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STUDY ON PARA-SASAKIAN MANIFOLDS ADMITTING η-EINSTEIN SOLITON

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Abstract. In this paper we explore the properties of para-Sasakian manifold with the help of Schouten-van Kampen connection. We examine para-Sasakian manifold admitting η-Einstein soliton with respect to this connection. Moreover, we investigate η-Einstein soliton on para-Sasakian manifolds admitting cyclic parallel Ricci tensor with respect to Schouten-van Kampen connection. Additionally, we investigate η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{S} = 0$, $\overline{K}(\xi, F_1).\overline{R} = 0$, for all smooth vector fields F_1, where $\overline{K}$, $\overline{S}$ and $\overline{R}$ are conharmonic curvature tensor, Ricci curvature tensor and Riemannian curvature tensor with respect to Schouten-van Kampen connection, respectively.

Keywords: Para-Sasakian manifold, Schouten-van Kampen connection, Einstein soliton, η-Einstein soliton.

MSC (2020): 53C15, 53C25.

1. Introduction

The concept of Schouten-van Kampen connection (abbreviated as SVK- connection) was introduced in the early twentieth century to study non-holomorphic manifolds [21, 26]. Later, in 2006, Bejancu [4] studied Schouten-van Kampen connection on Foliated manifolds. Recently, Biswas and Baisya [5] investigated some properties of pseudo symmetric Sasakian manifolds with respect to SVK-connection. Most recently, this connection has been introduced on para-Sasakian manifold by Sundriyal and Upreti [23]. They studied projective curvature tensor, concircular curvature tensor and Nijenhuis curvature tensor for the para-Sasakian manifold with respect to this connection. SVK-connection ($\overline{\nabla}$) for an almost contact metric manifold M of dimension n containing an almost contact metric structure (ϕ, ξ, η, g) composed of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\overline{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi, \tag{1.1}$$

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for all $X,\ Y \in \Gamma(M)$, where $\Gamma(M)$ is the set of all vector fields on M and ∇ being the Levi-Civita connection on M.

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds was introduced by Sato and Matsumoto [20]. Later, Adati and Matsumoto investigated some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold was studied by many authors. For instance, we see [2, 15, 17, 19, 22] and their references.

The concept of Ricci flow was introduced by Hamilton [9] in the early 1980s. That the Ricci flow is an excellent tool by which the structure of a manifold can be simplified was observed by Hamilton. The differential equation of Ricci flow is

$$\frac{\partial g}{\partial t} = -2S, \quad g(0) = g_0, \tag{1.2}$$

where g is a Riemannian metric, S is Ricci curvature tensor and t is time. The solitons for Ricci flow is identified with the self similar solutions of the above differential equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is usually represented by a triple $\left(g, \mathcal{V}^{/}, \lambda\right)$, where $\mathcal{V}^{/}$ is a vector field and λ is a constant satisfying

$$L_{\mathcal{V}^{/}} g + 2S + 2\lambda g = 0, \tag{1.3}$$

where S is Ricci curvature tensor and $L_{\mathcal{V}^{/}} g$ denotes the Lie derivative of g along the vector field $\mathcal{V}^{/}$. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. The vector field $\mathcal{V}^{/}$ is called potential vector field and if it is a gradient of a continuously differentiable function, then the Ricci soliton $\left(g, \mathcal{V}^{/}, \lambda\right)$ is called a gradient Ricci soliton and the associated function is said to be the potential function. Ricci soliton was further investigated by many researchers. For instance, we see [3, 14, 16, 18, 24, 25] and their references.

Catino and Mazzieri [7] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure (ϕ, ξ, η, g) is said to have an Einstein soliton $(g, \mathcal{V}, \lambda)$ if

$$L_{\mathcal{V}} g + 2S + (2\lambda - r)g = 0, \tag{1.4}$$

holds, where r being the scalar curvature. The Einstein soliton $(g, \mathcal{V}, \lambda)$ is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation given by

$$\frac{\partial g}{\partial t} = -2S + rg.$$

Again as a generalization of Einstein soliton, the η-Einstein soliton on manifold M was introduced by Blaga [6] and it is given by

$$L_{\mathcal{V}} g + 2S + (2\lambda - r)g + 2\beta \eta \otimes \eta = 0, \tag{1.5}$$

where, β is some constant. When $\beta = 0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\beta \neq 0$, the data $(g, \mathcal{V}, \lambda, \beta)$ is called proper η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$. This soliton was further studied by Mandal and Mallik [11, 12, 13] under various curvature conditions.

The conharmonic curvature tensor K [8, 10] of type (1,3) on an n-dimensional Riemannian manifold M is given by

$$\begin{aligned} K(F_1,F_2)F_3 &= R(F_1,F_2)F_3 - \frac{1}{n-2}\left[S(F_2,F_3)F_1 - S(F_1,F_3)F_2\right] \\ &\quad -\frac{1}{n-2}\left[g(F_2,F_3)QF_1 - g(F_1,F_3)QF_2\right], \end{aligned} \tag{1.6}$$

for all F_1, F_2, $F_3 \in \Gamma(M)$, where R is the Riemannian curvature tensor of type $(1,3)$ and r is the scalar curvature of M.

Definition 1.1. [13] *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S\left(Y,Z\right) = l_1 g\left(Y,Z\right) + l_2 \eta\left(Y\right)\eta\left(Z\right),$$

for all $Y,\ Z \in \Gamma\left(M\right),$ where l_1, l_2 are scalars.

Definition 1.2. [13] *A para-Sasakian manifold M is called a generalized η-Einstein manifold if its Ricci tensor is of the form*

$$S\left(Y,Z\right) = k_1 g\left(Y,Z\right) + k_2 \eta\left(Y\right)\eta\left(Z\right) + k_3 g\left(Y,\phi Z\right),$$

for all $Y,\ Z \in \Gamma\left(M\right),$ where k_1, k_2 and k_3 are scalars.

Definition 1.3. *A para-Sasakian manifold M is said to be ξ-conharmonically flat if and only if*

$$K(X,Y)\xi = 0,$$

for all $X,\ Y \in \Gamma\left(M\right).$

This paper is structured as follows:

First two sections of the paper have been kept for introduction and preliminaries. In **Section-3**, we study some properties of para-Sasakian manifold with respect to SVK-connection. In **Section-4**, we introduce η-Einstein soliton on para-Sasakian manifolds with respect to SVK-connection. In **Section-5**, we study η-Einstein soliton on para-Sasakian manifold admitting cyclic parallel Ricci tensor with respect to SVK-connection. **Section-6** deals with η-Einstein soliton on ξ-conharmonically flat para-Sasakian manifolds with respect to SVK-connection. **Section-7** deals with η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi,F_1).\overline{S} = 0$. Lastly, **Section-8** contains η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi,F_1).\overline{R} = 0$.

2. Preliminaries

An n-dimensional differentiable manifold M with structure (ϕ,ξ,η), where η is a 1-form, ξ is the structure vector field, ϕ is a $(1,1)$-tensor field satisfying [20]

$$\phi^2\left(F_1\right) = F_1 - \eta\left(F_1\right)\xi, \eta(\xi) = 1, \tag{2.7}$$

$$\phi\left(\xi\right) = 0, \eta\circ\phi = 0, \tag{2.8}$$

for all vector fields F_1 on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [27]

$$g(\phi F_1, \phi F_2) = -g(F_1, F_2) + \eta(F_1)\eta(F_2), \tag{2.9}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g). From (2.9) one can deduce that

$$g(F_1, \phi F_2) = -g(\phi F_1, F_2), \tag{2.10}$$

$$g(F_1, \xi) = \eta(\xi). \tag{2.11}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [27] if

$$g(F_1, \phi F_2) = d\eta(F_1, F_2),$$

for all vector fields F_1, F_2 on M, where

$$d\eta(F_1, F_2) = \frac{1}{2}\{F_1\eta(F_2) - F_2\eta(F_1) - \eta([F_1, F_2])\}.$$

The manifold M is called a para-Sasakian manifold if

$$(\nabla_{F_1}\varphi)F_2 = -g(F_1, F_2)\xi + \eta(F_2)F_1, \tag{2.12}$$

for any smooth vector fields F_1, F_2 on M.

In a para-Sasakian manifold the following relations also hold [27]:

$$(\nabla_{F_1}\eta)F_2 = g(F_1, \phi F_2), \nabla_{F_1}\xi = -\phi F_1, \tag{2.13}$$

$$\eta(R(F_1, F_2)F_3) = g(F_1, F_3)\eta(F_2) - g(F_2, F_3)\eta(F_1), \tag{2.14}$$

$$R(F_1, F_2)\xi = \eta(F_1)F_2 - \eta(F_2)F_1, \tag{2.15}$$

$$R(\xi, F_1)F_2 = -g(F_1, F_2)\xi + \eta(F_2)F_1, \tag{2.16}$$

$$R(F_1, \xi)F_2 = g(F_1, F_2)\xi - \eta(F_2)F_1, \tag{2.17}$$

$$R(\xi, F_1)\xi = F_1 - \eta(F_1)\xi, \tag{2.18}$$

$$S(F_1, \xi) = -(n-1)\eta(F_1), \tag{2.19}$$

$$S(\xi, \xi) = -(n-1), Q\xi = -(n-1)\xi, \tag{2.20}$$

$$S(\phi F_1, \phi F_2) = S(F_1, F_2) + (n-1)\eta(F_1)\eta(F_2). \tag{2.21}$$

for any smooth vector fields F_1, F_2, F_3 on M.

Example 2.1. *Let us consider* 3*-dimensional manifold*

$$M^3 = \{(x, y, z) \in R^3\},$$

where (x, y, z) *are the standard coordinates in* R^3. *We choose the linearly independent vector fields*

$$E_1 = e^x\frac{\partial}{\partial y}, E_2 = e^x\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right), E_3 = -\frac{\partial}{\partial x}.$$

Let g *be the pseudo Riemannian metric defined by* $g(E_i, E_j) = 0$, $if\ i \neq j\ for\ i, j = 1, 2, 3,$ *and* $g(E_1, E_1) = -1, g(E_2, E_2) = -1, g(E_3, E_3) = 1.$

Let η *be the* 1*-form defined by* $\eta\left(X\right)=g\left(X,E_3\right)$ *for any* $X\in\Gamma\left(M^3\right)$. *Let* ϕ *be the* $(1,1)$ *tensor field defined by*

$$\phi E_1 = E_1, \phi E_2 = E_2, \phi E_3 = 0, \tag{2.22}$$

$$where, \quad trace(\phi) = \sum_{i=1}^{3} g(E_i,\phi E_i) = -2. \tag{2.23}$$

Let X, Y, $Z\in\Gamma\left(M^3\right)$ *be given by*

$$\begin{aligned} X &= x_1E_1+x_2E_2+x_3E_3, \\ Y &= y_1E_1+y_2E_2+y_3E_3, \\ Z &= z_1E_1+z_2E_2+z_3E_3. \end{aligned}$$

Then, we have

$$\begin{aligned} g\left(X,Y\right) &= x_1y_1+x_2y_2+x_3y_3, \\ \eta\left(X\right) &= x_3, \\ g\left(\phi X,\phi Y\right) &= x_1y_1+x_2y_2. \end{aligned}$$

Using the linearity of g *and* ϕ, $\eta\left(E_3\right)=1, \phi^2X=X-\eta\left(X\right)E_3$ *and* $g\left(\phi X,\phi Y\right)=-g\left(X,Y\right)+\eta\left(X\right)\eta\left(Y\right)$ *for all* X, $Y\in\Gamma\left(M\right)$. *We have*

$$\begin{aligned} [E_1,E_2] &= 0, [E_1,E_3]=-E_1, [E_2,E_3]=E_2, \\ [E_2,E_1] &= 0, [E_3,E_1]=E_1, [E_3,E_2]=-E_2. \end{aligned}$$

Let the Levi-Civita connection with respect to g *be* ∇, *then using Koszul formula we get the following*

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above results we see that the structure (ϕ,ξ,η,g) *satisfies*

$$\left(\nabla_X\varphi\right)Y=-g\left(X,Y\right)\xi+\eta\left(Y\right)X,$$

for all X, $Y\in\Gamma\left(M^3\right)$, *where* $\eta\left(\xi\right)=\eta\left(E_3\right)=1$. *Hence* $M^3\left(\phi,\xi,\eta,g\right)$ *is a para-Sasakian manifold.*

The components of Riemannian curvature tensor of M^3 *are given by*

$$\begin{pmatrix} R(E_1,E_2)E_2 & R(E_1,E_3)E_3 & R(E_1,E_2)E_3 \\ R(E_2,E_1)E_1 & R(E_2,E_3)E_3 & R(E_2,E_3)E_1 \\ R(E_3,E_1)E_1 & R(E_3,E_2)E_2 & R(E_3,E_1)E_2 \end{pmatrix} = \begin{pmatrix} -E_1 & -E_1 & 0 \\ E_2 & E_2 & 0 \\ E_3 & E_3 & 0 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 *are given by*

$$S(E_1,E_1)=S(E_3,E_3)=0, S(E_2,E_2)=2. \tag{2.24}$$

Therefore the scalar curvature of M^3 *is*

$$r=\sum_{i=1}^{3}S(E_i,E_i)=2. \tag{2.25}$$

3. Schouten-van Kampen connection on para-Sasakian manifolds

In this section we get the relation between SVK-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SVK-connection. We also establish here the first Bianchi identity with respect to SVK-connection on M.

In view of (1.1), (2.13) and (2.11), we get the expression for SVK-connection in a para-Sasakian manifold M as

$$\overline{\nabla}_{F_1}F_2 = \nabla_{F_1}F_2 + g(F_1, \phi F_2)\xi + \eta(F_2)\phi F_1, \tag{3.26}$$

with torsion tensor

$$\overline{T}(F_1, F_2) = 2g(F_1, \phi F_2)\xi + \eta(F_2)\phi F_1 - \eta(F_1)\phi F_2.$$

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\overline{\nabla}_{F_1}\xi = 0, \left(\overline{\nabla}_{F_1}\eta\right)F_2 = g(\phi F_1, F_2), \tag{3.27}$$

$$\left(\overline{\nabla}_{F_1}g\right)(F_2, F_3) = g(\phi F_1, F_2)\eta(F_3) + g(\phi F_2, F_3)\eta(F_1). \tag{3.28}$$

for all $F_1, F_2 \in \Gamma(M)$.

Proposition 3.1. *The SVK-connection on a para-Sasakian manifold is non metric compatible connection.*

Proposition 3.2. *The SVK-connection on a para-Sasakian manifold is non symmetric connection.*

Proposition 3.3. *The structure vector field of a para-Sasakian manifold is parallel with respect to SVK-connection.*

Let $\overline{R}$ be the Riemannian curvature tensor with respect to SVK-connection on a para-Sasakian manifold defined as

$$\overline{R}(F_1, F_2)F_3 = \overline{\nabla}_{F_1}\overline{\nabla}_{F_2}F_3 - \overline{\nabla}_{F_2}\overline{\nabla}_{F_1}F_3 - \overline{\nabla}_{[F_1,F_2]}F_3. \tag{3.29}$$

In reference of (2.12), (2.13) and (3.26) we have

$$\begin{aligned}
\overline{\nabla}_{F_1}\overline{\nabla}_{F_2}F_3 &= \nabla_{F_1}\nabla_{F_2}F_3 + g(\nabla_{F_1}F_2, \phi F_3)\xi - g(F_1, F_3)\eta(F_2)\xi + g(F_1, F_2)\eta(F_3)\xi \\
&\quad + g(F_2, \phi\nabla_{F_1}F_3)\xi - g(F_2, \phi F_3)\phi F_1 + g(F_1, \phi F_3)\phi F_2 \\
&\quad + \eta(\nabla_{F_1}F_3)\phi F_2 - g(F_1, F_2)\eta(F_3)\xi + \eta(F_2)\eta(F_3)F_1 \\
&\quad + \eta(F_3)\phi\nabla_{F_1}F_2 + g(F_1, \phi\nabla_{F_2}F_3)\xi + g(F_1, F_2)\eta(F_3)\xi \\
&\quad - \eta(F_1)\eta(F_2)\eta(F_3)\xi + \eta(\nabla_{F_1}F_3)\phi F_2 + g(F_2, \phi F_3)\phi F_1,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
\overline{\nabla}_{[F_1,F_2]}F_3 &= \nabla_{[F_1,F_2]}F_3 + g(\nabla_{F_1}F_2, \phi F_3)\xi - g(\nabla_{F_2}F_1, \phi F_3)\xi \\
&\quad + \eta(F_3)\phi\nabla_{F_1}F_2 - \eta(F_3)\phi\nabla_{F_2}F_1.
\end{aligned} \tag{3.31}$$

Interchanging F_1 and F_2 in (3.30) and using it along with (3.30) and (3.31) in (3.29), we get

$$\begin{aligned} \overline{R}(F_1,F_2)F_3 &= R(F_1,F_2)F_3 + g(F_2,F_3)\eta(F_1)\xi - g(F_1,F_3)\eta(F_2)\xi \\ &\quad + g(F_1,\phi F_3)\phi F_2 - g(F_2,\phi F_3)\phi F_1 \\ &\quad + \eta(F_2)\eta(F_3)F_1 - \eta(F_1)\eta(F_3)F_2. \end{aligned} \tag{3.32}$$

Writing the equation (3.32) by cyclic permutations of F_1, F_2 and F_3 and using the fact that $R(F_1,F_2)F_3 + R(F_2,F_3)F_1 + R(F_3,F_1)F_2 = 0$, we have

$$\overline{R}(F_1,F_2)F_3 + \overline{R}(F_2,F_3)F_1 + \overline{R}(F_3,F_1)F_2 = 0,$$

for all $F_1, F_2, F_3 \in \Gamma(M)$.

Taking inner product of (3.32) with a vector field U and contracting over F_1 and U, we get

$$\overline{S}(F_2,F_3) = S(F_2,F_3) + (n-1)\eta(F_2)\eta(F_3) - \psi g(F_2,\phi F_3), \tag{3.33}$$

where $\overline{S}$ denotes Ricci curvature tensor with respect to $\overline{\nabla}$ and $\psi = trace(\phi)$.

Proposition 3.4. *The SVK-connection on para-Sasakian manifold satisfies the first Bianchi identity.*

Lemma 3.1. *Let M be an n-dimensional para-Sasakian manifold admitting SVK-connection, then*

$$\overline{R}(F_1,F_2)\xi = 0, \overline{R}(\xi,F_2)F_3 = 0, \overline{R}(F_1,\xi)F_3 = 0, \tag{3.34}$$

$$\overline{S}(F_1,\xi) = 0 = \overline{S}(\xi,F_2), \tag{3.35}$$

$$\overline{Q}F_1 = QF_1 + (n-1)\eta(F_1)\xi + \phi F_1\psi, \overline{Q}\xi = 0, \tag{3.36}$$

$$\overline{r} = r + (n-1) - \psi^2, \tag{3.37}$$

for all $F_1, F_2, F_3 \in \Gamma(M)$, where $\overline{R}$, $\overline{Q}$ and $\overline{r}$ denote Riemannian curvature tensor, Ricci operator and scalar curvature tensor with respect to $\overline{\nabla}$, respectively.

Remark 3.1. *Eigen value of Ricci operator with respect to SVK-connection corresponding to the eigen vector ξ is zero.*

4. η-Einstein soliton on para-Sasakian manifold with respect to SVK-connection

Writing equation (1.5) with respect to SVK-connection and expanding $\overline{L}_{F_4}$, we have

$$\begin{aligned} 0 &= \left(\overline{L}_{F_4}g\right)(F_1,F_2) + 2\overline{S}(F_1,F_2) + [2\lambda - \overline{r}]\,g(F_1,F_2) + 2\beta\eta(F_1)\eta(F_2) \\ &= g(\overline{\nabla}_{F_1}F_4,F_2) + g(F_1,\overline{\nabla}_{F_2}F_4) \\ &\quad + 2\overline{S}(F_1,F_2) + [2\lambda - \overline{r}]\,g(F_1,F_2) + 2\beta\eta(F_1)\eta(F_2). \end{aligned} \tag{4.38}$$

Using (3.33) and (3.37) in (4.38), we get

$$\begin{aligned} 0 &= \left(L_{F_4}g\right)(F_1,F_2) + 2S(F_1,F_2) + (2\lambda - r)g(F_1,F_2) + 2\beta\eta(F_1)\eta(F_2) \\ &\quad + g(F_1,\phi F_4)\eta(F_2) + g(F_2,\phi F_4)\eta(F_1) - \left[n-1-\psi^2\right]g(F_1,F_2) \\ &\quad - 2\psi g(F_1,\phi F_2) + 2(n-1)\eta(F_1)\eta(F_2), \end{aligned} \tag{4.39}$$

for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$.

Therefore we get the following theorem:

Theorem 4.1. *η-Einstein soliton on para-Sasakian manifold with respect to SVK-connection is invariant if and only if*

$$\begin{aligned} 0 &= g(F_1, \phi F_4)\eta(F_2) + g(F_2, \phi F_4)\eta(F_1) - 2\psi g(F_1, \phi F_2) \\ &\quad - \left[n - 1 - \psi^2\right] g(F_1, F_2) + 2(n-1)\eta(F_1)\eta(F_2), \end{aligned}$$

holds for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$, where $\psi = trace(\phi)$.

Setting $F_4 = \xi$ and using (3.27) in (4.38), we get

$$\overline{S}(F_1, F_2) = -\left[\lambda - \frac{\overline{r}}{2}\right] g(F_1, F_2) - \beta\eta(F_1)\eta(F_2). \tag{4.40}$$

Equation (4.40) gives

$$\overline{Q}F_1 = -\left[\lambda - \frac{\overline{r}}{2}\right] F_1 - \beta\eta(F_1)\xi. \tag{4.41}$$

Setting $F_2 = \xi$ in (4.40), we have

$$\overline{S}(F_1, \xi) = -\left[\lambda - \frac{\overline{r}}{2} + \beta\right] \eta(F_1). \tag{4.42}$$

Using (3.33) and (3.37) in (4.40), we obtain

$$\begin{aligned} S(F_1, F_2) &= -\left[\lambda - \frac{1}{2}\left\{r + (n-1) - \psi^2\right\}\right] g(F_1, F_2) \\ &\quad -(\beta + n - 1)\eta(F_1)\eta(F_2) + \psi g(F_1, \phi F_2). \end{aligned} \tag{4.43}$$

Hence we obtain the following theorem:

Theorem 4.2. *If an n-dimensional para-Sasakian manifold M contains η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection, then M is generalized η-Einstein manifold.*

Contracting (4.43) over F_1 and F_2, we get

$$r = \frac{2(n\lambda + \beta)}{n-2} - n + 1 + \psi^2. \tag{4.44}$$

Corollary 4.1. *If an n-dimensional para-Sasakian manifold M contains η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection, then the scalar curvature of M is given by equation (4.44).*

5. η-Einstein soliton on para-Sasakian manifold admitting cyclic parallel Ricci tensor with respect to SVK-connection

Suppose the para-Sasakian manifold M equipped with cyclic parallel Ricci tensor with respect to SVK-connection, then $\overline{S}$ satisfies

$$\left(\overline{\nabla}_{F_1}\overline{S}\right)(F_2, F_3) + \left(\overline{\nabla}_{F_2}\overline{S}\right)(F_3, F_1) + \left(\overline{\nabla}_{F_3}\overline{S}\right)(F_1, F_2) = 0, \tag{5.45}$$

for all F_1, F_2, $F_3 \in \Gamma(M)$.

Now, assuming $\overline{r}$ constant we get from (4.40) that

$$\left(\overline{\nabla}_{F_1}\overline{S}\right)(F_2,F_3) = -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_2,F_3)\eta(F_1)\right] -\beta\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_1,F_3)\eta(F_2)\right], \tag{5.46}$$

$$\left(\overline{\nabla}_{F_2}\overline{S}\right)(F_3,F_1) = -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_2,F_3)\eta(F_1)+g(\phi F_3,F_1)\eta(F_2)\right] -\beta\left[g(\phi F_2,F_3)\eta(F_1)+g(\phi F_2,F_1)\eta(F_3)\right], \tag{5.47}$$

$$\left(\overline{\nabla}_{F_3}\overline{S}\right)(F_1,F_2) = -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_3,F_1)\eta(F_2)+g(\phi F_1,F_2)\eta(F_3)\right] -\beta\left[g(\phi F_3,F_1)\eta(F_2)+g(\phi F_3,F_2)\eta(F_1)\right]. \tag{5.48}$$

Using (5.46), (5.47) and (5.48) in (5.45), we get

$$0=\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_2,F_3)\eta(F_1)+g(\phi F_3,F_1)\eta(F_2)\right]. \tag{5.49}$$

Contracting (5.49) over F_1 and F_2, we get

$$\left(\lambda-\frac{\overline{r}}{2}\right)\psi=0. \tag{5.50}$$

If $\psi\neq 0$, then (3.37) and (5.50) give

$$\lambda=\frac{1}{2}\left[r+(n-1)-\psi^2\right]. \tag{5.51}$$

From (4.44) and (5.51), we get

$$\beta=-\left[r+(n-1)-\psi^2\right]. \tag{5.52}$$

This implies the following theorem:

Theorem 5.1. *If an n-dimensional para-Sasakian manifold containing η-Einstein soliton (g,ξ,λ,β) admits cyclic parallel Ricci tensor with respect to SVK-connection, then the soliton constants are given by equations (5.51) and (5.52).*

6. η-Einstein soliton on ξ-conharmonically flat para-Sasakian manifold with respect to SVK-connection

The conharmonic curvature tensor with respect to SVK-connection is given by

$$\overline{K}(F_1,F_2)F_3 = \overline{R}(F_1,F_2)F_3-\frac{1}{n-2}\left[\overline{S}(F_2,F_3)F_1-\overline{S}(F_1,F_3)F_2\right] -\frac{1}{n-2}\left[g(F_2,F_3)\overline{Q}F_1-g(F_1,F_3)\overline{Q}F_2\right], \tag{6.53}$$

for all $F_1,\ F_2,\ F_3\in\Gamma(M)$.

The condition must be satisfied by $\overline{K}$ is

$$\overline{K}(F_1,F_2)\xi=0. \tag{6.54}$$

In view of (6.53) and (6.54), we get

$$\begin{aligned}\overline{R}(F_1,F_2)\xi &= \frac{1}{n-2}\left[\overline{S}(F_2,\xi)F_1-\overline{S}(F_1,\xi)F_2\right]\\ &\quad+\frac{1}{n-2}\left[\eta(F_2)\overline{Q}F_1-\eta(F_1)\overline{Q}F_2\right].\end{aligned} \tag{6.55}$$

In reference to (4.41), (4.42), (3.34) and (6.55), we have

$$(2\lambda-\overline{r}+\beta)\left[\eta(F_2)F_1-\eta(F_1)F_2\right]=0.$$

Since $\eta(F_2)F_1\neq\eta(F_1)F_2$, we have

$$2\lambda+\beta=\overline{r}. \tag{6.56}$$

In view of (3.37), (4.44) and (6.56), we have

$$\frac{2(\lambda+2)-n}{n-4}=\frac{\beta-2}{2}=\frac{r-(\psi^2-1)}{n-2}, \tag{6.57}$$

where $\psi=trace(\phi)$.

Hence we have the following theorem:

Theorem 6.1. *If a ξ-conharmonically flat para-Sasakian manifold of dimension n admits an η-Einstein soliton (g,ξ,λ,β) with respect to SVK-connection, then the soliton constants are given by (6.57).*

Setting $\beta=0$ in (6.57), we get

$$\lambda=\frac{1}{2}\left[r+(n-1)-\psi^2\right].$$

Corollary 6.1. *If an n-dimensional ξ-conharmonically flat para-Sasakian manifold admits an Einstein soliton (g,ξ,λ) with respect to SVK-connection, then the soliton is*

$$\begin{aligned} i)\ \textit{shrinking if } r &< -(n-1)+\psi^2,\\ ii)\ \textit{steady if } r &= -(n-1)+\psi^2,\\ iii)\ \textit{expanding if } r &> -(n-1)+\psi^2,\end{aligned}$$

where $\psi=trace(\phi)$.

7. η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi,F_1).\overline{S}=0$

Setting $F_1=\xi$ in (6.53), we get

$$\begin{aligned}\overline{K}(\xi,F_1)F_2 &= -\frac{1}{n-2}\left[\overline{S}(F_1,F_2)\xi-\overline{S}(\xi,F_2)F_1\right]\\ &\quad-\frac{1}{n-2}\left[g(F_1,F_2)\overline{Q}\xi-\eta(F_2)\overline{Q}F_1\right].\end{aligned} \tag{7.58}$$

Using (4.40). (4.41) and (4.42) in (7.58), we get

$$\overline{K}(\xi,F_1)F_2=\left[\frac{2\lambda-\overline{r}+\beta}{n-2}\right]\left[g(F_1,F_2)\xi-\eta(F_2)F_1\right]. \tag{7.59}$$

The condition that must be satisfied by $\overline{S}$ is

$$\overline{S}(\overline{K}(\xi,F_1)F_2,F_3)+\overline{S}(F_2,\overline{K}(\xi,F_1)F_3)=0, \tag{7.60}$$

for all $F_1, F_2, F_3 \in \Gamma(M)$.

Replacing the value of $\overline{S}$ from (4.40) and using (7.59) in (7.60), we get

$$\begin{aligned} 0 &= (2\lambda - \overline{r} + \beta)^2 \left[g(F_1, F_3)\eta(F_2) + g(F_1, F_2)\eta(F_3)\right] \\ &\quad + 2(2\lambda - \overline{r} + \beta)\beta\eta(F_1)\eta(F_2)\eta(F_3). \end{aligned} \tag{7.61}$$

Setting $F_1 = F_2 = \xi$ in (7.61), we have

$$(2\lambda - \overline{r} + \beta)\left[2(\lambda + \beta) - \overline{r}\right] = 0. \tag{7.62}$$

In view of (3.37) and (7.62), we have

$$2\lambda + \beta = r + (n-1) - \psi^2, \tag{7.63}$$

$$\text{or},$$

$$2(\lambda + \beta) = r + (n-1) - \psi^2, \tag{7.64}$$

where $\psi = trace(\phi)$.

From (4.44) and (7.63), we have

$$\lambda = \frac{1}{2}\left[\frac{n-4}{n-2}\right]\left(r + n - 1 - \psi^2\right), \beta = \left[\frac{2\left(r + n - 1 - \psi^2\right)}{n-2}\right]. \tag{7.65}$$

Again, in reference to (4.44) and (7.64), we get

$$\lambda = \frac{1}{2}\left[\frac{n-3}{n-1}\right]\left(r + n - 1 - \psi^2\right), \beta = \frac{1}{n-1}\left(r + n - 1 - \psi^2\right). \tag{7.66}$$

Hence we have the following theorem:

Theorem 7.1. *Let M be an n-dimensional para-Sasakian manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection. If M satisfies*

$$\overline{K}(\xi, F_1).\overline{S} = 0,$$

then the soliton scalars are given by (7.65) or (7.66).

8. η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{R} = 0$

The condition must be satisfied by $\overline{R}$ is

$$\left(\overline{K}(\xi, F_1)\overline{R}\right)(F_2, F_3)F_4 = 0, \tag{8.67}$$

for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$.

Equation (8.67) gives

$$\begin{aligned} 0 &= \overline{K}(\xi, F_1)\overline{R}(F_2, F_3)F_4 - \overline{R}(\overline{K}(\xi, F_1)F_2, F_3)F_4 \\ &\quad - \overline{R}(F_2, \overline{K}(\xi, F_1)F_3)F_4 - \overline{R}(F_2, F_3)\overline{K}(\xi, F_1)F_4. \end{aligned} \tag{8.68}$$

In reference to (3.32), (7.58) and (8.68), we have

$$\begin{aligned} 0 &= \left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right]\left[g(F_1, \overline{R}(F_2, F_3)F_4)\xi + \overline{R}(F_1, F_3)F_4)\eta(F_2)\right] \\ &\quad + \left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right]\left[\overline{R}(F_2, F_1)F_4)\eta(F_3) + \overline{R}(F_2, F_3)F_1)\eta(F_4)\right]. \end{aligned} \tag{8.69}$$

Setting $F_4 = \xi$ and using (3.34) in (8.69), we get

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \overline{R}(F_2, F_3)F_1 = 0. \tag{8.70}$$

Taking inner product of (8.70) with a vector field W and contracting over F_2 and W, we get

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \overline{S}(F_1, F_3) = 0. \tag{8.71}$$

By the help of (4.40) and (8.71), we obtain

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \left[\left(\lambda - \frac{\overline{r}}{2}\right) g(F_1, F_3) + \beta\eta\left(F_1\right)\eta\left(F_3\right)\right] = 0. \tag{8.72}$$

Replacing F_3 by ξ in and using (3.37) in (8.72), we have

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] [2(\lambda + \beta) - \overline{r}] = 0. \tag{8.73}$$

Using (3.37) in (8.73), we have

$$\lambda + \beta = \frac{1}{2}\left[r + (n-1) - \psi^2\right], \tag{8.74}$$

or,

$$2\lambda + \beta = \left[r + (n-1) - \psi^2\right], \tag{8.75}$$

where $\psi = trace(\phi)$.

In view of (4.44) and (8.74), we have

$$\frac{2\lambda - (n-3)}{2(n-3)} = \frac{\beta - 1}{2} = \frac{r - \psi^2}{2(n-1)}. \tag{8.76}$$

Again, in reference to (4.44) and (8.75), we get

$$\frac{2(\lambda + 2) - n}{n-4} = \frac{\beta - 2}{2} = \frac{r - \left(\psi^2 - 1\right)}{n-2}. \tag{8.77}$$

Hence we have the following theorem:

Theorem 8.1. *Let M be an n-dimensional para-Sasakian manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection. If M satisfies*

$$\overline{K}(\xi, F_1).\overline{R} = 0,$$

then the soliton scalars are given by (8.76) or (8.77).

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