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CHARACTERIZATION OF m-QUASI-EINSTEIN STRUCTURES IN LP-KENMOTSU MANIFOLDS

PUTTASIDDAPPA SOMASHEKHARA, ARASAIAH, BASVARAJU PHALAKSHA MURTHY, KODIDODDI CHIKKALINGAIAH AJEYAKASHI, AND KEMPAIAH ANANDA *

Abstract. In this paper, we investigate m-quasi Einstein metrics on LP-Kenmotsu manifolds, a recently introduced class of Lorentzian paracontact metric manifolds. We derive the curvature identity associated with a closed m-quasi Einstein structure and classify LP-Kenmotsu manifolds admitting such metrics. It is shown that if the potential vector field is conformal, collinear with the unit timelike vector field, or a strict infinitesimal contact transformation, then the manifold is either Einstein or η-Einstein under suitable conditions. Furthermore, we prove that the scalar curvature of such manifolds is necessarily constant.

Keywords: LP-Kenmotsu manifolds, m-Quasi-Einstein structures, Einstein manifolds, Conformal vector field.

MSC (2020): 53C18; 53C21; 53C25; 53C50.

1. Introduction

The study of Einstein manifolds has long been central to both mathematics and physics, especially in Riemannian geometry and the theory of general relativity. These are Riemannian manifolds (M, g) whose Ricci tensor satisfies $\mathrm{Ric} = \lambda g$ for some scalar $\lambda \in \mathbb{R}$. To explore geometric phenomena beyond this classical setting, various generalizations have been proposed among them, *Ricci solitons* and *m-quasi Einstein metrics* have gained significant attention due to their connections with the Ricci flow, warped product spaces, and weighted volume measures. For further information on Ricci solitons and their generalizations, the reader may refer to ([1, 5, 7, 12, 14, 15, 20, 26], [28]-[35]) and the sources cited within those works.

An *m-quasi Einstein manifold* (M, g, β_V, λ) is a Riemannian manifold equipped with a vector field β_V (called the potential vector field) that satisfies the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^{\sharp} \otimes \beta_V^{\sharp} = \lambda g, \tag{1.1}$$

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* Corresponding author

Puttasiddappa Somashekhara (somumathrishi@gmail.com) ⋄ ORCID:0009-0002-4177-8217

Arasaiah (ars.gnr94@gmail.com) ⋄ ORCID:0000-0002-1755-9789

Basvaraju Phalaksha Murthy (pmurthymath@gmail.com)⋄ ORCID:0000-0001-7125-6749

Kodidoddi Chikkalingaiah Ajeyakashi (ajaykashikc@gmail.com) ⋄ ORCID:0000-0001-5098-0965

Kempaiah Ananda (anandaknhce@gmail.com) ⋄ ORCID:0000-0003-3848-0997.

where $\mathcal{L}_{\beta_V} g$ denotes the Lie derivative of the metric g along β_V, $\beta_V^\sharp$ is the 1-form metrically dual to β_V, and $m > 0$ (possibly $m = \infty$) (For further details, see [17, 29]).

The expression

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{m}\, df \otimes df = \mathrm{Ric}_f^m$$

is known as the *m-Bakry-Émery Ricci tensor*. It arises naturally in the study of *warped product manifolds* $(M \times N, \bar{g})$, where (M^n, g) and (N^m, h) are Riemannian manifolds, and the warped metric is given by $\bar{g} = g + e^{-\frac{2f}{m}} h$.

Furthermore, when $m = \infty$, equation (1.1) reduces to the classical Ricci soliton equation, and when $\beta_V = 0$, it coincides with the Einstein condition. Thus, the m-quasi Einstein framework unifies several important geometric structures, including Einstein manifolds, both gradient and non-gradient Ricci solitons, and certain warped product spaces governed by Bakry-Émery Ricci tensors.

The theory of m-quasi Einstein metrics has been extended to various geometric settings. Case et al. [9, 10] studied such metrics in the context of warped product manifolds, while Barros and Ribeiro Jr. [3, 4] investigated conditions for their existence on compact and homogeneous spaces. Limoncu [22] introduced a formulation involving vector fields, generalizing the m-Bakry-Émery Ricci tensor. In contact geometry, Ghosh [16] explored m-quasi Einstein structures on contact and K-contact manifolds, showing that collinearity between the potential vector field β_V and the Reeb vector field ξ leads to η-Einstein or Sasakian manifolds. Venkatesha et al. [36] extended the study to almost Kenmotsu and Kenmotsu $(\kappa, \mu)'$-manifolds. Recently, De et al. [13] investigated m-quasi Einstein metrics in the framework of paracontact geometry, focusing on K-paracontact and (k, μ)-paracontact metric manifolds. They showed that a K-paracontact metric manifold admitting an m-quasi Einstein metric must have constant scalar curvature, and provided a classification and explicit examples of such structures, highlighting the compatibility of paracontact geometry with generalized Einstein-type conditions. Parallel to these developments, another significant branch of geometric study has emerged in *Lorentzian almost paracontact geometry* a natural generalization of classical contact geometry to the pseudo-Riemannian setting. Among these, *LP-Kenmotsu manifolds* (Lorentzian para-Kenmotsu manifolds) form a distinct and rich subclass. Originally, the concept of *LP-Sasakian manifolds* was introduced by Matsumoto [23] in 1989, and later independently studied by Mihai and Rosca [25], who obtained foundational results concerning their curvature and structure. As a continuation of this line of inquiry, Haseeb and Prasad [18] introduced *LP-Kenmotsu manifolds* in 2021, which generalize Kenmotsu manifolds to the Lorentzian setting. These manifolds are endowed with a Lorentzian paracontact structure (ϕ, ξ, η, g), satisfying the conditions:

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \quad \nabla_{\beta_X}\xi = \beta_X - \eta(\beta_X)\xi,$$

$$d\eta = 0, \quad d\Phi = 2\eta \wedge \Phi, \quad \text{where } \Phi(\beta_X, \beta_Y) = g(\beta_X, \phi\beta_Y).$$

These conditions mimic those of Kenmotsu geometry but are framed in the *Lorentzian* context. The geometric behavior of LP-Kenmotsu manifolds shares several structural features with both Kenmotsu and para-Sasakian manifolds, while also exhibiting distinct curvature properties due to the indefinite metric. The interplay of curvature, contact-type structure, and Lorentzian signature makes LP-Kenmotsu manifolds promising candidates for exploring generalized Einstein-type structures.

Despite the geometric richness and recent introduction of LP-Kenmotsu manifolds, the study of m-quasi Einstein metrics within this framework remains largely unexplored. This research gap forms the central motivation for the present work. Inspired by previous developments in contact and Kenmotsu geometry, this paper aims to investigate the existence, structural properties, and geometric implications of m-quasi Einstein structures on LP-Kenmotsu manifolds.

The paper is organized as follows: Section 2 presents a detailed overview of the fundamental definitions and properties of LP-Kenmotsu manifolds. In Section 3, we construct explicit examples of LP-Kenmotsu manifolds admitting m-quasi Einstein structures. Section 4 is devoted to the development and analysis of the theory of m-quasi Einstein metrics in the context of LP-Kenmotsu geometry.

2. Lorentzian Para-Kenmotsu Manifolds

Let M be a smooth manifold of dimension n. This manifold is referred to as a *Lorentzian almost paracontact metric manifold* if it is equipped with a tensor field ϕ of type $(1,1)$, a vector field ξ of type $(1,0)$, a 1-form η, and a Lorentzian metric g, which together satisfy the following conditions [23]:

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi\beta_X, \phi\beta_Y) = g(\beta_X, \beta_Y) + \eta(\beta_X)\eta(\beta_Y), \quad g(\beta_X, \xi) = \eta(\beta_X), \tag{2.3}$$

for all vector fields β_X, β_Y on M, where I is the identity transformation on the tangent bundle of M. The structure (ϕ, ξ, η, g) is known as a *Lorentzian almost paracontact structure*.

If the manifold additionally satisfies

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \tag{2.4}$$

for any vector fields β_X, β_Y, where ∇ is the Levi-Civita connection associated with g, then M is called a *Lorentzian para-Kenmotsu manifold* (briefly, LP-Kenmotsu manifold).

From Equation (2.4), the following relations are obtained:

$$\nabla_{\beta_X}\xi = -\beta_X - \eta(\beta_X)\xi, \tag{2.5}$$

$$(\nabla_{\beta_X}\eta)(\beta_Y) = -g(\beta_X, \beta_Y) - \eta(\beta_X)\eta(\beta_Y). \tag{2.6}$$

In addition, the LP-Kenmotsu manifold satisfies the following curvature properties [18]:

$$\eta(R(\beta_X, \beta_Y)\beta_Z) = g(\beta_Y, \beta_Z)\eta(\beta_X) - g(\beta_X, \beta_Z)\eta(\beta_Y), \tag{2.7}$$

$$R(\beta_X, \beta_Y)\xi = \eta(\beta_Y)\beta_X - \eta(\beta_X)\beta_Y, \tag{2.8}$$

$$R(\xi, \beta_X)\beta_Y = g(\beta_X, \beta_Y)\xi - \eta(\beta_Y)\beta_X, \tag{2.9}$$

$$R(\xi, \beta_X)\xi = \beta_X + \eta(\beta_X)\xi, \tag{2.10}$$

$$\mathrm{Ric}(\beta_X, \xi) = (n-1)\eta(\beta_X), \tag{2.11}$$

$$Q\xi = (n-1)\xi, \tag{2.12}$$

for all $\beta_X, \beta_Y, \beta_Z \in \chi(M)$, where R, Ric, and Q denote the Riemannian curvature tensor, the Ricci tensor, and the Ricci operator, respectively.

A Lorentzian para-Kenmotsu manifold M is said to be *Einstein* if the Ricci tensor Ric is proportional to the metric tensor g, that is, $Ric = fg$ for some smooth function f. According to Lemma 1 in [21], the Ricci operator Q in such a manifold satisfies:

$$(\nabla_{\beta_X} Q)\xi = QX - 2n\beta_X, \tag{2.13}$$

$$(\nabla_\xi Q)\beta_X = 2QX - 4n\beta_X. \tag{2.14}$$

3. Examples

Example 3.1. *Let $M = \mathbb{R}^3$ be a 3-dimensional manifold with global coordinates (x, y, z), and consider the Lorentzian metric*

$$g = dx^2 + e^{2x}dy^2 - dz^2.$$

Define a vector field $\xi = \frac{\partial}{\partial z}$, a 1-form $\eta = dz$, and a (1,1)-tensor field ϕ by:

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

Then (ϕ, ξ, η, g) defines an LP-Kenmotsu structure on M, satisfying

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \quad \nabla_{\beta_X}\xi = \beta_X - \eta(\beta_X)\xi.$$

Define the potential vector field $\beta_V = f\xi$, where $f(x, y, z) = z$, and take $m = 2$. Then the manifold M satisfies the m-quasi Einstein equation

$$\text{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g,$$

for a suitable scalar function λ. Thus, M admits a non-trivial m-quasi Einstein metric in the LP-Kenmotsu setting.

Example 3.2. *Let $M = \mathbb{R}^5$ be a 5-dimensional manifold with coordinates (x, y, z, u, v), and define a Lorentzian metric*

$$g = dx^2 + dy^2 + e^{2x}(dz^2 + du^2) - dv^2.$$

Let $\xi = \frac{\partial}{\partial v}$, $\eta = dv$, and define the (1,1)-tensor field ϕ by:

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x},$$
$$\phi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial u}, \quad \phi\left(\frac{\partial}{\partial u}\right) = -\frac{\partial}{\partial z}, \quad \phi(\xi) = 0.$$

Then the structure (ϕ, ξ, η, g) defines an LP-Kenmotsu manifold. Let the potential vector field be $\beta_V = a\xi$ for some constant $a \neq 0$, and choose $m = 3$. It can be verified that the manifold M satisfies the m-quasi Einstein equation

$$\text{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g,$$

for a suitable constant λ. Therefore, M admits a constant-potential m-quasi Einstein metric within the LP-Kenmotsu framework.

4. Main Results

In this section, we consider the LP-Kenmotsu metric satisfying the m-quasi-Einstein metric. Before establishing our main results, we prove the following:

Lemma 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an LP-Kenmotsu manifold admitting a closed m-quasi Einstein structure with potential vector field β_V. Then, the curvature tensor satisfies the following identity:*

$$R(\beta_X, \beta_Y)\beta_V = (\nabla_{\beta_Y} Q)\beta_X - (\nabla_{\beta_X} Q)\beta_Y + \frac{1}{m}\left\{\beta_V^\sharp(\beta_X)\, QY - \beta_V^\sharp(\beta_Y)\, QX\right\} + \frac{\lambda}{m}\left\{\beta_V^\sharp(\beta_Y)\, \beta_X - \beta_V^\sharp(\beta_X)\, \beta_Y\right\}, \tag{4.15}$$

for all vector fields $\beta_X, \beta_Y \in \Gamma(TM)$, where Q is the Ricci operator corresponding to the Ricci tensor, and $\beta_V^\sharp$ denotes the 1-form metrically dual to β_V.

Proof. Since the manifold admits a closed m-quasi Einstein structure, the associated metric satisfies

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g.$$

The closedness of $\beta_V^\sharp$ implies the local existence of a smooth function f such that $\beta_V = \nabla f$. Consequently, the Lie derivative term simplifies to $\mathcal{L}_{\beta_V} g = 2\,\mathrm{Hess}_f$, though we proceed considering β_V as a general vector field with closed dual 1-form.

Rewriting the m-quasi Einstein equation gives:

$$\nabla_{\beta_X}\beta_V + QX = \lambda\beta_X + \frac{1}{m}\beta_V^\sharp(\beta_X)\,\beta_V. \tag{4.16}$$

We now apply this identity to compute the curvature operator using:

$$R(\beta_X, \beta_Y)\beta_V = \nabla_{\beta_X}\nabla_{\beta_Y}\beta_V - \nabla_{\beta_Y}\nabla_{\beta_X}\beta_V - \nabla_{[\beta_X, \beta_Y]}\beta_V.$$

Substituting the expression for $\nabla\beta_V$, and simplifying using the symmetry of Q and properties of the Levi-Civita connection, we obtain:

$$R(\beta_X, \beta_Y)\beta_V = (\nabla_{\beta_Y} Q)\beta_X - (\nabla_{\beta_X} Q)\beta_Y + \frac{1}{m}\left\{\beta_V^\sharp(\beta_X)\, QY - \beta_V^\sharp(\beta_Y)\, QX\right\} + \frac{\lambda}{m}\left\{\beta_V^\sharp(\beta_Y)\, \beta_X - \beta_V^\sharp(\beta_X)\, \beta_Y\right\}.$$

Hence, the required identity holds. □

Definition 4.1. *Let M be an almost contact metric manifold. A vector field β_V on M is said to be an* infinitesimal contact transformation *if the Lie derivative of the contact 1-form η along β_V satisfies*

$$\mathcal{L}_{\beta_V}\eta = \alpha\eta,$$

for some smooth function α on M. In particular, if $\alpha = 0$, i.e., $\mathcal{L}_{\beta_V}\eta = 0$, then β_V is called a strict infinitesimal contact transformation.

Theorem 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an LP-Kenmotsu manifold admitting a closed m-quasi Einstein structure with $m \neq 1$. Then, one of the following holds:*

(1) *The potential vector field β_V is pointwise collinear with the unit timelike vector field ξ, and M is η-Einstein;*

(2) β_V *is a strict infinitesimal contact transformation, i.e.,* $\mathcal{L}_{\beta_V}\eta = 0$;
(3) M *is Einstein.*

Proof. First, by replacing β_Y with ξ in equation (4.15) and using equations (2.13) and (2.14), we obtain

$$R(\beta_X,\xi)\beta_V = QX - 2n\beta_X + \frac{2n-\lambda}{m}\beta_V^{\sharp}(\beta_X)\,\xi + \frac{\eta(\beta_V)}{m}(\lambda\beta_X - QX). \tag{4.17}$$

Now, taking the inner product of both sides of the curvature identity (2.8) yields

$$g(R(\beta_X,\xi)\xi,\beta_V) = -g(\beta_X,\beta_V) - \eta(\beta_X)\eta(\beta_V).$$

Together with equation (4.17), the preceding result leads to

$$QX - 2n\beta_X + \frac{2n+m-\lambda}{m}\beta_V^{\sharp}(\beta_X)\xi + \frac{\eta(\beta_V)}{m}(\lambda\beta_X - m\beta_X - Q\beta_X) = 0. \tag{4.18}$$

Next, by taking the inner product with ξ and using equation (2.12), we get

$$\left(\frac{2n+m-\lambda}{m}\right)(g(\beta_X,\beta_V) - \eta(\beta_X)\eta(\beta_V)) = 0. \tag{4.19}$$

As (4.19) holds for all β_X, we are led to two possibilities:

(i) $\beta_V = \eta(\beta_V)\xi$, i.e., β_V is collinear with ξ;
(ii) $\lambda = (2n+m)$.

Case (i): If $\beta_V = \eta(\beta_V)\xi$, then differentiating this along any vector field β_X and applying equation (2.5), we obtain

$$\nabla_{\beta_X}\beta_V = \eta(\nabla_{\beta_X}\beta_V)\xi - \eta(\beta_V)(\beta_X + \eta(\beta_X)\xi).$$

Substitute into (4.16), we get:

$$QX - \lambda\beta_X - \frac{1}{m}\beta_V^{\sharp}(\beta_X)\beta_V = 2n\,\eta(\beta_X)\,\xi - \lambda\eta(\beta_X)\xi - \frac{1}{m}\beta_V^{\sharp}(\beta_X)\,\eta(\beta_V)\,\xi + \eta(\beta_V)(\beta_X + \eta(\beta_X)\xi).$$

It follows from aforesaid equation, using the condition $\phi\beta_V = 0$, that

$$\phi QX = [\lambda + \eta(\beta_V)]\phi\beta_X. \tag{4.20}$$

Substituting $\beta_X = \phi\beta_X$ into equation (4.20), and recalling that the Ricci operator Q and ϕ commute on M, we obtain

$$QX = (\lambda + \eta(\beta_V))\beta_X + (\lambda + \eta(\beta_V) - 2n)\,\eta(\beta_X)\xi. \tag{4.21}$$

Taking the trace of equation (4.21) over β_X, we immediately obtain $\lambda + \eta(\beta_V) = \frac{r}{2n} - 1$. Substituting this result back into equation (4.21), we arrive at

$$QX = \left(\frac{r}{2n} - 1\right)\beta_X - \left(\frac{r}{2n} - (2n+1)\right)\eta(\beta_X)\xi,$$

which shows that M is an η-Einstein manifold. This completes the proof of part (1).

Case (ii): If $\lambda = (2n+m)$, substitute this into (4.18) and simplify:

$$\left(1 - \frac{\eta(\beta_V)}{m}\right)[Q\beta_X - 2n\beta_X] = 0. \tag{4.22}$$

Subcase 1: If $\eta(\beta_V) = m$, compute the Lie derivative:

$$\mathcal{L}_{\beta_V}\eta(\beta_X) = \mathcal{L}_{\beta_V}g(\beta_X,\xi) - g(\beta_X,\mathcal{L}_{\beta_V}\xi).$$

Since $\mathcal{L}_{\beta_V} g(\beta_X, \xi) = 2(\eta(\beta_X)\eta(\beta_V) - g(\beta_X, \beta_V)) = 0$, and $\mathcal{L}_{\beta_V}\xi = 0$, it follows that:

$$\mathcal{L}_{\beta_V}\eta = 0,$$

so β_V is a strict infinitesimal contact transformation.

Subcase 2: If $\eta(\beta_V) \neq m$, then from (4.22):

$$QX = 2n\beta_X,$$

which implies that M is Einstein. This completes the proof. □

Ghosh [16] demonstrated that a K-contact manifold admitting a conformal m-quasi Einstein metric is necessarily a non-trivial η-Einstein manifold, and the potential vector field β_V in this case is Killing. In a related study, Venkatesha et al. [36] showed that a Kenmotsu manifold equipped with a conformal m-quasi Einstein metric is Einstein with constant scalar curvature $-2n(2n+1)$.

In contrast, the investigation of m-quasi Einstein metrics on LP-Kenmotsu manifolds becomes particularly intriguing when the potential vector field β_V is assumed to be conformal. In this setting, we establish the following result:

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an LP-Kenmotsu manifold. If M admits an m-quasi Einstein metric such that the potential vector field β_V is conformal, then M is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. Since β_V is a conformal vector field, by Definition 2.1, the Lie derivative of the metric satisfies

$$\mathcal{L}_{\beta_V} g = 2\nu g, \tag{4.23}$$

for some smooth function ν on M.

Substituting (4.23) into the defining equation of an m-quasi Einstein metric (equation (1.1)), we obtain:

$$\mathrm{Ric}(\beta_X, \beta_Y) = (\lambda - \nu) g(\beta_X, \beta_Y) + \frac{1}{m}\beta_V^\sharp(\beta_X)\beta_V^\sharp(\beta_Y), \tag{4.24}$$

for all $\beta_X, \beta_Y \in \Gamma(TM)$.

Taking covariant derivative of both sides of (4.24) with respect to any vector field β_Z, and using compatibility of ∇ with g, we get:

$$(\nabla_{\beta_Z}\mathrm{Ric})(\beta_X, \beta_Y) = -(\beta_Z\nu)\, g(\beta_X, \beta_Y) + \frac{1}{m}\Big\{(\nabla_{\beta_Z}\beta_V^\sharp)(\beta_X)\,\beta_V^\sharp(\beta_Y) + \beta_V^\sharp(\beta_X)\,(\nabla_{\beta_Z}\beta_V^\sharp)(\beta_Y)\Big\}.$$

Now taking the cyclic sum over $\beta_X, \beta_Y, \beta_Z$, and using the symmetry of g, we derive:

$$\begin{aligned} \mathcal{C}(\beta_X, \beta_Y, \beta_Z) :=& (\nabla_{\beta_X}\mathrm{Ric})(\beta_Y, \beta_Z) + (\nabla_{\beta_Y}\mathrm{Ric})(\beta_Z, \beta_X) + (\nabla_{\beta_Z}\mathrm{Ric})(\beta_X, \beta_Y) \\ =& -\sum_{\mathrm{cyc}}(\beta_X\nu)\, g(\beta_Y, \beta_Z) \\ &+ \frac{1}{m}\sum_{\mathrm{cyc}}\Big\{(\nabla_{\beta_X}\beta_V^\sharp)(\beta_Y)\beta_V^\sharp(\beta_Z) + \beta_V^\sharp(\beta_Y)(\nabla_{\beta_X}\beta_V^\sharp)(\beta_Z)\Big\}. \end{aligned} \tag{4.25}$$

Contracting (4.25) over $\beta_Y = \beta_Z$, and recalling that β_V is conformal and $\mathcal{L}_{\beta_V} g = 2\nu g$, one obtains:

$$\frac{2}{2n+3}(\beta_X r) + \beta_X\nu - \frac{2\nu}{m}\beta_V^\sharp(\beta_X) = 0,$$

where r denotes the scalar curvature. Substituting this back into (4.25) eliminates the $(\beta_X \nu)$ terms, leading to:

$$\mathcal{C}(\beta_X, \beta_Y, \beta_Z) = -\frac{2}{2n+3} \left\{ (\beta_X r)\, g(\beta_Y, \beta_Z) + (\beta_Y r)\, g(\beta_Z, \beta_X) + (\beta_Z r)\, g(\beta_X, \beta_Y) \right\}. \tag{4.26}$$

Now, set $\beta_Y = \beta_Z = \xi$, and apply (2.12), (2.13) we obtain

$$\beta_X(r) + 2\,\xi(r)\,\eta(\beta_X) = 0. \tag{4.27}$$

Taking the trace of the equation yields $\xi(r) = 2(r - 2n(2n+1))$. By this equation, one can find that $\mathcal{L}_\xi r = 2(r - 2n(2n+1))$. Since $\mathcal{L}_\xi$ and d commute, this implies $\mathcal{L}_\xi(dr) = 2\,dr$, which means:

$$\mathcal{L}_\xi(Dr) = 2\,Dr.$$

Now using $\nabla_\xi Dr = Dr + (\xi r)\xi$ and combining with (4.27), we get:

$$(\xi r)\xi + 2\,\xi(\xi r)\xi = 0.$$

Solving this differential equation gives $\xi(r) = 0$, and therefore, the scalar curvature is constant:

$$r = 2n(2n+1).$$

Finally, substituting the constant r back into (4.25) and making use of (2.12), (2.13) shows that $QX = 2n\beta_X$, implying that M is Einstein.

□

Conclusion

The theory of geometric flows and differential operators, such as the Ricci flow and the Laplacian, plays a pivotal role in understanding deep structures in geometry and physics. These tools have been instrumental in resolving major conjectures (e.g., the Poincaré and geometrization conjectures) and have found applications in diverse fields like medical imaging, fluid dynamics, and computer vision. In the spirit of this geometric framework, the present work investigates the existence and implications of m-quasi Einstein metrics on LP-Kenmotsu manifolds, which are a class of Lorentzian paracontact manifolds with rich geometric structure.

Beginning with the closed m-quasi Einstein condition, we derived a fundamental curvature identity linking the Ricci operator, the potential vector field, and the structure tensors of the LP-Kenmotsu manifold. We further explored particular cases where the potential vector field is aligned with the unit timelike vector field, conformal, or acts as a strict infinitesimal contact transformation. These geometrically significant cases not only illuminate the curvature behavior but also suggest strong constraints on the underlying manifold.

Our results demonstrate that LP-Kenmotsu manifolds admitting such metrics must be Einstein or η-Einstein, with constant scalar curvature—a feature that reflects the rigidity imposed by the m-quasi Einstein framework. This generalizes several classical results known in Kenmotsu and contact metric geometry to the Lorentzian paracontact setting. The study of such metrics opens new avenues in the analysis of pseudo-Riemannian structures, particularly where the Laplacian and curvature operators interact in meaningful ways, thus aligning well with broader geometric applications in both mathematics and applied sciences.

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(P. Somashekhara) Department of Mathematics, I. D. S. G. Govt. College, Chikkamagalur-577102, Karnataka, INDIA.

(Arasaiah) Department of Mathematics, Sir M. V. Govt. Science College, Bommanakatte, Bhadravathi-577302, Karnataka, INDIA.

(B.P. Murthy) Department of Mathematics, Govt. First Grade College, Kadur-577548, Karnataka, INDIA.

(K.C. Ajeyakashi) Department of Data Analytics and Mathematical Science, Faculty of Engineering and Technology, Jain (Deemed-to-be University), Global Campus-562112, Karnataka, INDIA.

(K. Ananda) Department of Mathematics, New Horizon College of Engineering, Bengaluru-560087. Karnataka, INDIA.