International Journal of Maps in Mathematics

*Volume **9**, Issue **1**, 2026, Pages:**81–99***

E-ISSN: 2636-7467

www.simadp.com/journalmim

ON BIPOLAR FUZZY IMPLICATIVE IDEALS IN SHEFFER STROKE BG-ALGEBRAS

NEELAMEGARAJAN RAJESH, TAHSIN ONER *, IBRAHIM SENTURK, AND KANNAN GEETHA

Abstract. This paper presents a comprehensive investigation into the algebraic integration of bipolar fuzzy logic and Sheffer stroke BG-algebras, introducing the concept of bipolar fuzzy Sheffer stroke BG-algebras. By embedding bipolar fuzzy sets—distinguished by their dual positive and negative membership degrees—into the Sheffer stroke BG-algebraic framework, we systematically examine the structure and properties of bipolar fuzzy subalgebras and various classes of SBG-ideals. Central to our approach is the analysis of level sets associated with bipolar fuzzy subsets, through which we establish rigorous correspondences between these fuzzy constructs and classical subalgebras and ideals. Our results reveal that the level sets of bipolar fuzzy SBG-subalgebras and SBG-ideals naturally inherit the underlying algebraic structure, subject to well-defined conditions. This theoretical advancement not only enhances the algebraic modeling of systems characterized by uncertainty and duality but also lays a solid foundation for further applications in logical inference, information processing, and computational intelligence within bipolar fuzzy contexts.

Keywords: Sheffer stroke (BG-algebra), BG-ideal, fuzzy SBG-ideal.

MSC (2020): 06F05, 03G25, 03G10.

1. Introduction

In [2], published in 2008 by C. B. Kim and H. S. Kim, the concept of BG-algebras was introduced as a generalization of the notion of B-algebras (as defined in [4]). BG-algebras, generalize Boolean algebras by introducing advanced operations and properties, thereby enabling the analysis of complex logical and scientific systems beyond the scope of classical Boolean logic. Rooted in the foundations of Boolean algebra, BG-algebras provide a versatile framework that has proven valuable in areas such as mathematical logic and fuzzy logic, where nuanced forms of inference must be represented.

A central operation in logic, the Sheffer stroke (NAND), was introduced by Henry Maurice Sheffer [10]. This operation is functionally complete, meaning any logical expression or axiom

Received: 2025.07.11 *Accepted: 2025.10.19*

* Corresponding author

Neelamegarajan Rajesh (nrajesh_topology@yahoo.co.in) ⋄ ORCID:0000-0003-0921-6054

Tahsin Oner (tahsin.oner@ege.edu.tr) ⋄ ORCID:0000-0002-6514-4027

Ibrahim Senturk (ibrahim.senturk@ege.edu.tr) ⋄ ORCID:0000-0001-8296-2796

Kannan Geetha (geeethaak@gmail.com) ⋄ ORCID:0009-0004-9137-3239

can be constructed using only the Sheffer stroke [3]. This property not only simplifies the formulation of logical systems but also underscores the Sheffer stroke's fundamental role in both algebraic and logical contexts. As a result, the Sheffer stroke has inspired extensive research, including studies on its application to basic algebras and their congruences [6], MLT-algebras [9], ortholattices [1], new state operators and their filters in basic algebras [7], Riečan and Bosbach state operators on MTL-algebras [8], and BG-algebras [5, 11].

Sheffer stroke BG-algebras are of particular interest due to their ability to unify various logical operations within a single structure, thereby broadening the analytical scope for logical deduction and facilitating the exploration of relationships among diverse algebraic frameworks. These algebras have significant implications for both theoretical research and practical applications, especially in computational models relevant to quantum computing and fuzzy logic.

Fuzzy set theory, pioneered by Zadeh [13], marked a significant advancement in modeling uncertainty and vagueness, providing a robust framework for approximate reasoning and decision-making. Building on this, bipolar fuzzy sets—introduced by Zhang [14]—extend the concept by simultaneously representing positive and negative information, thus reflecting real-world situations where data may be contradictory or dual in nature.

This paper contributes to the field by investigating bipolar fuzzy Sheffer stroke BG-algebras, a novel algebraic system that synthesizes bipolar fuzzy logic with Sheffer stroke operations. We aim to generalize classical notions of subalgebras and ideals to the bipolar fuzzy context, with a particular focus on their level sets. These level sets provide a means to connect membership functions with algebraic properties, offering deeper insight into the structure of fuzzy subsets.

The motivation for this research lies in the potential of algebraic models equipped with bipolar fuzzy logic to address contradictions and uncertainties prevalent in decision-making, data analysis, and computational intelligence. By establishing rigorous correspondences between bipolar fuzzy subalgebras and ideals and their level sets, this study lays the groundwork for future applications and extensions within fuzzy algebraic systems.

The list of acronyms is provided in Table 1.1.

Table 1.1. List of acronyms

Acronyms	Representation
SBG-algebra	Sheffer stroke BG-algebra
BFISBG-ideal	bipolar fuzzy implicative SBG-ideal
BVFISBG-ideal	bipolar-valued fuzzy implicative SBG-ideal

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke BG-algebras and bipolar fuzzy structures concept.

Definition 2.1. [10] *Let $(H, |)$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$$(S1)\ \omega \mid \zeta = \zeta \mid \omega$$
$$(S2)\ (\omega \mid \omega) \mid (\omega \mid \zeta) = \omega$$
$$(S3)\ \omega \mid ((\zeta \mid \upsilon) \mid (\zeta \mid \upsilon)) = ((\omega \mid \zeta) \mid (\omega \mid \zeta)) \mid \upsilon$$
$$(S4)\ (\omega \mid ((\omega \mid \omega) \mid (\zeta \mid \zeta))) \mid (\omega \mid ((\omega \mid \omega) \mid (\zeta \mid \zeta))) = \omega.$$

Definition 2.2. [5] *A Sheffer stroke BG-algebra (briefly, SBG-algebra) is a structure $(A, |)$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x, y, z \in A$*

$$(SBG-1)\ (\omega \mid (\omega \mid \omega)) \mid (x \mid (\omega \mid \omega)) = 0$$
$$(SBG-2)\ (0 \mid (\zeta \mid \zeta)) \mid (\omega \mid (\zeta \mid \zeta)) \mid (\omega \mid (\zeta \mid \zeta)) = \omega \mid \omega.$$

To enhance the readability of this manuscript about Sheffer stroke BG-algebras, we will consistently use the following notation:

$$\omega \mid (\zeta \mid \zeta) = \omega^{\zeta},$$

$$\omega \mid (\zeta \mid (\omega \mid \omega)) = \omega_{\zeta}.$$

Proposition 2.1. [5] *Let $(A, |)$ be an SBG-algebra. Then the binary relation $\omega \leq \zeta$ if and only if $\zeta^{\omega} \mid y^{x} = 0$ is a partial order on A.*

Definition 2.3. [5] *Let $(A, |)$ be an SBG-algebra. A nonempty subset G of A is called an SBG-subalgebra of A if $\omega^{\zeta} \mid \omega^{\zeta} \in G$ for all $\omega, \zeta \in G$.*

Definition 2.4. [5] *Let $(A, |)$ be an SBG-algebra. A nonempty subset I of a A is called an SBG-ideal of A if for all $\omega, \zeta \in I$*

(1) $0 \in I$,
(2) $\omega^{\zeta} \mid \omega^{\zeta} \in I$ *and* $\zeta \in I \Rightarrow \omega \in I$.

Definition 2.5 ([14])**.** *Let X be a nonempty set. A* bipolar fuzzy set *B on X is defined as*

$$B = \{(\omega, \Psi^{-}(\omega), \Psi^{+}(\omega)) \mid \omega \in X\},$$

where $\Psi^{+} : X \to [0, 1]$ and $\Psi^{-} : X \to [-1, 0]$ are two functions. The value $\Psi^{+}(\omega)$ represents the degree to which the element x satisfies the property associated with the bipolar fuzzy set B (the positive membership degree), while $\Psi^{-}(\omega)$ represents the degree to which x satisfies an implicit counter-property of B (the negative membership degree).

If $\Psi^{+}(\omega) \neq 0$ and $\Psi^{-}(\omega) = 0$, then x is interpreted as satisfying only the positive aspect of B. Conversely, if $\Psi^{+}(\omega) = 0$ and $\Psi^{-}(\omega) \neq 0$, then x is viewed as not satisfying the main property of B but partially satisfying its counter-property. It is also possible that both $\Psi^{+}(\omega) = 0$ and $\Psi^{-}(\omega) = 0$, which indicates an overlap or neutrality between the property and its counter-property at ω.

For simplicity, we denote the bipolar fuzzy set B by the pair of functions $\Psi = (\Psi^{+}, \Psi^{-})$.

Lemma 2.1 ([12])**.** *Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then the following identities hold:*

(1) $\alpha - \min\{\beta, \gamma\} = \max\{\alpha - \beta, \alpha - \gamma\}$,
(2) $\alpha - \max\{\beta, \gamma\} = \min\{\alpha - \beta, \alpha - \gamma\}$.

3. Bipolar fuzzy implicative SBG-ideals

This section is devoted to the study of bipolar fuzzy implicative SBG-ideals in SBG-algebras. We begin by formally defining the notion of a bipolar fuzzy implicative SBG-ideal and investigate its fundamental properties and characterizations. Several key results are established, including necessary and sufficient conditions for a bipolar-valued fuzzy set to be a BVFISBG-ideal in terms of its level sets. The relationships between various classes of bipolar fuzzy ideals—such as sub-implicative, completely closed, and p-ideals—are explored, and their connections with implicative SBG-ideals are clarified. Moreover, closure properties under infimum and translation operators are examined. The section also provides illustrative theorems and propositions that characterize these ideals within specific algebraic settings, such as medial and implicative SBG-algebras, thereby enriching the structural understanding of bipolar fuzzy ideals in this context.

Definition 3.1. *A bipolar fuzzy set $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ on an SBG-algebra $\mathcal{L}$ is called a bipolar fuzzy implicative SBG-ideal of $\mathcal{L}$ if*

$$(\forall \omega, \zeta \in \mathcal{L}) \left(\begin{array}{l} \Psi^-(0) \leq \Psi^-(\omega) \leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \\ \Psi^+(0) \geq \Psi^+(\omega) \geq \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \end{array} \right). \quad (3.1)$$

Proposition 3.1. *Every BFISBG-ideal of an SBG-algebra $\mathcal{L}$ is also a bipolar fuzzy SBG-ideal of $\mathcal{L}$.*

Proof. Let Ψ be a BFISBG-ideal of an SBG-algebra $\mathcal{L}$. Then, the following inequalities hold for all $\omega \in \mathcal{L}$:

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

Also,

$$\begin{array}{rcl} \Psi^-(\omega) & \leq & \max\{\Psi^-(\zeta), \Psi^-((\omega_\omega \mid \omega_\zeta)^\zeta \mid (\omega_\omega \mid \omega_\zeta)^\zeta)\} \\ & = & \max\{\Psi^-(\zeta), \Psi^-((\omega^0 \mid \omega^0)^\zeta \mid (\omega^0 \mid \omega^0)^\zeta)\} \\ & = & \max\{\Psi^-(\zeta), \Psi^-(\omega^\zeta \mid \omega^\zeta)\}, \end{array}$$

and

$$\begin{array}{rcl} \Psi^+(\omega) & \geq & \min\{\Psi^+(\zeta), \Psi^+((\omega_\omega \mid \omega_\zeta)^\zeta \mid (\omega_\omega \mid \omega_\zeta)^\zeta)\} \\ & = & \min\{\Psi^+(\zeta), \Psi^+((\omega^0 \mid \omega^0)^\zeta \mid (\omega^0 \mid \omega^0)^\zeta)\} \\ & = & \min\{\Psi^+(\zeta), \Psi^+(\omega^\zeta \mid \omega^\zeta)\}. \end{array}$$

Therefore, Ψ is a bipolar fuzzy SPG-ideal of $\mathcal{L}$. □

Theorem 3.1. *Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar-valued fuzzy set in $\mathcal{L}$. Then Ψ is a BVFISBG-ideal of $(\mathcal{L}, |)$ if and only if, for all $(\mathfrak{p}, \mathfrak{q}) \in [-1, 0] \times [0, 1]$, the negative $\mathfrak{p}$-cut and the positive $\mathfrak{q}$-cut of Ψ (whenever they are nonempty) are implicative SBG-ideals of $\mathcal{L}$.*

Proof. Assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $(\mathcal{L}, |)$, and that for all $(\mathfrak{p}, \mathfrak{q}) \in [-1, 0] \times [0, 1]$, the negative $\mathfrak{p}$-cut $\mathcal{L}(\Psi^-, \mathfrak{p})$ and the positive $\mathfrak{q}$-cut $U(\Psi^+, \mathfrak{q})$ are nonempty.

Let $\omega, \alpha \in \mathcal{L}$ be such that $(\omega, \alpha) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$, i.e.,

$$\Psi^-(\omega) \leq \mathfrak{p} \quad \text{and} \quad \Psi^+(\alpha) \geq \mathfrak{q}.$$

Since $\Psi^-(0) \leq \Psi^-(\omega) \leq \mathfrak{p}$ and $\Psi^+(0) \geq \Psi^+(\alpha) \geq \mathfrak{q}$, it follows that $(0, 0) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$.

Now, let $\omega, \zeta, \upsilon, \alpha, \beta, \gamma \in \mathcal{L}$ satisfy

$$(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \in L(\Psi^-, \mathfrak{p}),$$

$$(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma) \in U(\Psi^+, \mathfrak{q}),$$

and $(\upsilon, \gamma) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$. Then we have

$$\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \leq s, \quad \Psi^-(\upsilon) \leq \mathfrak{p},$$

$$\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \geq \mathfrak{q}, \quad \Psi^+(\gamma) \geq \mathfrak{q}.$$

From this, it follows that

$$\Psi^-(\omega) \leq \max\{\Psi^-(\omega^\zeta \mid \omega^\zeta), \Psi^-(\upsilon)\} \leq \mathfrak{p},$$

and

$$\Psi^+(\alpha) \geq \min\{\Psi^+(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^+(c)\} \geq \mathfrak{q},$$

which shows that $\omega, \alpha \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$. Hence, $L(\Psi^-, \mathfrak{p})$ and $U(\Psi^+, \mathfrak{q})$ are implicative SBG-ideals of $\mathcal{L}$.

Conversely, assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a bipolar-valued fuzzy set in $\mathcal{L}$ such that its negative $\mathfrak{p}$-cuts and positive $\mathfrak{q}$-cuts are implicative SBG-ideals of $\mathcal{L}$ whenever they are nonempty.

Suppose there exists $\alpha \in \mathcal{L}$ such that $\Psi^-(0) > \Psi^-(\alpha)$. Then $\alpha \in L(\Psi^-, \Psi^-(\alpha))$, but $0 \notin L(\Psi^-, \Psi^-(\alpha))$, contradicting the fact that $L(\Psi^-, \Psi^-(\alpha))$ is an implicative SBG-ideal. Hence,

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{for all } \omega \in \mathcal{L}.$$

Similarly, suppose there exists $\omega \in \mathcal{L}$ such that $\Psi^+(0) < \Psi^+(\omega)$. Then $\omega \in U(\Psi^+, \Psi^+(\omega))$, but $0 \notin U(\Psi^+, \Psi^+(\omega))$, a contradiction. Therefore,

$$\Psi^+(0) \geq \Psi^+(\alpha) \quad \text{for all } \alpha \in \mathcal{L}.$$

Now, suppose there exist $\alpha, \beta, \gamma, \omega, \zeta, \upsilon \in \mathcal{L}$ such that

$$\Psi^-(\alpha) > \max\left\{\Psi^-(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^-(\gamma)\right\}$$

or

$$\Psi^+(\omega) < \min\left\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\right\}.$$

Define

$$\mathfrak{p} = \max\left\{\Psi^-(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^-(\gamma)\right\},$$

$$\mathfrak{q} = \min\left\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\right\}.$$

Since $a \notin L(\Psi^-, s)$ or $x \notin U(\Psi^+, t)$, this contradicts the assumption that these cuts are implicative SBG-ideals.

Therefore,

$$\Psi^-(\omega) \leq \max\left\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\right\}$$

and

$$\Psi^+(\omega) \geq \min\left\{\Psi^+\left(x^y \mid x^y\right), \Psi^+(\upsilon)\right\}$$

for all $\omega, \zeta, \upsilon \in \mathcal{L}$. Hence, Ψ is a BVFISBG-ideal of $\mathcal{L}$. □

Theorem 3.2. *A bipolar-valued fuzzy set* $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ *in* $\mathcal{L}$ *is a BVFISBG-ideal of* $(\mathcal{L}, |)$ *if and only if the fuzzy sets* Ψ^-_γ *and* Ψ^+ *are fuzzy implicative SBG-ideals of* $(\mathcal{L}, |)$*, where* $\Psi^-_\gamma : \mathcal{L} \to [0,1]$ *is defined by* $\Psi^-_\gamma(\omega) = 1 - \Psi^-(\omega)$ *for all* $\omega \in \mathcal{L}$*.*

Proof. Assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $(\mathcal{L}, |)$. Then Ψ satisfies the following: For every $\omega, \zeta \in \mathcal{L}$,

$$\begin{aligned} \Psi^-_\gamma(0) &= 1 - \Psi^-(0) \\ &\geq 1 - \Psi^-(\omega) \\ &= \Psi^-_\gamma(\omega), \end{aligned}$$

$$\begin{aligned} \Psi^-_\gamma(\omega) &= 1 - \Psi^-(\omega) \\ &\geq 1 - \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \\ &= \min\{1 - \Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), 1 - \Psi^-(\upsilon)\} \\ &= \min\{\Psi^-_c(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_c(\upsilon)\}. \end{aligned}$$

Thus, Ψ^-_γ satisfies the conditions of a fuzzy implicative SBG-ideal. Since Ψ^+ is already assumed to be one, the claim holds.

Conversely, Conversely, suppose that Ψ^-_γ and Ψ^+ are fuzzy implicative SBG-ideals of $(\mathcal{L}, |)$. Then for $\omega, \zeta, \upsilon \in \mathcal{L}$ we have $\Psi^-(0) \leq \Psi^-(\omega)$ and

$$\begin{aligned} 1 - \Psi^-(\omega) &= \Psi^-_\gamma(\omega) \\ &\geq \min\{\Psi^-_\gamma(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_\gamma(\upsilon)\} \\ &= \min\{1 - \Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), 1 - \Psi^-(\upsilon)\} \\ &= 1 - \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\}, \\ \Psi^-(\omega) &\leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\}. \end{aligned}$$

Hence $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $\mathcal{L}$. □

Theorem 3.3. *Let* $\mathcal{F}$ *be a nonempty subset of* $\mathcal{L}$*. Define the bipolar-valued fuzzy set* $\Psi_\mathcal{F} = (L, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ *in* $\mathcal{L}$ *as follows:*

$$\Psi^-_\mathcal{F} : \mathcal{L} \to [-1,0], \alpha \mapsto \begin{cases} \mathfrak{p}^- & \text{if } \alpha \in \mathcal{F}, \\ \mathfrak{q}^- & \text{otherwise}, \end{cases} \quad \Psi^+_\mathcal{F} : \mathcal{L} \to [0,1], \omega \mapsto \begin{cases} \mathfrak{p}^+ & \text{if } \omega \in \mathcal{F}, \\ \mathfrak{q}^+ & \text{otherwise}, \end{cases}$$

where $\mathfrak{p}^- < \mathfrak{q}^-$ *in* $[-1,0]$ *and* $\mathfrak{p}^+ > \mathfrak{q}^+$ *in* $[0,1]$*. Then* $\Psi_\mathcal{F} = (\mathcal{L}, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ *is a BVFISBG-ideal of* $(\mathcal{L}, |)$ *if and only if* $\mathcal{F}$ *is an implicative SBG-ideal of* $\mathcal{L}$*.*

Proof. Assume that $\Psi_\mathcal{F} = (\mathcal{L}, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ is a BVFISBG-ideal of $(\mathcal{L}, |)$. Let $\omega, \zeta, \upsilon \in \mathcal{L}$ be such that $\omega, \zeta \in \mathcal{F}$. Then, by the definition of $\Psi^-_\mathcal{F}$ and $\Psi^+_\mathcal{F}$, we have $\Psi^-_\mathcal{F}(\omega) = \mathfrak{p}^-$ and $\Psi^+_\mathcal{F}(\omega) = \mathfrak{p}^+$. Since $\Psi_\mathcal{F}$ is a BVFISBG-ideal, it satisfies the conditions $\Psi^-_\mathcal{F}(0) \leq \Psi^-_\mathcal{F}(\omega)$ and $\Psi^+_\mathcal{F}(0) \geq \Psi^+_\mathcal{F}(\omega)$, which gives $\Psi^-_\mathcal{F}(0) \leq \mathfrak{p}^-$ and $\Psi^+_\mathcal{F}(0) \geq \mathfrak{p}^+$. However, by the definition of $\Psi^-_\mathcal{F}$ and $\Psi^+_\mathcal{F}$, we also know that $\Psi^-_\mathcal{F}(0)$ can only be $\mathfrak{p}^-$ or $\mathfrak{q}^-$, and $\Psi^+_\mathcal{F}(0)$ can only be $\mathfrak{p}^+$ or $\mathfrak{q}^+$. But since $^- < \mathfrak{q}^-$ and $\mathfrak{p}^+ > \mathfrak{q}^+$, the inequalities above can hold only if $\Psi^-_\mathcal{F}(0) = \mathfrak{p}^-$, and $\Psi^+_\mathcal{F}(0) = \mathfrak{p}^+$, which implies that $0 \in \mathcal{F}$. Also $\Psi^-(\omega) \leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\zeta)\} = \mathfrak{p}^-$, $\Psi^+(\omega) \geq \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon)), \Psi^+(\zeta)\} = s^+$, and so $\Psi^-(\omega) = \mathfrak{p}^-$ and $\Psi^+(\omega) = \mathfrak{p}^+$. Therefore, $\omega \in \mathcal{F}$. Hence, $\mathcal{F}$ is closed under the implicative SBG-ideal condition, and since $0 \in \mathcal{F}$, we conclude that $\mathcal{F}$ is an implicative SBG-ideal of $\mathcal{L}$.

Conversely, assume that $\mathcal{F}$ is an implicative SBG-ideal of $(\mathcal{L}, |)$. Let $\omega, \zeta, \upsilon \in \mathcal{L}$. If $\omega \in \mathcal{F}$, then $0 \in \mathcal{F}$, and thus $\Psi^-_{\mathcal{F}}(0) = \mathfrak{p}^- = \Psi^-_{\mathcal{F}}(\omega)$, and $\Psi^+_{\mathcal{F}}(0) = \mathfrak{p}^+ = \Psi^+_{\mathcal{F}}(\omega)$. If $0 \notin \mathcal{F}$, then $\Psi^-_{\mathcal{F}}(0) = \mathfrak{q}^- > \mathfrak{p}^- = \Psi^-_{\mathcal{F}}(\omega)$, and $\Psi^+_{\mathcal{F}}(0) = \mathfrak{q}^+ < s^+ = \Psi^+_{\mathcal{F}}(\omega)$, which still satisfies the required inequality conditions for a BVFISBG-ideal.

Furthermore, suppose $\alpha = (((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \in \mathcal{F}$ and $\upsilon \in \mathcal{F}$. Then, since $\mathcal{F}$ is an implicative SBG-ideal, we have $\omega \in \mathcal{F}$. Hence,

$$\Psi^-_{\mathcal{F}}(\omega) = \mathfrak{p}^- = \max\{\Psi^-_{\mathcal{F}}(\eta), \Psi^-_{\mathcal{F}}(\upsilon)\}, \quad \Psi^+_{\mathcal{F}}(\omega) = \mathfrak{p}^+ = \min\{\Psi^+_{\mathcal{F}}(\eta), \Psi^+_{\mathcal{F}}(\upsilon)\}.$$

If either $\eta \notin \mathcal{F}$ or $\upsilon \notin \mathcal{F}$, then at least one of $\Psi^-_{\mathcal{F}}(\eta)$ or $\Psi^-_{F}(\upsilon)$ equals t^-, and so

$$\Psi^-_{\mathcal{F}}(\omega) \leq \mathfrak{q}^- = \max\{\Psi^-_{\mathcal{F}}(\eta), \Psi^-_{\mathcal{F}}(\upsilon)\}, \quad \Psi^+_{\mathcal{F}}(\omega) \geq \mathfrak{q}^+ = \min\{\Psi^+_{\mathcal{F}}(\eta), \Psi^+_{\mathcal{F}}(\upsilon)\},$$

which still satisfies the defining conditions.

Thus, $\Psi_{\mathcal{F}} = (\mathcal{L}, \Psi^-_{F}, \Psi^+_{\mathcal{F}})$ is a BVFISBG-ideal of $\mathcal{L}$. □

Proposition 3.2. *Let $\{\Psi_\mathfrak{k} = (\Psi^-_\mathfrak{k}, \Psi^+_\mathfrak{k}) : \mathfrak{k} \in \Delta\}$ be a family of bipolar-valued fuzzy implicative SBG-ideals of a SBG-algebra $\mathcal{L}$. Then the infimum*

$$\bigwedge_{\mathfrak{k}\in\Delta} \Psi_\mathfrak{k} = \left(\bigvee_{\mathfrak{k}\in\Delta} \Psi^-_\mathfrak{k}, \bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k} \right)$$

is also a BVFISBG-ideal of $\mathcal{L}$.

Proof. Let $\Psi_\mathfrak{k} = \{(\Psi^+_\mathfrak{k}, \Psi^-_\mathfrak{k}) : \mathfrak{k} \in \Delta\}$ be a family of bipolar fuzzy implicative SBG-ideals of a SBG-algebra $\mathcal{L}$. Let $\omega, \zeta \in \mathcal{L}$. Then we have

$$(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k})(0) = \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_\mathfrak{k}(0)\} \geq \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_\mathfrak{k}(\omega)\} = (\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k})(\omega),$$

$$(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^-_\mathfrak{k})(0) = \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_\mathfrak{k}(0)\} \leq \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_\mathfrak{k}(\omega)\} = (\bigwedge_{i\in\Delta} \Psi^-_\mathfrak{k})(\omega).$$

Let $\omega, \zeta \in \mathcal{L}$. Then we have

$$\begin{aligned} (\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k})(\omega) &= \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_\mathfrak{k}(\omega)\} \\ &\geq \inf_{\mathfrak{k}\in\Delta}\{\min\{\Psi^+_\mathfrak{k}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+_\mathfrak{k}(\upsilon)\}\} \\ &= \min\{\inf_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \inf_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k}(\upsilon)\} \\ &= \min\{(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k})(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), (\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_\mathfrak{k})(\upsilon)\}, \end{aligned}$$

and

$$\begin{aligned} (\bigwedge_{i\in\Delta} \Psi^-_\mathfrak{k})(\omega) &= \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_\mathfrak{k}(\omega)\} \\ &\leq \sup_{\mathfrak{k}\in\Delta}\{\max\{\Psi^-_\mathfrak{k}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_\mathfrak{k}(\upsilon)\}\} \\ &= \max\{\sup_{\mathfrak{k}\in\Delta} \Psi^-_\mathfrak{k}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \sup_{\mathfrak{k}\in\Delta} \Psi^-_\mathfrak{k}(\upsilon)\} \\ &= \max\{(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^-_\mathfrak{k})(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), (\bigwedge_{i\in\Delta} \Psi^-_\mathfrak{k})(\upsilon)\}. \end{aligned}$$

Hence $\bigwedge_{\mathfrak{k}\in\Delta} \Psi_\mathfrak{k}$ is a BFISBG-ideal of a SBG-algebra $\mathcal{L}$. □

Definition 3.2. *A bipolar fuzzy subset $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ of a SBG-algebra $\mathcal{L}$ is called a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ if it satisfies the following conditions:*

$$(\forall\omega,\zeta,\upsilon\in\mathcal{L})\left(\begin{array}{l}\Psi^-(0)\leq\Psi^-(\omega),\\ \Psi^+(0)\geq\Psi^+(\omega),\\ \Psi^-((\zeta|\zeta^\omega)|(\zeta|\zeta^\omega))\leq\max\{\Psi^-(((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon\mid((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon),\Psi^-(\upsilon)\},\\ \Psi^+((\zeta|\zeta^\omega)|(\zeta|\zeta^\omega))\geq\min\{\Psi^+(((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon\mid((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon),\Psi^+(\upsilon)\},\end{array}\right). \tag{3.2}$$

Proposition 3.3. *Let $\mathcal{L}$ be a SBG-algebra. Then every bipolar fuzzy subimplicative SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi=(\mathcal{L},\Psi^-,\Psi^+)$ be a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. Then $\Psi^-(0)\leq\Psi^-(\omega),\ \Psi^+(0)\geq\Psi^+(\omega)$,

$$\begin{aligned}\Psi^-(\omega)&=\Psi^-(\omega^0\mid\omega^0)\\&=\Psi^-((\omega|\omega^\omega)|(\omega|\omega^\omega))\\&\leq\max\{\Psi^-(((((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)\mid((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)),\Psi^-(\upsilon)\}\\&=\max\{\Psi^-((\omega^0\mid\omega^0)^\upsilon\mid(\omega^0\mid\omega^0)^\upsilon),\Psi^-(\upsilon)\}\\&=\max\{\Psi^-(\omega^\upsilon\mid\omega^\upsilon),\Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega)&=\Psi^+(\omega^0\mid\omega^0)\\&=\Psi^+((\omega|\omega^\omega)|(\omega|\omega^\omega))\\&\geq\min\{\Psi^+(((((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)\mid((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)),\Psi^+(\upsilon)\}\\&=\min\{\Psi^+((\omega^0\mid\omega^0)^\upsilon\mid(\omega^0\mid\omega^0)^\upsilon),\Psi^+(\upsilon)\}\\&=\min\{\Psi^+(\omega^\upsilon\mid\omega^\upsilon),\Psi^+(\upsilon)\}.\end{aligned}$$

Therefore, $\Psi=(\mathcal{L},\Psi^-,\Psi^+)$ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. □

Theorem 3.4. *Let $\Psi=(\mathcal{L},\Psi^-,\Psi^+)$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then $\Psi=(\mathcal{L},\Psi^-,\Psi^+)$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ if and only if*

$$\Psi^-((\zeta\mid\zeta^\omega)\mid(\zeta\mid\zeta^\omega))\leq\Psi^-((\omega\mid\omega^\zeta)\mid(\omega\mid\omega^\zeta))$$

and

$$\Psi^+((\zeta\mid\zeta^\omega)\mid(\zeta\mid\zeta^\omega))\geq\Psi^+((\omega\mid\omega^\zeta)\mid(\omega\mid\omega^\zeta)).$$

Proof. Assume that $\Psi=(\mathcal{L},\Psi^-,\Psi^+)$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. Then we have

$$\begin{aligned}&\Psi^-((\zeta\mid\zeta^\omega)\mid(\zeta\mid\zeta^\omega))\\&\leq\max\left\{\Psi^-(((\zeta\mid\zeta^\omega)\mid(\zeta\mid\zeta^\omega))^0\mid((\zeta\mid\zeta^\omega)\mid(\zeta\mid\zeta^\omega))^0),\Psi^-(0)\right\}\\&=\max\left\{\Psi^-((\omega\mid\omega^\zeta)\mid(\omega\mid\omega^\zeta)),\Psi^-(0)\right\}\\&=\Psi^-((\omega\mid\omega^\zeta)\mid(\omega\mid\omega^\zeta)).\end{aligned}$$

Similarly,

$$\begin{aligned}&\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\&\geq \min\left\{\Psi^+(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0 \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0), \Psi^+(0)\right\}\\&= \min\left\{\Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^+(0)\right\}\\&= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)).\end{aligned}$$

Conversely, assume that the given inequalities hold and that Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Then we have

$$\Psi^-(0) \leq \Psi^-(\omega), \quad \Psi^+(0) \geq \Psi^+(\omega) \quad \text{for all } \omega \in \mathcal{L}.$$

Now, from the assumption, we obtain

$$\begin{aligned}\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &\leq \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))\\&\leq \max\left\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon), \Psi^-(\upsilon)\right\},\end{aligned}$$

and similarly,

$$\begin{aligned}\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &\geq \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))\\&\geq \min\left\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon), \Psi^+(\upsilon)\right\}.\end{aligned}$$

Hence, Ψ satisfies the condition for being a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Theorem 3.5. *Let $\mathcal{L}$ be an implicative SBG-algebra. Then every bipolar fuzzy SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Then it satisfies $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. By the properties of bipolar fuzzy SBG-ideals, we have

$$\begin{aligned}&\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\&\quad\leq \max\{\Psi^-(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon), \Psi^-(\upsilon)\}\\&\quad= \max\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon, \Psi^-(\upsilon)\},\end{aligned}$$

$$\begin{aligned}&\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\&\quad\geq \min\{\Psi^+(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon), \Psi^+(\upsilon)\}\\&\quad= \min\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon, \Psi^+(\upsilon)\}.\end{aligned}$$

Thereby, Ψ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Lemma 3.1. *In an SBG-algebra $\mathcal{L}$, the following identity holds for all $\omega, \zeta \in \mathcal{L}$:*

$$((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \mid \zeta^\omega = \omega \mid \omega^\zeta.$$

Theorem 3.6. *Every bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$. Then by definition, for all $\omega \in \mathcal{L}$ we have $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$. Note that in a medial SBG-algebra, the following identity holds:

$$(\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega}) = \omega.$$

Therefore,

$$\Psi^{-}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) = \Psi^{-}(\omega),$$

and

$$\Psi^{+}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) = \Psi^{+}(\omega).$$

Next, we observe that

$$\begin{aligned}\Psi^{-}(\omega) &\leq \max\{\Psi^{-}(\omega^{\upsilon} \mid \omega^{\upsilon}),\ \Psi^{-}(\upsilon)\}\\ &= \max\left\{\Psi^{-}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}),\ \Psi^{-}(\upsilon)\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Psi^{+}(\omega) &\geq \min\{\Psi^{+}(\omega^{\upsilon} \mid \omega^{\upsilon}),\ \Psi^{+}(\upsilon)\}\\ &= \min\left\{\Psi^{+}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}),\ \Psi^{+}(\upsilon)\right\}.\end{aligned}$$

Thus, the conditions for Ψ to be a bipolar fuzzy sub-implicative SBG-ideal are satisfied. Hence, every bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$ is indeed a bipolar fuzzy sub-implicative SBG-ideal. □

Theorem 3.7. *Let $\mathcal{L}$ be an SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^{-}(\zeta^{\upsilon} \mid \zeta^{\upsilon}) \leq \Psi^{-}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}) \\ \Psi^{+}(\zeta^{\upsilon} \mid \zeta^{\upsilon}) \geq \Psi^{+}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}) \end{pmatrix}. \tag{3.3}$$

Then every bipolar fuzzy SBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.

Proof. We are given that the following inequality holds:

$$\begin{aligned}\Psi^{-}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) &\leq \Psi^{-}((((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}) \mid (((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}))\\ &= \Psi^{-}((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Psi^{+}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) &\geq \Psi^{+}((((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}) \mid (((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}))\\ &= \Psi^{+}((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})).\end{aligned}$$

Hence, Ψ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Theorem 3.8. *Every medial SBG-algebra is an implicative SBG-algebra.*

Theorem 3.9. *Let $\Psi = (\mathcal{L}, \Psi^{-}, \Psi^{+})$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then Ψ is a BFISBG-ideal of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall \omega, \zeta \in \mathcal{L}) \quad \begin{cases} \Psi^{-}(\omega) \leq \Psi^{-}(\omega_{\zeta} \mid \omega_{\zeta}), \\ \Psi^{+}(\omega) \geq \Psi^{+}(\omega_{\zeta} \mid \omega_{\zeta}). \end{cases} \tag{3.4}$$

Proof. Let Ψ be a BFISBG-ideal of $\mathcal{L}$. Then

$$\begin{aligned}\Psi^-(\omega) &\leq \max\{\Psi^-(0), \Psi^-((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0)\}\\ &= \max\{\Psi^-(0), \Psi^-(\omega_\zeta \mid \omega_\zeta)\}\\ &= \Psi^-(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &\geq \min\{\Psi^+(0), \Psi^+((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0)\}\\ &= \min\{\Psi^+(0), \Psi^+(\omega_\zeta \mid \omega_\zeta)\}\\ &= \Psi^+(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

for all $\omega, \zeta \in \mathcal{L}$.

Conversely, let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of $\mathcal{L}$ satisfying the inequality (3.4). Then it is evident that $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. Since

$$\begin{aligned}\Psi^-(\omega) &\leq \Psi^-(\omega_\zeta \mid \omega_\zeta)\\ &\leq \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &\geq \Psi^+(\omega_\zeta \mid \omega_\zeta)\\ &\geq \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\},\end{aligned}$$

for all $\omega, \zeta, \upsilon \in \mathcal{L}$. Therefore, we conclude that Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Theorem 3.10. *Let $\mathcal{L}$ be a medial SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta \in \mathcal{L}) \left(\begin{array}{c} \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \leq \Psi^-(\omega_\zeta \mid \omega_\zeta) \\ \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \geq \Psi^+(\omega_\zeta \mid \omega_\zeta) \end{array} \right). \tag{3.5}$$

Then every bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy implicative ideal of $\mathcal{L}$.

Proof. Let Ψ be a bipolar fuzzy sub-implicative SBG-ideal of a medial SBG-algebra $\mathcal{L}$ satisfying the inequality (3.5). Then we obtain that

$$\begin{aligned}\Psi^-(\omega) &= \Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\ &\leq \max\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0 \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0), \Psi^-(0)\}\\ &= \max\{\Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^-(0)\}\\ &= \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))\\ &\leq \Psi^-(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &= \Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\ &\geq \min\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0 \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0), \Psi^+(0)\}\\ &= \min\{\Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^+(0)\}\\ &= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))\\ &\geq \Psi^+(\omega_\zeta \mid \omega_\zeta).\end{aligned}$$

Thus, Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Theorem 3.11. *Let $\mathcal{L}$ be an implicative SBG-algebra. Then every BFISBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a BFISBG-ideal of an implicative SBG-algebra $\mathcal{L}$. By definition, Ψ is also a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Therefore, we have $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. Hence, for all $\omega, \zeta, \upsilon \in \mathcal{L}$, we obtain

$$\begin{aligned}\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &= \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\leq \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\geq \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\},\end{aligned}$$

Thus, Ψ satisfies the defining conditions of a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Corollary 3.1. *Let $\mathcal{L}$ be a medial SBG-algebra. Then every BFISBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Definition 3.3. *A bipolar fuzzy SBG-ideal $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ of a SBG-algebra $\mathcal{L}$ is said to be* bipolar fuzzy closed *if it satisfies the following condition:*

$$(\forall \omega \in \mathcal{L}) \begin{pmatrix} \Psi^-(0^\omega \mid 0^\omega) \leq \Psi^-(\omega) \\ \Psi^+(0^\omega \mid 0^\omega) \geq \Psi^+(\omega) \end{pmatrix}. \tag{3.6}$$

Definition 3.4. *Let $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then Ψ is called a* bipolar fuzzy completely closed SBG-ideal *of $\mathcal{L}$ if it satisfies the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^-(\omega^\zeta \mid \omega^\zeta) \leq \max\{\Psi^-(\omega), \Psi^-(\zeta)\}, \\ \Psi^+(\omega^\zeta \mid \omega^\zeta) \geq \min\{\Psi^+(\omega), \Psi^+(\zeta)\}, \end{pmatrix}. \tag{3.7}$$

Theorem 3.12. *Let $\mathcal{L}$ be a SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \quad (((\omega^\zeta \mid \omega^\zeta) \mid \omega^\upsilon) \mid ((\omega^\zeta \mid \omega^\zeta) \mid \omega^\upsilon)) \mid \upsilon^\zeta = 0 \mid 0. \tag{3.8}$$

Then $\mathcal{L}$ is implicative if and only if every bipolar fuzzy closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal of $\mathcal{L}$.

Proof. Let $\mathcal{L}$ be a SBG-algebra satisfying (3.8). Assume that $\mathcal{L}$ is implicative and let $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ be a bipolar fuzzy closed SBG-ideal of $\mathcal{L}$. Then Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Thus, for all $\omega \in \mathcal{L}$, we have

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

Also, for all $\omega, \zeta, \upsilon \in \mathcal{L}$, we have

$$\begin{aligned}\Psi^-(\omega) &\leq \max\left\{\Psi^-(\upsilon), \Psi^-(\omega^\upsilon \mid \omega^\upsilon)\right\} \\ &= \max\left\{\Psi^-(\upsilon), \Psi^-((((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon) \mid (((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon)\right) \\ &= \max\left\{\Psi^-(\upsilon), \Psi^-(((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon \mid ((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^x))^\upsilon)\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Psi^+(\omega) &\geq \min\left\{\Psi^+(\upsilon), \Psi^+(\omega^\upsilon \mid \omega^\upsilon)\right\} \\ &= \min\left\{\Psi^+(\upsilon), \Psi^+((((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon) \mid (((\omega \mid \omega)^\omega \mid \omega^\zeta)^\upsilon \mid \upsilon)) \mid\right. \\ &= \min\left\{\Psi^+(\upsilon), \Psi^+(((x \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon \mid ((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon)\right\}.\end{aligned}$$

This implies that Ψ is a BFISBG-ideal of $\mathcal{L}$.

Conversely, suppose that every bipolar fuzzy closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal. Then, from equation (3.8), we have

$$\upsilon^\zeta = \omega^\upsilon \mid (\omega^\zeta \mid \omega^\zeta).$$

Using identities (S1) and (S3), we also have

$$\upsilon^\zeta = ((\omega \mid \omega^\upsilon) \mid (\omega \mid \omega^\upsilon))^\zeta.$$

Then, applying (S2) and Lemma 2.1(3), it follows that

$$\upsilon = (\omega \mid \omega^\upsilon) \mid (\omega \mid \omega^\upsilon).$$

Now, we derive

$$\begin{aligned}(\omega \mid \omega^\zeta) &= (((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)))^y \\ &= ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \mid (y \mid (((\zeta \mid \zeta) \mid \zeta^\omega) \mid ((\zeta \mid \zeta) \mid \zeta^\omega))) \\ &= ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \mid y^y \\ &= ((\zeta^\zeta \mid (\zeta \mid \zeta)^\zeta) \mid (\zeta^\zeta \mid (\zeta \mid \zeta)^\zeta)) \mid \zeta^\omega \\ &= \zeta \mid \zeta^\omega,\end{aligned}$$

for all $\omega, \zeta \in \mathcal{L}$, which means that $\mathcal{L}$ is implicative. □

Proposition 3.4. *Let A be an implicative SBG-algebra satisfying equation (3.8). Then every bipolar fuzzy completely closed SBG-ideal of A is also a BFISBG-ideal.*

Proof. Let Ψ be a bipolar fuzzy completely closed SBG-ideal of an implicative SBG-algebra $\mathcal{L}$. By definition, Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Note that

$$\Psi^-(0^\zeta \mid 0^\zeta) \leq \max\{\Psi^-(0), \Psi^-(\zeta)\} = \Psi^-(\zeta)$$

and

$$\Psi^+(0^\zeta \mid 0^\zeta) \geq \min\{\Psi^+(0), \Psi^+(\zeta)\} = \Psi^+(\zeta).$$

These inequalities imply that Ψ is a bipolar fuzzy closed SBG-ideal of $\mathcal{L}$. Hence, Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Corollary 3.2. *Let $\mathcal{L}$ be a medial SBG-algebra satisfying equation (3.8). Then every bipolar fuzzy completely closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal.*

Definition 3.5. *A bipolar fuzzy set Ψ on an SBG-algebra $\mathcal{L}$ is said to be a bipolar fuzzy p-ideal of $\mathcal{L}$ if it satisfies the condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^-(0) \leq \Psi^-(\omega) \\ \Psi^+(0) \geq \Psi^+(\omega) \\ \Psi^-(\omega) \leq \max\{\Psi^-(((\omega^\upsilon \mid \omega^\upsilon) \mid \zeta^\upsilon) \mid ((\omega^\upsilon \mid \omega^\upsilon) \mid \zeta^\upsilon)), \Psi^-(\zeta)\} \\ \Psi^+(\omega) \geq \min\{\Psi^+(((\omega^\upsilon \mid \omega^\upsilon) \mid \zeta^\upsilon) \mid ((\omega^\upsilon \mid \omega^\upsilon) \mid |\zeta^\upsilon)), \Psi^+(\zeta)\} \end{pmatrix}. \quad (3.9)$$

Definition 3.6. *Let $\mathcal{L}$ be an SBG-algebra. The subset*

$$\mathcal{A}^+ = \{x \in A : 0^\omega \mid 0^\omega = 0\}$$

is called the BCA-part *of $\mathcal{L}$.*

Theorem 3.13. *Let $\mathcal{A} = \mathcal{A}^+$ be an SBG-algebra. Then every bipolar fuzzy p-ideal of $\mathcal{L}$ is a BFISBG-ideal of $\mathcal{L}$.*

Proof. Let Ψ be a bipolar fuzzy p-ideal of $\mathcal{L}$. Then

$$\begin{array}{rcl}\Psi^-(\omega) & \leq & \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega)) \mid ((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega))), \Psi^-(0)\} \\ & = & \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0), \Psi^-(0)\} \\ & = & \max\{\Psi^-(\omega_\zeta \mid \omega_\zeta), \Psi^-(0)\} \\ & = & \Psi^-(\omega_\zeta \mid \omega_\zeta),\end{array}$$

and

$$\begin{array}{rcl}\Psi^+(\omega) & \geq & \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega)) \mid ((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega))), \Psi^+(0)\} \\ & = & \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0), \Psi^+(0)\} \\ & = & \min\{\Psi^+(\omega_\zeta \mid \omega_\zeta), \Psi^+(0)\} \\ \omega & = & \Psi^+(\omega_\zeta \mid \omega_\zeta).\end{array}$$

Hence Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Theorem 3.14. *Let $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ be a BFISBG-ideal of $\mathcal{A}$. Then, for every $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$, the bipolar fuzzy (θ, ϑ)-translation*

$$\Psi^{T_2}_{(\theta,\vartheta)} = (A, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$$

is also a BFISBG-ideal of $\mathcal{A}$.

Proof. Assume that $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$. Then for all $\omega \in \mathcal{A}$ and $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$, we have

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

It follows that

$$\Psi^-_{(\theta,T_2)}(0) = \Psi^-(0) - \theta \leq \Psi^-(\omega) - \theta = \Psi^-_{(\theta,T_2)}(\omega),$$
$$\Psi^+_{(\vartheta,T_2)}(0) = \Psi^+(0) - \vartheta \geq \Psi^+(\omega) - \vartheta = \Psi^+_{(\vartheta,T_2)}(\omega).$$

Now let $\omega, \zeta, \upsilon \in \mathcal{A}$. Since Ψ is a BFISBG-ideal, we have

$$\Psi^-(\omega) \leq \max\left\{\Psi^-(\mathfrak{q}), \Psi^-(\upsilon)\right\}, \quad \Psi^+(\omega) \geq \min\left\{\Psi^+(\mathfrak{q}), \Psi^+(\upsilon)\right\},$$

where

$$\mathfrak{q} = (\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon.$$

Subtracting θ and ϑ respectively, we obtain

$$\Psi^-_{(\theta,T_2)}(\omega) = \Psi^-(\omega) - \theta \leq \max\left\{\Psi^-(\mathfrak{q}) - \theta, \Psi^-(\upsilon) - \theta\right\} = \max\left\{\Psi^-_{(\theta,T_2)}(\mathfrak{q}), \Psi^-_{(\theta,T_2)}(\upsilon)\right\},$$

$$\Psi^+_{(\vartheta,T_2)}(\omega) = \Psi^+(\omega) - \vartheta \geq \min\left\{\Psi^+(\mathfrak{q}) - \vartheta, \Psi^+(\upsilon) - \vartheta\right\} = \min\left\{\Psi^+_{(\vartheta,T_2)}(\mathfrak{q}), \Psi^+_{(\vartheta,T_2)}(\upsilon)\right\}.$$

Therefore, the (θ, ϑ)-translation $f^{T_2}_{(\theta,\vartheta)} = (\mathcal{A}, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$ satisfies the conditions of a BFISBG-ideal of $\mathcal{A}$. □

Theorem 3.15. *Let $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ be a bipolar fuzzy set on an SBG-algebra $\mathcal{A}$. If there exists a pair $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$ such that the (θ, ϑ)-translation*

$$\Psi^{T_2}_{(\theta,\vartheta)} = (\mathcal{A}, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$$

is a BFISBG-ideal of $\mathcal{A}$, then Ψ itself is a BFISBG-ideal of $\mathcal{A}$.

Proof. Suppose there exist $(\theta,\vartheta)\in[\pm,0]\times[0,\mp]$ such that the (θ,ϑ)-translation

$$\Psi^{T_2}_{(\theta,\vartheta)}=(\mathcal{A},\Psi^-_{(\theta,T_2)},\Psi^+_{(\vartheta,T_2)})$$

is a BFISBG-ideal of $\mathcal{A}$. Then, for all $\omega,\zeta,\upsilon\in\mathcal{A}$, we have
$\Psi^-(0)\le\Psi^-(\omega)$ and $\Psi^+(0)\ge\Psi^+(\omega)$.

$$\Psi^-(0)-\theta=\Psi^-_{(\theta,T_2)}(0)\le\Psi^-_{(\theta,T_2)}(\omega)=\Psi^-(\omega)-\theta,$$

$$\Psi^+(0)-\vartheta=\Psi^+_{(\vartheta,T_2)}(0)\ge\Psi^+_{(\vartheta,T_2)}(\omega)=\Psi^+(\omega)-\vartheta.$$

Now let $x,y,z\in A$. Then

$$\begin{aligned}\Psi^-(\omega)-\theta &= \Psi^-_{(\theta,T_2)}(\omega)\\ &\le \max\{\Psi^-_{(\theta,T_2)}((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^-_{(\theta,T_2)}(\zeta)\}\\ &= \max\{\Psi^-((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon)-\theta,\Psi^-(\zeta)-\theta\}\\ &= \max\{\Psi^-((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^-(\zeta)\}-\theta,\end{aligned}$$

$$\begin{aligned}\Psi^+(\omega)-\vartheta &= \Psi^+_{(\vartheta,T_2)}(\omega)\\ &\ge \min\{\Psi^+_{(\vartheta,T_2)}((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^+_{(\vartheta,T_2)}(\upsilon)\}\\ &= \min\{\Psi^+((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon)-\vartheta,\Psi^+(\upsilon)-\vartheta\}\\ &= \min\{\Psi^+((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^+(\upsilon)\}-\vartheta,\end{aligned}$$

Hence $\Psi^-(\omega)\le\max\{\Psi^-((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^-(\upsilon)\}$ and $\Psi^+(\omega)\ge\min\{\Psi^+((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon),\Psi^+(\upsilon)\}$. Hence $\Psi=(\mathcal{A},\Psi^-,\Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$. □

Theorem 3.16. *Let $\overline{\Psi}=(\mathcal{A},\overline{\Psi^-},\overline{\Psi^+})$ be a bipolar fuzzy set in $\mathcal{A}$. Then, $\Psi=(\mathcal{A},\Psi^-,\Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$ if and only if for every $(\mathfrak{q}^-,\mathfrak{q}^+)\in[-1,0]\times[0,1]$, the sets $N_U(\Psi,\mathfrak{q}^-)$ and $P_L(\Psi,\mathfrak{q}^+)$ are implicative SBG-ideals of $\mathcal{A}$, whenever these sets are nonempty.*

Proof. Assume that $\overline{\Psi}=(\mathcal{A},\overline{\Psi^-},\overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$. Let $(\mathfrak{q}^-,\mathfrak{q}^+)\in[-1,0]\times[0,1]$ be such that the level sets $N_U(\Psi,\mathfrak{q}^-)$ and $P_L(\Psi,\mathfrak{q}^+)$ are nonempty. Consider any $\omega\in\mathcal{A}$ with $\omega\in N_U(\Psi,\mathfrak{q}^-)$, which implies that $\Psi^-(\omega)\ge\mathfrak{q}^-$. Since $\Psi=(\mathcal{A},\Psi^-,\Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$, it follows that

$$\begin{aligned}\overline{\Psi^-}(0) &\le \overline{\Psi^-}(\omega)\\ 1-\Psi^+(0) &\le 1-\Psi^+(\omega)\\ \Psi^+(0) &\ge \Psi^+(\omega)\\ &\ge \mathfrak{q}^-.\end{aligned}$$

Hence, we have $0\in N_U(\Psi,\mathfrak{q}^-)$.

Next, let $\omega,\zeta,\upsilon\in\mathcal{A}$ be such that

$$\big((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon\big),\quad \upsilon\in N_U(\Psi,\mathfrak{q}^-).$$

Then

$$\Psi^-\big((\omega_\zeta\mid\omega_\zeta)^\upsilon\mid(\omega_\zeta\mid\omega_\zeta)^\upsilon\big)\ge t^-,\quad\text{and}\quad\Psi^-(\upsilon)\ge\mathfrak{q}^-.$$

Since $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$, it follows that

$$\begin{aligned}
\overline{\Psi^-}(\omega) &\leq \max\left\{\overline{\Psi^-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \overline{\Psi^-}(\upsilon)\right\},\\
1-\Psi^-(\omega) &\leq \max\left\{1-\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, 1-\Psi^-(\upsilon)\right\},\\
1-\Psi^-(\omega) &\leq 1-\min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \Psi^-(\upsilon)\right\},\\
\Psi^-(\omega) &\geq \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \Psi^-(\upsilon)\right\} \geq \mathfrak{q}^-.
\end{aligned}$$

Hence, $\omega \in N_U(\Psi, \mathfrak{q}^-)$. Therefore, $N_U(\Psi, \mathfrak{q}^-)$ is an implicative SBG-ideal of $\mathcal{A}$.

Let $\omega \in \mathcal{A}$ be such that $\omega \in P_L(\Psi, \mathfrak{q}^+)$. Then $\Psi^+(\omega) \geq \mathfrak{q}^+$. Since $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$, we have

$$\begin{aligned}
\overline{\Psi^+}(0) &\geq \overline{\Psi^+}(\omega),\\
1-\Psi^+(0) &\geq 1-\Psi^+(\omega),\\
\Psi^+(0) &\leq \Psi^+(\omega) \leq \mathfrak{q}^+.
\end{aligned}$$

Hence, $0 \in P_L(\Psi, \mathfrak{q}^+)$.

Next, let $\omega, \zeta, \upsilon \in \mathcal{A}$ be such that $\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \upsilon \in P_L(f, t^+)$. Then $\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \geq \mathfrak{q}^+$ and $\Psi^+(\upsilon) \geq \mathfrak{q}^+$. Since $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$, we have

$$\begin{aligned}
\overline{\Psi^+}(\omega) &\leq \max\{\overline{\Psi^+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^+}(\upsilon)\}\\
1-\Psi^+(\omega) &\leq \max\{1-\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), 1-\Psi^+(\upsilon)\}\\
1-\Psi^+(\omega) &\leq 1-\min\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\
\Psi^+(\omega) &\geq \min\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\
&\geq \mathfrak{q}^+.
\end{aligned}$$

Hence, $\omega \in P_L(\Psi, \mathfrak{q}^+)$. Therefore, $P_L(\Psi, \mathfrak{q}^+)$ is an implicative SBG-ideal of A.

Conversely, suppose that for all $(\mathfrak{q}^-, \mathfrak{q}^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(\Psi, \mathfrak{q}^-)$ and $P_L(\Psi, \mathfrak{q}^+)$ are implicative SBG-ideals of $\mathcal{A}$, provided they are nonempty.

Let $\omega \in \mathcal{A}$. Since $\Psi^-(\omega) \in [-1, 0]$, choose $\mathfrak{q}^- = \Psi^-(\omega)$. Then

$$\Psi^-(0) \geq \Psi^-(\omega) = \mathfrak{q}^-,$$

which implies $0 \in N_U(\Psi, \mathfrak{q}^-) \neq \emptyset$. By assumption, $N_U(\Psi, \mathfrak{q}^-)$ is an implicative SBG-ideal of $\mathcal{A}$, and therefore $0 \in N_U(\Psi, \mathfrak{q}^-)$ holds. Consequently,

$$\Psi^-(0) \geq \mathfrak{q}^- = \Psi^-(\omega),$$

and so

$$\overline{\Psi^-}(0) = -1 - \Psi^-(0) \leq -1 - \Psi^-(\omega) = \overline{\Psi^-}(\omega).$$

Now let $x, y, z \in A$. Note that

$$\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \Psi^-(\upsilon) \in [-1, 0].$$

Choose

$$\mathfrak{q}^- = \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^-(\upsilon)\right\}.$$

Then

$$\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \geq t^-,$$

and

$$\Psi^{-}(\upsilon) \geq \mathfrak{q}^{-},$$

so

$$\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \upsilon \in N_U(\Psi, \mathfrak{q}^{-}) \neq \emptyset.$$

By assumption, $N_U(\Psi, \mathfrak{q}^{-})$ is an implicative SBG-ideal of $\mathcal{A}$, and hence $\omega \in N_U(\Psi, \mathfrak{q}^{-})$, which implies

$$\Psi^{-}(\omega) \geq \mathfrak{q}^{-} = \min\left\{\Psi^{-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^{-}(\upsilon)\right\}.$$

By Lemma 2.1 (1), we have

$$\begin{aligned}
\overline{\Psi^{-}}(\omega) &= -1 - \Psi^{-}(\omega) \\
&\leq -1 - \min\{\Psi^{-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^{-}(\upsilon)\} \\
&= \max\{-1 - \Psi^{-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), -1 - \Psi^{-}(\upsilon)\} \\
&= \max\{\overline{\Psi^{-}}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^{-}}(\upsilon)\}.
\end{aligned}$$

Let $\omega, \zeta \in \mathcal{A}$. Since $\Psi^{+}(\omega) \in [0,1]$, choose $\mathfrak{q}^{+} = \Psi^{+}(\omega)$. Then, as

$$\Psi^{+}(0) \leq \Psi^{+}(\omega) = \mathfrak{q}^{+},$$

it follows that $\omega \in P_L(\Psi, \mathfrak{q}^{+}) \neq \emptyset$. By assumption, $P_L(f, t^{+})$ is an implicative SBG-ideal of $\mathcal{A}$, hence $0 \in P_L(\Psi, \mathfrak{q}^{+})$.

Therefore,

$$\Psi^{+}(0) \geq \mathfrak{q}^{+} = \Psi^{+}(\omega),$$

and thus

$$\overline{\Psi^{+}}(0) = 1 - \Psi^{+}(0) \geq 1 - \Psi^{+}(\omega) = \overline{\Psi^{+}}(\omega).$$

Let $\omega, \zeta, \upsilon \in \mathcal{A}$. Note that

$$\Psi^{+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \Psi^{+}(\upsilon) \in [0,1].$$

Choose

$$t^{+} = \max\left\{\Psi^{+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^{+}(\upsilon)\right\}.$$

Then,

$$\Psi^{+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \leq \mathfrak{q}^{+}$$

and

$$\Psi^{+}(\upsilon) \leq t^{+},$$

which implies

$$\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \upsilon \in P_L(\Psi, \mathfrak{q}^{+}) \neq \emptyset.$$

By assumption, $P_L(\Psi, \mathfrak{q}^{+})$ is an implicative SBG-ideal of $\mathcal{A}$, and therefore $x \in P_L(\Psi, \mathfrak{q}^{+})$, which yields

$$\Psi^{+}(\omega) \geq \mathfrak{q}^{+} = \max\left\{\Psi^{+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^{+}(\upsilon)\right\}.$$

By Lemma 2.1 (1), we have

$$\begin{aligned}\overline{\Psi^+}(\omega) &= 1-\Psi^+(\omega)\\ &\leq 1-\max\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\ &= \min\{1-\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), 1-\Psi^+(\upsilon)\}\\ &= \min\{\overline{\Psi^+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^+}(\upsilon)\}.\end{aligned}$$

Hence $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$. □

4. Conclusion

In this paper, we have systematically developed the theory of bipolar fuzzy Sheffer stroke BG-algebras, focusing on the interplay between bipolar fuzzy sets and algebraic structures defined by the Sheffer stroke operation. By introducing and analyzing the notions of bipolar fuzzy subalgebras and various classes of SBG-ideals, we have established clear criteria under which the level sets of bipolar fuzzy subsets correspond to conventional algebraic substructures and ideals. This approach provides a rigorous algebraic foundation for representing and manipulating dual (positive and negative) information within SBG-algebras.

Our results demonstrate that the framework of bipolar fuzzy sets not only extends the expressive power of fuzzy algebraic systems, but also enables a more nuanced treatment of uncertainty and contradiction in logical and computational contexts. The equivalence conditions and closure properties we have proven for bipolar fuzzy (implicative, sub-implicative, completely closed, and p-) SBG-ideals contribute to a deeper understanding of the algebraic behavior of these fuzzy structures.

The integration of bipolar fuzzy logic with Sheffer stroke BG-algebras, as presented here, paves the way for further research into the algebraic modeling of complex systems characterized by ambiguity, duality, or inconsistent information. Future investigations may include the study of categorical aspects, the extension to other algebraic systems, or the application of these results to areas such as soft computing, knowledge representation, and decision support systems.

In summary, this work highlights the foundational role of level set techniques in relating bipolar fuzzy membership functions to algebraic properties, thereby enriching both the theoretical landscape and the potential applications of fuzzy algebraic systems within mathematics and computer science.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

Conflict of interest. The authors declare no potential conflict of interests.

References

[1] Chajda, I. (2005). Sheffer operation in ortholattices. *Acta Universitatis Palackianae Olomucensis Facultas Rerum Naturalium Mathematica*, *44*(1), 19–23.

[2] Kim, C. B., & Kim, H. S. (2008). On BG-algebras. *Demonstratio Mathematica*, *41*(3), 497–505. https://doi.org/10.1515/dema-2013-0098

[3] McCune, W., Veroff, R., Fitelson, B., Harris, K., Feist, A., & Wos, L. (2002). Short single axioms for Boolean algebra. *Journal of Automated Reasoning*, *29*(1), 1–16.

[4] Neggers, J., & Kim, H. S. (2002). On B-algebras. *Matematički vesnik*, *54*(1), 21–29.

[5] Oner, T., Kalkan, T., & Kircali Gursoy, N. (2021). Sheffer stroke BG-algebras. *International Journal of Maps in Mathematics*, *4*(1), 27–30.

[6] Oner, T., & Senturk, I. (2017). The Sheffer stroke operation reducts of basic algebra. *Open Mathematics*, *15*, 926–935.

[7] Senturk, I. (2021). A new on state operators in Sheffer stroke basic algebras. *Soft Computing*, *25*, 11471–11484.

[8] Senturk, I. (2022). Riecan and Bosbach state operators on Sheffer stroke MTL-algebras. *Bulletin of the International Mathematical Virtual Institute*, *12*(1), 181–193.

[9] Senturk, I., & Oner, T. (2021). A construction of very true operator on Sheffer stroke MLT-algebras. *International Journal of Maps in Mathematics*, *4*, 93–106.

[10] Sheffer, H. M. (1913). A set of five independent postulates for Boolean algebras, with application to logical constants. *Transactions of the American Mathematical Society*, *14*(4), 481–488.

[11] Oner, T., Kalkan, T., Katican, T., & Rezaei, A. (2021). Fuzzy implicative ideals of Sheffer stroke BG-algebras. *Facta Universitatis (Niš), Series: Mathematics and Informatics*, *36*(4), 913–926.

[12] Udten, N., Songseang, N., & Iampan, A. (2019). Translation and density of a bipolar-valued bipolar fuzzy set in UP-algebra. *Italian Journal of Pure and Applied Mathematics*, *41*, 469–496.

[13] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.

[14] Zhang, W. R. (1998). Bipolar bipolar fuzzy sets. In *Proceedings of FUZZ-IEEE* (pp. 835–840).

(N. Rajesh) Department of Mathematics, Rajah Serfoji Government College, Thanjavur, 613005, Tamilnadu, India

(T. Oner) Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey

(I. Senturk) Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey

(K. Geetha) Department of Mathematics, Rajah Serfoji Government College, Thanjavur, Tamilnadu, India