^f({}^V\eta, {}^V\mu)\big) + {}^V\eta\big(g^f({}^V\mu, {}^HX)\big) - {}^V\mu\big(g^f({}^HX, {}^V\eta)\big) \\
&\quad + g^f({}^V\mu, [{}^HX, {}^V\eta]) + g^f({}^V\eta, [{}^V\mu, {}^HX]) - g^f({}^HX, [{}^V\eta, {}^V\mu]) \\
&= g^f({}^V(\nabla_X\eta), {}^V\mu) + g^f({}^V\eta, {}^V(\nabla_X\mu)) + g^f({}^V\mu, {}^V(\nabla_X\eta)) - g^f({}^V\eta, {}^V(\nabla_X\mu)) \\
&= 2g^f({}^V(\nabla_X\eta), {}^V\mu).
\end{aligned}$$

from which we find the expression of $\widetilde{\nabla}_{^HX}{}^V\eta$.
The remaining formulas can be derived through similar calculations. □

Lemma 3.3. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then, the following relations hold:*

$$\begin{array}{rrcl} (1) & \widetilde{\nabla}_{{}^HX}{}^Vp & = & \dfrac{rf'(r)}{f(r)}{}^HX, \\ (2) & \widetilde{\nabla}_{{}^Vp}{}^HX & = & \dfrac{rf'(r)}{f(r)}{}^HX, \\ (3) & \widetilde{\nabla}_{{}^V\omega}{}^Vp & = & {}^V\omega, \\ (4) & \widetilde{\nabla}_{{}^Vp}{}^V\omega & = & 0, \\ (5) & \widetilde{\nabla}_{{}^Vp}{}^Vp & = & {}^Vp, \end{array}$$

for each vector field $X \in \Im^1_0(M)$ and covector field $\omega \in \Im^0_1(M)$.

Proof. In the proof, we use the formulas (2.3) and (2.4), as well as the Theorem 3.1. Through simple calculations, we obtain:

$$\begin{array}{rrcl} (1) & \widetilde{\nabla}_{{}^HX}{}^Vp & = & \widetilde{\nabla}_{{}^HX}(p_k{}^V(dx^k)) = {}^HX(p_k){}^V(dx^k) + p_k\widetilde{\nabla}_{{}^HX}{}^V(dx^k) \\ & & = & X^i\partial_i(p_k){}^V(dx^k) + p_h\Gamma^h_{ij}X^j\partial_{\bar{i}}(p_k){}^V(dx^k) + p_k{}^V(\nabla_X dx^k) \\ & & & +p_k\dfrac{f'(r)}{f(r)}g^{-1}(dx^k, p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp(dx^k))X) \\ & & = & X(p_k){}^V(dx^k) + p_h\Gamma^h_{kj}X^j{}^V(dx^k) + p_k{}^V(\nabla_X dx^k) \\ & & & +\dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp p)X) \\ & & = & {}^V(\nabla_X(p_k dx^k)) - {}^V(\nabla_X p) + \dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX \\ & & = & \dfrac{rf'(r)}{f(r)}{}^HX. \end{array}$$

$$\begin{array}{rrcl} (2) & \widetilde{\nabla}_{{}^Vp}{}^HX & = & \widetilde{\nabla}_{p_k{}^V(dx^k)}{}^HX = p_k\widetilde{\nabla}_{{}^V(dx^k)}{}^HX \\ & & = & p_k\dfrac{f'(r)}{f(r)}g^{-1}(dx^k, p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp(dx^k))X) \\ & & = & \dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX + \dfrac{1}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp p)X) \\ & & = & \dfrac{rf'(r)}{f(r)}{}^HX. \\ (3) & \widetilde{\nabla}_{{}^V\omega}{}^Vp & = & \widetilde{\nabla}_{{}^V\omega}(p_k{}^V(dx^k)) = {}^V\omega(p_k){}^V(dx^k) + p_k\widetilde{\nabla}_{{}^V\omega}{}^V(dx^k) \\ & & = & \omega_k{}^V(dx^k) \\ & & = & {}^V\omega. \\ (4) & \widetilde{\nabla}_{{}^Vp}{}^V\omega & = & \widetilde{\nabla}_{p_k{}^V(dx^k)}{}^V\omega = p_k\widetilde{\nabla}_{{}^V(dx^k)}{}^V\omega = 0. \\ (5) & \widetilde{\nabla}_{{}^Vp}{}^Vp & = & \widetilde{\nabla}_{p_h{}^V(dx^h)}(p_k{}^V(dx^k)) = p_h{}^V(dx^h)(p_k){}^V(dx^k) + p_hp_k\widetilde{\nabla}_{{}^V(dx^h)}{}^V(dx^k) \\ & & = & p_k{}^V(dx^k) \\ & & = & {}^Vp. \end{array}$$

□

Definition 3.2. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. For any $(1,1)$-tensor field F on M, we can define associated horizontal and vertical vector fields ${}^H F$ and ${}^V F$ of F to T^*M as follows:*

$$\begin{array}{rcl} {}^H F : T^*M & \to & TT^*M \\ (x,p) & \mapsto & {}^H F(x,p) = {}^H(F_x(\sharp p)), \\ {}^V F : T^*M & \to & TT^*M \\ (x,p) & \mapsto & {}^V F(x,p) = {}^V(pF_x). \end{array}$$

Locally we have

$$ {}^H(F(\sharp p)) = (\sharp p)^i \, {}^H(F(\partial_i)), \tag{3.15}$$

$$ {}^V(pF) = p_i \, {}^V(dx^i F), \tag{3.16}$$

where $p = p_j dx^j$ and $\sharp p = (\sharp p)^i \partial_i = g^{ij} p_j \partial_i$.

From Theorem 3.1, we obtain

Proposition 3.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. For any $(1,1)$-tensor field F on M. Then, the following relations hold:*

$$\begin{array}{rcl} \widetilde{\nabla}_{{}^H X} {}^H(F(\sharp p)) & = & {}^H((\nabla_X F)(\sharp p)) - f'(r) g(X, F(\sharp p)) \, {}^V p + \dfrac{1}{2} \, {}^V(p\mathrm{R}(X, F(\sharp p))), \\ \widetilde{\nabla}_{{}^H X} {}^V(pF) & = & {}^V(p(\nabla_X F)) + \dfrac{f'(r)}{f(r)} g^{-1}(pF, p) \, {}^H X + \dfrac{1}{2f(r)} \, {}^H(\mathrm{R}(\sharp p, \sharp(pF))X), \\ \widetilde{\nabla}_{{}^V \omega} {}^H(F(\sharp p)) & = & {}^H(F(\sharp \omega)) + \dfrac{f'(r)}{f(r)} g^{-1}(\omega, p) \, {}^H(F(\sharp p)) + \dfrac{1}{2f(r)} \, {}^H(\mathrm{R}(\sharp p, \sharp \omega) F(\sharp p)), \\ \widetilde{\nabla}_{{}^V \omega} {}^V(pF) & = & {}^V(\omega F), \end{array}$$

for each vector field $X \in \Im^1_0(M)$ and covector field $\omega \in \Im^0_1(M)$, where $\sharp(pF) = g^{-1} \circ (pF)$.

4. The Riemannian curvature tensors of the horizontally deformed Sasaki metric

We now consider the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ associated with the cotangent bundle T^*M equipped with the horizontally deformed Sasaki metric g^f. The theorem below presents the various forms of the curvature tensor in explicit form.

Theorem 4.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated Riemannian* *curvature tensor* $\widetilde{\mathrm{R}}$ *with respect to this metric satisfies the following properties:*

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^{H}X, {}^{H}Y){}^{H}Z &= {}^{H}(\mathrm{R}(X,Y)Z) + \frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(X,Y)\sharp p)Z) \\
&\quad + \frac{1}{4f(r)}\big({}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(X,Z)\sharp p)Y) - {}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(Y,Z)\sharp p)X)\big) \\
&\quad + \frac{r(f'(r))^2}{f(r)}\big(g(X,Z){}^{H}Y - g(Y,Z){}^{H}X\big) \\
&\quad - \frac{1}{2}{}^{V}\big(p(\nabla_Z \mathrm{R})(X,Y)\big), \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^{H}X, {}^{V}\eta){}^{H}Z &= \frac{1}{2f(r)}{}^{H}((\nabla_X \mathrm{R})(\sharp p, \sharp\eta)Z) + \frac{1}{4f(r)}{}^{V}(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\eta)Z)) \\
&\quad - \frac{1}{2}{}^{V}(\eta \mathrm{R}(X,Z)) + \frac{f'(r)}{2f(r)}g^{-1}(\eta,p){}^{V}(p\mathrm{R}(X,Z)) \\
&\quad + f'(r)g(X,Z){}^{V}\eta + \frac{f'(r)}{2f(r)}g(\mathrm{R}(\sharp p, \sharp\eta)X, Z){}^{V}p \\
&\quad + \big(2f''(r) - \frac{(f'(r))^2}{f(r)}\big)g^{-1}(\eta,p)g(X,Z){}^{V}p, \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^{H}X, {}^{H}Y){}^{V}\mu &= \frac{1}{2f(r)}\big({}^{H}((\nabla_X \mathrm{R})(\sharp p, \sharp\mu)Y) - {}^{H}((\nabla_Y \mathrm{R})(\sharp p, \sharp\mu)X)\big) \\
&\quad + \frac{1}{4f(r)}\big({}^{V}(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\mu)Y)) - {}^{V}(p\mathrm{R}(Y, \mathrm{R}(\sharp p, \sharp\mu)X))\big) \\
&\quad + \frac{f'(r)}{f(r)}\big(g^{-1}(\mu,p){}^{V}(p\mathrm{R}(X,Y)) + g(\mathrm{R}(\sharp p, \sharp\mu)X, Y){}^{V}p\big) \\
&\quad - {}^{V}(\mu \mathrm{R}(X,Y)), \qquad (4.19)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^{H}X, {}^{V}\eta){}^{V}\mu &= \Big(\big(\big(\frac{f'(r)}{f(r)}\big)^2 - 2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)g^{-1}(\mu,p) - \frac{f'(r)}{f(r)}g^{-1}(\eta,\mu)\Big){}^{H}X \\
&\quad + \frac{f'(r)}{2(f(r))^2}\big(g^{-1}(\eta,p){}^{H}(\mathrm{R}(\sharp p, \sharp\mu)X) - g^{-1}(\mu,p){}^{H}(\mathrm{R}(\sharp p, \sharp\eta)X)\big) \\
&\quad - \frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp\eta, \sharp\mu)X) - \frac{1}{4(f(r))^2}{}^{H}(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\mu)X), \qquad (4.20)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^{V}\omega, {}^{V}\eta){}^{H}Z &= \frac{1}{4(f(r))^2}\big({}^{H}(\mathrm{R}(\sharp p, \sharp\omega)\mathrm{R}(\sharp p, \sharp\eta)Z) - {}^{H}(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\omega)Z)\big) \\
&\quad + \frac{f'(r)}{(f(r))^2}\big(g^{-1}(\eta,p){}^{H}(\mathrm{R}(\sharp p, \sharp\omega)Z) - g^{-1}(\omega,p){}^{H}(\mathrm{R}(\sharp p, \sharp\eta)Z)\big) \\
&\quad + \frac{1}{f(r)}{}^{H}(\mathrm{R}(\sharp\omega, \sharp\eta)Z), \qquad (4.21)
\end{aligned}$$

$$\widetilde{\mathrm{R}}({}^{V}\omega, {}^{V}\eta){}^{V}\mu = 0, \qquad (4.22)$$

for each vector fields $X, Y, Z \in \Im_0^1(M)$ *and covector fields* $\omega, \eta, \mu \in \Im_1^0(M)$.

Proof. The proof makes use of Lemmas 2.1, 3.1, and 3.2, together with Theorem 3.1 and Proposition 3.1.

(1) $\widetilde{\mathrm{R}}({}^{H}X,{}^{H}Y){}^{H}Z=\widetilde{\nabla}_{{}^{H}X}\widetilde{\nabla}_{{}^{H}Y}{}^{H}Z-\widetilde{\nabla}_{{}^{H}Y}\widetilde{\nabla}_{{}^{H}X}{}^{H}Z-\widetilde{\nabla}_{[{}^{H}X,{}^{H}Y]}{}^{H}Z.$

Let $F:TM\to TM$ be the bundle endomorphism given by $pF=p\mathrm{R}(Y,Z)$.
Using (2.8) and direct calculations, we have:

$$\begin{aligned}\widetilde{\nabla}_{{}^{H}X}\widetilde{\nabla}_{{}^{H}Y}{}^{H}Z &= \widetilde{\nabla}_{{}^{H}X}\big({}^{H}(\nabla_Y Z)-f'(r)g(Y,Z){}^{V}p+\frac{1}{2}{}^{V}(pF)\big)\\ &= {}^{H}(\nabla_X\nabla_Y Z)-f'(r)g(X,\nabla_Y Z){}^{V}p+\frac{1}{2}{}^{V}(p\mathrm{R}(X,\nabla_Y Z))\\ &\quad -f'(r)g(\nabla_X Y,Z){}^{V}p-f'(r)g(Y,\nabla_X Z){}^{V}p\\ &\quad -\frac{r(f'(r))^2}{f(r)}g(Y,Z){}^{H}X+\frac{1}{2}{}^{V}(\nabla_X(p\mathrm{R}(Y,Z)))\\ &\quad -\frac{1}{2}{}^{V}((\nabla_X p)\mathrm{R}(Y,Z))-\frac{1}{4f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(Y,Z)\sharp p)X). \end{aligned} \tag{4.23}$$

From which, with permutation of X by Y, we get:

$$\begin{aligned}\widetilde{\nabla}_{{}^{H}Y}\widetilde{\nabla}_{{}^{H}X}{}^{H}Z &= {}^{H}(\nabla_Y\nabla_X Z)-f'(r)g(Y,\nabla_X Z){}^{V}p+\frac{1}{2}{}^{V}(p\mathrm{R}(Y,\nabla_X Z))\\ &\quad -f'(r)g(\nabla_Y X,Z){}^{V}p-f'(r)g(X,\nabla_Y Z){}^{V}p\\ &\quad -\frac{r(f'(r))^2}{f(r)}g(X,Z){}^{H}Y+\frac{1}{2}{}^{V}(\nabla_Y(p\mathrm{R}(X,Z)))\\ &\quad -\frac{1}{2}{}^{V}((\nabla_Y p)\mathrm{R}(X,Z))-\frac{1}{4f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(X,Z)\sharp p)Y). \end{aligned} \tag{4.24}$$

Also, we find:

$$\begin{aligned}\widetilde{\nabla}_{[{}^{H}X,{}^{H}Y]}{}^{H}Z &= \widetilde{\nabla}_{{}^{H}[X,Y]}{}^{H}Z+\widetilde{\nabla}_{{}^{V}(p\mathrm{R}(X,Y))}{}^{H}Z\\ &= {}^{H}(\nabla_{[X,Y]}Z)-f'(r)g([X,Y],Z){}^{V}p+\frac{1}{2}{}^{V}(p\mathrm{R}([X,Y],Z))\\ &\quad -\frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Z). \end{aligned} \tag{4.25}$$

From the formulas (4.23), (4.24), (4.25) and using (2.10) we get:

$$\begin{aligned}\widetilde{\mathrm{R}}({}^{H}X,{}^{H}Y){}^{H}Z &= {}^{H}(\mathrm{R}(X,Y)Z)+\frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Z)\\ &\quad +\frac{1}{4f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(X,Z)\sharp p)Y)-\frac{1}{4f(r)}{}^{H}(\mathrm{R}(\sharp p,\mathrm{R}(Y,Z)\sharp p)X)\\ &\quad +\frac{r(f'(r))^2}{f(r)}\big(g(X,Z){}^{H}Y-g(Y,Z){}^{H}X\big)\\ &\quad +\frac{1}{2}{}^{V}\big(p(\nabla_X\mathrm{R})(Y,Z)\big)-\frac{1}{2}{}^{V}\big(p(\nabla_Y\mathrm{R})(X,Z)\big). \end{aligned}$$

Using (2.11) and the second Bianchi identity, we obtain the formula (4.17).
The other formulas (4.18)-(4.22) are obtained by a similar calculation. □

Proposition 4.1. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then, the following assertions holds:*
(i) If M is flat and $f=const$ then, T^*M is flat.
(ii) Conversely, if T^*M is flat then, M is flat.

Proof.
(i) This follows directly from Theorem 4.1, which implies that $\widetilde{\mathrm{R}} = 0$.
(ii) Suppose that $\widetilde{\mathrm{R}} = 0$. Evaluating the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ at a point $(x, 0) \in T^*M$, and using equation (4.19), we deduce that $\mathrm{R} = 0$. □

Corollary 4.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. If $f \neq const$, then T^*M is never flat.*

Let $\{e_i\}_{i=\overline{1,m}}$ (resp. $\{\omega^i\}_{i=\overline{1,m}}$) be a local orthonormal frame (resp. coframe on M), then

$$\{\frac{1}{\sqrt{f(r)}}{}^He_i, {}^V\omega^i\}_{i=\overline{1,m}} \tag{4.26}$$

is a local orthonormal frame on (T^*M, g^f). Moreover, we have

$$\sharp\omega^a = e_a \tag{4.27}$$

Indeed

$$\begin{aligned}
\sharp\omega^a &= \sum_{i=1}^m g(\sharp\omega^a, e_i)e_i = \sum_{i,h,k=1}^m g_{hk}(\sharp\omega^a)^h e_i^k e_i = \sum_{i,h,k,j=1}^m g_{hk}g^{jh}\omega_j^a e_i^k e_i \\
&= \sum_{i,k,j=1}^m \delta_k^j \omega_j^a e_i^k e_i = \sum_{i,j=1}^m \omega_j^a e_i^j e_i = \sum_{i=1}^m \omega^a(e_i)e_i = \sum_{i=1}^m \delta_i^a e_i = e_a.
\end{aligned}$$

Theorem 4.2. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated Ricci curvature $\widetilde{\mathrm{Ric}}$ with respect to this metric satisfies the following properties:*

$$\begin{aligned}
1.\ \widetilde{\mathrm{Ric}}({}^HX, {}^HY) &= \mathrm{Ric}(X, Y) - \frac{1}{2f(r)}\sum_{i=1}^m g(\mathrm{R}(e_i, X)\sharp p, \mathrm{R}(e_i, Y)\sharp p) \\
&\quad -\big(2rf''(r) + mf'(r) + (m-2)r\frac{(f'(r))^2}{f(r)}\big)g(X, Y), \qquad (4.28) \\
2.\ \widetilde{\mathrm{Ric}}({}^HX, {}^V\eta) &= \frac{1}{2f(r)}\sum_{i=1}^m g((\nabla_{e_i}\mathrm{R})(\sharp p, \sharp\eta)X, e_i), \qquad (4.29) \\
3.\ \widetilde{\mathrm{Ric}}({}^V\omega, {}^V\eta) &= \frac{1}{4(f(r))^2}\sum_{i=1}^m g(\mathrm{R}(\sharp p, \tilde{\omega})e_i, \mathrm{R}(\sharp p, \sharp\eta)e_i) - \frac{mf'(r)}{f(r)}g^{-1}(\omega, \eta) \\
&\quad +\big(m(\frac{f'(r)}{f(r)})^2 - 2m\frac{f''(r)}{f(r)}\big)g^{-1}(\omega, p)g^{-1}(\eta, p), \qquad (4.30)
\end{aligned}$$

for each vector fields $X, Y \in \Im_0^1(M)$ and covector fields $\omega, \eta \in \Im_1^0(M)$, where Ric *represents the Ricci curvature associated with the Riemannian manifold (M^m, g).*

Proof. Using (4.26), we have

$$\begin{aligned}
\widetilde{\mathrm{Ric}}({}^HX, {}^HY) &= \sum_{i=1}^m g^f(\widetilde{\mathrm{R}}(\frac{1}{\sqrt{f(r)}}{}^He_i, {}^HX){}^HY, \frac{1}{\sqrt{f(r)}}{}^He_i) + \sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^V\omega^i, {}^HX){}^HY, {}^V\omega^i) \\
&= \frac{1}{f(r)}\sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^He_i, {}^HX){}^HY, {}^He_i) - \sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^HX, {}^V\omega^i){}^HY, {}^V\omega^i)
\end{aligned}$$

Using (4.17), (4.18) and (4.27), and performing direct calculations, we find

$$\begin{aligned}\widetilde{\mathrm{Ric}}({}^{H}X,{}^{H}Y) &= \mathrm{Ric}(X,Y)-\frac{3}{4f(r)}\sum_{i=1}^{m}g(\mathrm{R}(e_i,X)\sharp p,\mathrm{R}(e_i,Y)\sharp p)\\ &\quad+\frac{1}{4f(r)}\sum_{i=1}^{m}g(\mathrm{R}(e_i,\sharp p)X,\mathrm{R}(e_i,\sharp p)Y)\\ &\quad-\big(2rf''(r)+mf'(r)+(m-2)r\frac{(f'(r))^2}{f(r)}\big)g(X,Y).\end{aligned}$$

To simplify this expression, we utilize

$$\sum_{i=1}^{m}g(\mathrm{R}(e_i,X)\sharp p,\mathrm{R}(e_i,Y)\sharp p)=\sum_{i=1}^{m}g(\mathrm{R}(e_i,\sharp p)X,\mathrm{R}(e_i,\sharp p)Y).$$

This completes the proof.
The remaining identities, namely (4.29) and (4.30), can be derived through analogous computations. □

We proceed to evaluate the sectional curvature of the cotangent bundle (T^*M,g^f). The sectional curvature $\widetilde{\mathrm{K}}(V,W)$ corresponding to the plane P spanned by two linearly independent tangent vectors $V,W\in TT^*M$ is given by the expression:

$$\widetilde{\mathrm{K}}(V,W)=\frac{g^f(\widetilde{\mathrm{R}}(V,W)W,V)}{g^f(V,V)g^f(W,W)-g^f(V,W)^2}, \tag{4.31}$$

where $\widetilde{\mathrm{R}}$ denotes the Riemann curvature tensor associated with the Levi-Civita connection of the metric g^f.

We examine three specific cases of sectional curvature on (T^*M,g^f)

- $\widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y)$ for the horizontal plane spanned by $\{{}^{H}X,{}^{H}Y\}$.
- $\widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta)$ for the mixed horizontal-vertical plane spanned by $\{{}^{H}X,{}^{V}\eta\}$.
- $\widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta)$ or the vertical plane spanned by $\{{}^{V}\omega,{}^{V}\eta\}$.

where X,Y are orthonormal vector fields and ω,η are orthonormal covector fields on the base manifold M.

Theorem 4.3. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated sectional curvature $\widetilde{\mathrm{K}}$ with respect to this metric satisfies the following properties:*

$$\begin{aligned}\widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y) &= \frac{1}{f(r)}\mathrm{K}(X,Y)-\frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)\sharp p|^2-r\big(\frac{f'(r)}{f(r)}\big)^2,\\ \widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta) &= \frac{1}{4(f(r))^2}|\mathrm{R}(\sharp p,\sharp\eta)X|^2+\big(\big(\frac{f'(r)}{f(r)}\big)^2-2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)},\\ \widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta) &= 0,\end{aligned}$$

where K represents the sectional curvature associated with the Riemannian manifold (M^m,g).

Proof. Using Theorem 4.1,(4.31) and direct calculations we have,

$$\begin{aligned}
1.\ \widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y) &= \frac{g^f(\widetilde{\mathrm{R}}({}^{H}X,{}^{H}Y){}^{H}Y,{}^{H}X)}{g^f({}^{H}X,{}^{H}X)g^f({}^{H}Y,{}^{H}Y)-g^f({}^{H}X,{}^{H}Y)^2}\\
&= \frac{1}{(f(r))^2}\Big(f(r)g(\mathrm{R}(X,Y)Y,X)+\frac{1}{2}g(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Y,X)\\
&\quad +\frac{1}{4}g(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Y,X)-r(f'(r))^2\Big)\\
&= \frac{1}{f(r)}\mathrm{K}(X,Y)-\frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)\sharp p|^2-r\big(\frac{f'(r)}{f(r)}\big)^2.
\end{aligned}$$

$$\begin{aligned}
2.\ \widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta) &= \frac{g^f(\widetilde{\mathrm{R}}({}^{H}X,{}^{V}\eta){}^{V}\eta,{}^{H}X)}{g^f({}^{H}X,{}^{H}X)g^f({}^{V}\eta,{}^{V}\eta)-g^f({}^{H}X,{}^{V}\eta)^2}\\
&= \frac{1}{f(r)}\Big(\big(\frac{(f'(r))^2}{f(r)}-2f''(r)\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)}g^{-1}(\eta,\eta)\\
&\quad +\frac{1}{4f(r)}|\mathrm{R}(\sharp p,\sharp\eta)X|^2\Big)\\
&= \frac{1}{4(f(r))^2}|\mathrm{R}(\sharp p,\sharp\eta)X|^2+\big(\big(\frac{f'(r)}{f(r)}\big)^2-2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)}.
\end{aligned}$$

$$3.\ \widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta) = \frac{g^f(\widetilde{\mathrm{R}}({}^{V}\omega,{}^{V}\eta){}^{V}\eta,{}^{V}\omega)}{g^f({}^{V}\omega,{}^{V}\omega)g^f({}^{V}\eta,{}^{V}\eta)-g^f({}^{V}\omega,{}^{V}\eta)^2}=0.$$

□

Theorem 4.4. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated scalar curvature $\widetilde{\sigma}$ with respect to this metric satisfies the following:*

$$\begin{aligned}
\widetilde{\sigma} &= \frac{1}{f(r)}\sigma-\frac{1}{4(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2-4mr\frac{f''(r)}{f(r)}-2m^2\frac{f'(r)}{f(r)}\\
&\quad -(m^2-3m)r\big(\frac{f'(r)}{f(r)}\big)^2. \qquad (4.32)
\end{aligned}$$

where σ represents the scalar curvature associated with the Riemannian manifold (M^m,g).

Proof. Let $\{\frac{1}{\sqrt{f(r)}}{}^{H}e_i,{}^{V}\omega^i\}_{i=1,\ldots,m}$ be a local orthonormal frame on (T^*M,g^f) defined by (4.26). We will compute the scalar curvature of (T^*M,g^f), as follows:

$$\widetilde{\sigma} = \frac{1}{f(r)}\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{H}e_j,{}^{H}e_j)+\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{V}\omega^j,{}^{V}\omega^j). \qquad (4.33)$$

Using (4.28), we have

$$\begin{aligned}
\frac{1}{f(r)}\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{H}e_j,{}^{H}e_j) &= \frac{1}{f(r)}\sigma-\frac{1}{2(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2\\
&\quad -2mr\frac{f''(r)}{f(r)}-m^2\frac{f'(r)}{f(r)}-m(m-2)r\big(\frac{f'(r)}{f(r)}\big)^2. \qquad (4.34)
\end{aligned}$$

Using (4.30), we have

$$\begin{aligned}\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{V}\omega^{j},{}^{V}\omega^{j}) &= \frac{1}{4(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(\sharp p,\sharp\omega^{j})e_i|^2\\ &\quad -2mr\frac{f''(r)}{f(r)}-m^2\frac{f'(r)}{f(r)}+mr\Big(\frac{f'(r)}{f(r)}\Big)^2. \end{aligned}\tag{4.35}$$

On the other hand, we have

$$\sum_{i,j=1}^{m}|\mathrm{R}(\sharp p,e_j)e_i|^2=\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2.\tag{4.36}$$

Substituting (4.34), (4.35) and (4.36) into (4.33), we find (4.32). □

From Theorem 4.4, we obtain

Proposition 4.2. *Let (M^m,g) be a Riemannian manifold of constant sectional curvature κ, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then scalar curvature $\widetilde{\sigma}$ of (T^*M,g^f) is given by:*

$$\widetilde{\sigma}=\frac{m(m-1)\kappa}{f(r)}-\frac{\kappa^2(m-1)r}{2(f(r))^2}-4mr\frac{f''(r)}{f(r)}-2m^2\frac{f'(r)}{f(r)}-(m^2-3m)r\Big(\frac{f'(r)}{f(r)}\Big)^2.$$

5. Horizontally deformed Sasaki metric on hypersurface

Let us denote by T^*M_0 the cotangent bundle of a manifold M, excluding the zero section. Consider a smooth real-valued function $f:[0,+\infty[\to]0,+\infty[$ satisfying $f'(r)\neq 1$, where $r=g^{-1}(p,p)$.

We define a hypersurface of T^*M_0 by

$$T^*_{f(r)}M=\big\{(x,p)\in T^*M_0,\ f(r)=r\big\}.$$

To describe this hypersurface, consider the smooth function $\phi:T^*M_0\to\mathbb{R}$ defined by

$$\phi(x,p)=f(r)-r,$$

Then the hypersurface $T^*_{f(r)}M$ is given as the zero level set of ϕ:

$$T^*_{f(r)}M=\{(x,p)\in T^*M_0,\quad \phi(x,p)=0\}.$$

The gradient gradϕ computed with respect to the horizontally deformed Sasaki metric g^f, provides a normal vector field to the hypersurface $T^*_{f(r)}M$. From Lemma 3.1, we obtain:

$$\begin{aligned} g^f({}^{H}Z,\mathrm{grad}\phi) &= {}^{H}Z(\phi)={}^{H}Z(f(r)-r)=0,\\ g^f({}^{V}\omega,\mathrm{grad}\phi) &= {}^{V}\omega(\phi)={}^{V}\omega(f(r)-r)=2(f'(r)-1)g^{-1}(\omega,p)=2(f'(r)-1)g^f({}^{V}\omega,{}^{V}p). \end{aligned}$$

for each vector field $Z\in\Im^1_0(M)$ and covector field $\omega\in\Im^0_1(M)$. Thus, we conclude

$$\mathrm{grad}\phi=2(f'(r)-1)^{V}p.$$

Accordingly, a unit normal vector field $\mathcal{N}$ to $T^*_{f(r)}M$ is expressed by

$$\mathcal{N}=\frac{\mathrm{grad}\phi}{|\mathrm{grad}\phi|_{g^f}}=\frac{{}^{V}p}{\sqrt{g^f({}^{V}p,{}^{V}p)}}=\frac{1}{\sqrt{r}}{}^{V}p.$$

We define the tangential lift ${}^T\omega$ of a covector $\omega \in T_x^*M$ at the point $(x,p) \in T^*_{f(r)}M$ with respect to the metric g^f as the orthogonal projection of the vertical lift ${}^V\omega$ onto the tangent space of the hypersurface, that is:

$$ {}^T\omega = {}^V\omega - g^f_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} = {}^V\omega - \frac{1}{r} g_x^{-1}(\omega, p)^V p. $$

Using the decomposition given by Equation (2.2), any tangent vector $W \in T_{(x,p)}T^*M$ at a point (x,p) can be expressed in terms of horizontal and vertical lifts. That is, there exist $v \in T_xM$ and $w \in T_x^*M$, such that

$$ \begin{aligned} W &= {}^Hv + {}^Vw \\ &= {}^Hv + {}^Tw + g^f_{(x,p)}({}^Vw, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} \\ &= {}^Hv + {}^Tw + \frac{1}{r} g_x^{-1}(w,p)^V p, \end{aligned} $$

where $(x,p) \in T^*_{f(r)}M$.This expression leads directly to the orthogonal decomposition:

$$ T_{(x,p)}T^*M = T_{(x,p)}T^*_{f(r)}M \oplus span\{\mathcal{N}_{(x,p)}\}, \tag{5.37} $$

which implies that the tangent space to the hypersurface $T_{(x,p)}T^*M$ at (x,p) is expressed by:

$$ T_{(x,p)}T^*_{f(r)}M = \{{}^Hv + {}^Tw \,/\, v, \in T_xM, \omega \in p^\perp\}, $$

where

$$ p^\perp = \big\{ w \in T_x^*M,\ g^{-1}(w,p) = 0 \big\} \subset T_x^*M. $$

denotes the orthogonal complement of p with respect to the inverse metric g^{-1}.

Given a covector field ω on M, its tangential lift to the hypersurface $T_{(x,p)}T^*M$ is expressed by:

$$ {}^T\omega_{(x,p)} = \big({}^V\omega - g^f({}^V\omega, \mathcal{N})\mathcal{N}\big)_{(x,p)} = {}^V\omega_{(x,p)} - \frac{1}{r} g_x^{-1}(\omega_x, p)^V p. $$

To simplify notation, we define the projected covector

$$ \overline{\omega} = \omega - \frac{1}{r} g^{-1}(\omega,p)p, $$

so that ${}^T\omega = {}^V\overline{\omega}$.

Now, consider the hypersurface $T^*_{f(r)}M$ endowed with the induced metric from the horizontally deformed Sasaki metric g^f on T^*M. With this induced metric, also denoted by g^f, $T^*_{f(r)}M$ becomes a Riemannian submanifold. Its Levi-Civita connection $\widehat{\nabla}$ is described by the Gauss formula:

$$ \widehat{\nabla}_P Q = \widetilde{\nabla}_P Q - g^f(\widetilde{\nabla}_P Q, \mathcal{N})\mathcal{N}, \tag{5.38} $$

for all vector fields P and Q tangent to $T^*_{f(r)}M$, where $\widetilde{\nabla}$ is the Levi-Civita connection on (T^*M, g^f).

The following theorem provides explicit formulas for $\widehat{\nabla}$ in terms of the horizontal and tangential lifts.

Theorem 5.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then the associated Levi-Civita connection $\widehat{\nabla}$ satisfies the following identities:*

$$\begin{aligned}
\widehat{\nabla}_{^HX}{}^HY &= {}^H(\nabla_X Y) + \frac{1}{2}{}^T(p\mathrm{R}(X,Y)),\\
\widehat{\nabla}_{^HX}{}^T\eta &= {}^T(\nabla_X \eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X),\\
\widehat{\nabla}_{^T\omega}{}^HY &= \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\omega)Y),\\
\widehat{\nabla}_{^T\omega}{}^T\eta &= \frac{-1}{r}g^{-1}(\eta, p){}^T\omega,
\end{aligned}$$

for each vector fields $X, Y \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$.

Proof. The proof relies on Theorem 3.1, Lemma 3.3, and the Gauss formula (5.38):
• From Theorem 3.1, we compute:

$$\begin{aligned}
\widehat{\nabla}_{^HX}{}^HY &= \widetilde{\nabla}_{^HX}{}^HY - g^f(\widetilde{\nabla}_{^HX}{}^HY, \mathcal{N})\mathcal{N}\\
&= {}^H(\nabla_X Y) - f'(r)g(X,Y){}^Vp + \frac{1}{2}{}^V(p\mathrm{R}(X,Y)) + f'(r)g(X,Y)g^f({}^Vp, \mathcal{N})\mathcal{N}\\
&\quad -\frac{1}{2}g^f({}^V(p\mathrm{R}(X,Y)), \mathcal{N})\mathcal{N}.
\end{aligned}$$

Noting that

$$g^f({}^Vp, \mathcal{N}) = \sqrt{r} \quad \textit{and} \quad g^f({}^V(p\mathrm{R}(X,Y)), \mathcal{N}) = \frac{1}{\sqrt{r}}g^{-1}(p\mathrm{R}(X,Y), p),$$

we obtain:

$$\widehat{\nabla}_{^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^T(p\mathrm{R}(X,Y)).$$

• We apply the Gauss formula:

$$\widehat{\nabla}_{^HX}{}^T\eta = \widetilde{\nabla}_{^HX}{}^T\eta - g^f(\widetilde{\nabla}_{^HX}{}^T\eta, \mathcal{N})\mathcal{N},$$

From Theorem 3.1 and Lemma 3.3, we obtain

$$\widetilde{\nabla}_{^HX}{}^T\eta = {}^T(\nabla_X \eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X),$$

and since both terms are tangent to $T^*_{f(r)}M$, the scalar product with $\mathcal{N}$ vanishes:

$$g^f(\widetilde{\nabla}_{^HX}{}^T\eta, \mathcal{N})\mathcal{N} = 0.$$

Hence:

$$\widehat{\nabla}_{^HX}{}^T\eta = {}^T(\nabla_X \eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X).$$

• The remaining formulas for $\widehat{\nabla}_{^T\omega}{}^HY$ and $\widehat{\nabla}_{^T\omega}{}^T\eta$ follow analogously, by direct computation from the formulas for $\widetilde{\nabla}$ and applying the orthogonal projection via the Gauss formula. □

We now proceed to compute the Riemannian curvature tensor of the hypersurface $(T^*_{f(r)}M, g^f)$. Let $\widehat{\mathrm{R}}$ denote the Riemannian curvature tensor associated with the Levi-Civita connection $\widehat{\nabla}$ of $(T^*_{f(r)}M, g^f)$. By applying the Gauss equation for hypersurfaces, we obtain the following expression for $\widehat{\mathrm{R}}(P,Q)W$:

$$\widehat{\mathrm{R}}(P,Q)W = {}^{t}(\widetilde{\mathrm{R}}(P,Q)W) - B(P,W).A_{\mathcal{N}}Q + B(Q,W).A_{\mathcal{N}}P, \tag{5.39}$$

for all vector fields P,Q and W vector fields on $T^*_{f(r)}M$. Here

- $\widetilde{\mathrm{R}}$ denotes the curvature tensor of the ambient manifold (T^*M, g^f),
- ${}^{t}(\widetilde{\mathrm{R}}(P,Q)W)$ is the tangential component of $\widetilde{\mathrm{R}}(P,Q)W$, taken with respect to the orthogonal decomposition given in Equation (5.37),
- $A_{\mathcal{N}}$ is the shape operator (or Weingarten operator) associated with the unit normal vector field $\mathcal{N}$,
- B denotes the second fundamental form of the hypersurface $T^*_{f(r)}M$ with respect to $\mathcal{N}$.

The shape operator $A_{\mathcal{N}}$ is defined via the Weingarten formula:

$$\widetilde{\nabla}_P\mathcal{N} = -A_{\mathcal{N}}P + \nabla^{\mathcal{N}}_P\mathcal{N},$$

where $\nabla^{\mathcal{N}}$ denotes the normal connection on $T^*_{f(r)}M$. Consequently, we have

$$A_{\mathcal{N}}P = -{}^{t}(\widetilde{\nabla}_P\mathcal{N}). \tag{5.40}$$

The second fundamental form B is given by the Gauss formula:

$$\widetilde{\nabla}_PQ = \widehat{\nabla}_PQ + B(P,Q)\mathcal{N}.$$

from which we deduce the explicit expression:

$$B(P,Q) = g^f(\widetilde{\nabla}_PQ, \mathcal{N}). \tag{5.41}$$

Lemma 5.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then, the following relation holds:*

$$\begin{aligned}
A_{\mathcal{N}}{}^{H}X &= \frac{-f'(r)}{\sqrt{r}}{}^{H}X,\\
A_{\mathcal{N}}{}^{T}\omega &= \frac{-1}{\sqrt{r}}{}^{T}\omega,\\
B({}^{H}X, {}^{H}Y) &= -\sqrt{r}f'(r)g(X,Y),\\
B({}^{H}X, {}^{T}\eta) &= B({}^{T}\omega, {}^{H}Y) = 0,\\
B({}^{T}\omega, {}^{T}\eta) &= \frac{-1}{\sqrt{r}}g^{-1}(\overline{\omega}, \overline{\eta}),
\end{aligned}$$

for each vector fields $X, Y \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$.

Proof. The result is obtained directly by applying Theorem 3.1, Lemma 3.3, together with formulas (5.40) and (5.41). □

Theorem 5.2. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then, the following identities hold:*

1. $$\begin{aligned}\widehat{\mathrm{R}}({}^{H}X, {}^{H}Y){}^{H}Z &= {}^{H}(\mathrm{R}(X,Y)Z) + \frac{1}{2r}{}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(X,Y)\sharp p)Z) + \frac{1}{4r}{}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(X,Z)\sharp p)Y)\\ &\quad -\frac{1}{4r}{}^{H}(\mathrm{R}(\sharp p, \mathrm{R}(Y,Z)\sharp p)X) - \frac{1}{2}{}^{T}\big(p(\nabla_Z R)(X,Y)\big).\end{aligned}$$

$$
\begin{aligned}
2.\ \widehat{\mathrm{R}}({}^{H}X, {}^{T}\eta)^{H}Z &= \frac{1}{2r}{}^{H}((\nabla_X R)(\sharp p, \sharp\eta)Z) + \frac{1}{4r}{}^{T}(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\eta)Z)) - \frac{1}{2}{}^{T}(\overline{\eta}\mathrm{R}(X, Z)).\\
3.\ \widehat{\mathrm{R}}({}^{H}X, {}^{H}Y)^{T}\mu &= \frac{1}{2r}{}^{H}((\nabla_X R)(\sharp p, \sharp\mu)Y) - \frac{1}{2r}{}^{H}((\nabla_Y R)(\sharp p, \sharp\mu)X) - {}^{T}(\overline{\mu}\mathrm{R}(X, Y))\\
&\quad + \frac{1}{4r}{}^{T}(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\mu)Y)) - \frac{1}{4r}{}^{T}(p\mathrm{R}(Y, \mathrm{R}(\sharp p, \sharp\mu)X)).\\
4.\ \widehat{\mathrm{R}}({}^{H}X, {}^{T}\eta)^{T}\mu &= -\frac{1}{2r}{}^{H}(\mathrm{R}(\sharp\overline{\eta}, \sharp\overline{\mu})X) - \frac{1}{4r^2}{}^{H}(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\mu)X).\\
5.\ \widehat{\mathrm{R}}({}^{T}\omega, {}^{T}\eta)^{H}Z &= \frac{1}{4r^2}{}^{H}(\mathrm{R}(\sharp p, \sharp\omega)\mathrm{R}(\sharp p, \sharp\eta)Z) - \frac{1}{4r^2}{}^{H}(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\omega)Z)\\
&\quad + \frac{1}{r}{}^{H}(\mathrm{R}(\sharp\overline{\omega}, \sharp\overline{\eta})Z).\\
6.\ \widehat{\mathrm{R}}({}^{T}\omega, {}^{T}\eta)^{T}\mu &= \frac{1}{r}g(\overline{\eta}, \overline{\mu})^{T}\omega - \frac{1}{r}g(\overline{\omega}, \overline{\mu})^{T}\eta.
\end{aligned}
$$

for each $X, Y, Z \in \Im^1_0(M)$ *and* $\omega, \eta, \mu \in \Im^0_1(M)$, *where* $\overline{\omega} = \omega - \frac{1}{r}g^{-1}(\omega, p)p$ *and* $\sharp\overline{\omega} = \sharp\omega - \frac{1}{r}g^{-1}(\omega, p)\sharp p$.

Proof. The proof follows directly from Theorem 4.1, Gauss's equation (5.39) and Lemma 3.2. □

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