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Hsu-UNIFIED STRUCTURE MANIFOLDS COUPLED WITH A GENERALIZED WINTGEN TYPE INEQUALITY

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Abstract. The goal of this research note is that, Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection has been investigated. The Riemannian curvature tensor, Ricci curvature tensor, and scalar curvature of the Hsu-unified structure manifold with a semi-symmetric non-metric S-connection are transformed into their respective formulations. Mainly, we derive the generalized Wintgen inequalities for submanifolds in Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection. In addition, we discuss the Wintgen inequality for totally umbilical submanifolds of Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection. Also, we deduce the same inequality for the almost complex manifold, almost tangent manifold, almost product manifold and GF-manifold, and π-structure ambient manifolds. Finally, we deduced a universal lower bound of the norms of specific functions involving scalar curvature, normalized scalar curvature, and mean curvature to establish topological obstructions for submanifolds in Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection.

Keywords: Hsu-unified structure manifold, semi-symmetric non-metric S-connection, Submanifolds, Generalized Wintgen inequalities, Totally umbilical submanifolds.

MSC (2020): 53C15, 53C20, 53C25, 53C44.

1. Introduction

In the year 1924, Friedmann and Schouten [9] opened a new door for researchers by working on differentiable manifolds with semi-symmetric linear connection ∇. Later on, such connections were given geometrical meaning by Bartolotti [3] and Hayden [14] produced the discourse of semi-symmetric metric connection if $\nabla \mathsf{g} = 0$. After a lengthy break, Prvanonic [28] started the research of a semi-symmetric connection ∇ satisfying $\nabla \mathsf{g} \neq 0$ under the name pseudo-metric semi-symmetric connection, and Andonie [2] followed. A semi-symmetric non-metric connection was introduced and explored by Agashe and Chafle [1] in 1992. Yano [39] improved the semi-symmetric non-metric connection study on Riemannian manifolds in a systematic manner, and other geometers looked into it further (see [15], [13], [29], [30], [37] for details). Moreover, a quarter symmetric or pseudo-metric connection satisfies the specific

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form of torsion mentioned below:

$$\mathcal{T}(\mho_1, \mho_2) = \eta(\mho_1)\varphi\mho_2 - \eta(\mho_2)\varphi\mho_1, \tag{1.1}$$

where η is a 1-form and φ is a (1,1) tensor field. A quarter symmetric or pseudo-metric connection can be further classified as either metric or non-metric, as we have mentioned above. If in the quarter-symmetric condition, we have $\varphi\mho = \mho$ (the identity transformation), the connection becomes a semi-symmetric connection. Therefore, in the present study, we have focused on this specific case with the condition of non-metric $\nabla \mathsf{g} \neq 0$.

The Wintgen inequality is a sharp geometric inequality for surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ that involves major extrinsic invariants like normal curvature, square mean curvature, and shape operator, as well as intrinsic invariants like sectional curvatures, Ricci curvatures, and scalar curvatures.

P. Wintgen [38] derived an inequality for any surface $\mathbb{S}^2$ in $\mathbb{E}^4$ by the following way

$$||\mathbb{H}||^2 \geq \Omega + |\Omega_{Nor}|,$$

where the normal curvature is represented by Ω_{Nor} and the Gaussian curvature by Ω. To indicate the squared mean curvature, use $||\mathbb{H}||^2$.

In 1983, Guadalupe and Rodriguez [11] developed an extension for arbitrary codimension m in real space forms $\overline{M}^{m+2}(c)$ by

$$||\mathbb{H}||^2 + c \geq \Omega + |\Omega_{Nor}|. \tag{1.2}$$

In real space form $N^{n+2}(c)$ with $n \geq 2$, De Smet, Dillen, Verstraelen, and Vrancken conjectured the generalized Wintgen inequality for submanifolds [8]. Ge and Tang [10] also established this conjecture, which is also known as the DDVV conjecture for real space form. Several geometers have recently proven DDVV inequality for various classes of submanifolds in various ambient manifolds (see [21, 7, 31, 32, 33, 34]). Recently, Siddiqi et al. [32] established a generalized Wintgen inequality for quaternionic bi-slant submanifolds and QR-submanifolds (with minimal codimension) in quaternion space forms.

In 1960 C. Hsu [16] constructed a new structure on differentiable manifold of smooth class, in terms of any tensor field φ of type $(0, 2)$-type and a vector field $\mathcal{U}$, named Hsu-unified structure manifold ($\mathcal{HUS}$-manifolds). Then Singh also discussed hypersurfaces of an Hsu-unified structure [35]. Nivas et al. ([22], [23]) are also explored some properties of $\mathcal{HUS}$-manifolds endowed with a non-metric connection. After that, other authors also studied $\mathcal{HUS}$-manifolds endowed with a induced non-metric connection (see [4], [19]).

On the other hand, in this research paper, we have derived the generalized Wintgen inequality for totally umbilical submanifolds of geometric structures such as almost complex, tangent, product, GF-manifolds, providing a broad framework unified under H*su*-unified structure with a semi-symmetric non-metric S-connection. In addition, exploring some topological obstructions through integral inequalities involving Betti numbers that connect the scalar curvature (an intrinsic property), the normal scalar curvature (an extrinsic property), and the squared mean curvature (also an extrinsic property), with a semi-symmetric non-metric S-connection.

The purpose of this monograph is to further investigate submanifolds in of type $(0, 2)$-type and a vector field $\mathcal{U}$ endowed with a semi-symmetric non-metric S-connection ($SSNMs$) by means of generalized Wintgen inequalities.

2. Preliminaries

Let $\mathcal{H}^n$ be C^∞ class, $(n = 2p)$-even dimensional differentiable manifold. Let φ be a vector valued real linear function of differentiable class C^∞ satisfies

$$\varphi^2 \mho_1 = \alpha^r I(\mho_1) \tag{2.3}$$

for any vector field $\mho_1 \in \chi(\mathcal{H}^n)$.
In addition, a Riemannian metric g, such that

$$\mathsf{g}(\varphi\mho_1, \varphi\mho_2) = \alpha^r \mathsf{g}(\mho_1, \mho_2), \tag{2.4}$$

where α is a real or complex number and $0 \leq r \leq n$ and I indicates the identity operator. Then $(\mathcal{H}^n, \varphi, \mathsf{g}, r, \alpha)$ is said to be a $\mathcal{HUS}$-manifolds [16].

Let Φ in $\mathcal{HUS}$-manifolds $\mathcal{H}^n$ is a $(0, 2)$-type symmetric tensor field expressed as

$$\Phi(\mho_1, \mho_2) = \mathsf{g}(\varphi\mho_1, \mho_2) = \Phi(\mho_1, \mho_2) = \mathsf{g}(\mho_1, \mho_2), \tag{2.5}$$

and we have

$$\Phi(\varphi\mho_1, \mho_2) = \alpha^r \mathsf{g}(\mho_1, \mho_2). \tag{2.6}$$

Also, an $\mathcal{HUS}$-manifolds $(\mathcal{H}^n, \varphi, \mathsf{g}, r, \alpha)$ is satisfied [16]

$$\nabla_{\mho_1}\zeta = \varphi\mho_1. \tag{2.7}$$

Different structures manifolds are presented by equation (2.3) and (2.4) according to the values of r and α [16, 4, 27].

Value of r	Value of α	Manifolds type
$r = \pm 1$	$\alpha = -1 \quad or \quad \alpha = \pm i$	almost complex manifold
$r \neq 0$	$\alpha = 0$	almost tangent manifold
$r = 0$	$\alpha = +1$	almost product manifold
$r = 2$	$\alpha \neq 0$	GF-manifold [27] and π-structure

3. Semi-symmetric non-metric s-connection

An affine metric connection $\nabla^\sharp$ obeys

$$(\nabla^\sharp{}_{\mho_1}\varphi)\mho_2 = \eta(\mho_2)\mho_1 - \mathsf{g}(\mho_1, \mho_2)\zeta, \tag{3.8}$$

is said to be s-connection [25], where ζ is a 1-form and η is a vector field associated with ζ.

A s-connection $\nabla^\sharp$ is said to be a $SSNMs$-connection [24] if and only if

$$(\nabla^\sharp{}_{\mho_1}\varphi)\mho_2 = \nabla_{\mho_1}\mho_2 - \eta(\mho_2)\mho_1 - \mathsf{g}(\mho_1, \mho_2)\zeta. \tag{3.9}$$

Moreover,

$$(\nabla^{\sharp}{}_{\mho_1}\varphi)(\mho_2,\mho_3) = 2\eta(\mho_2)\mathsf{g}(\mho_1,\mho_3) + 2\eta(\mho_3)\mathsf{g}(\mho_1,\mho_2), \tag{3.10}$$

adopting (3.9), the torsion tensor $\mathcal{T}$ of $\mathcal{H}^n$ with respect to the connection $\nabla^{\sharp}$, is represented by

$$\mathcal{T}(\mho_1,\mho_2) = \eta(\mho_1)\mho_2 - \eta(\mho_2)\mho_1, \tag{3.11}$$

where

$$\mathcal{T}(\mho_1,\mho_2) = \nabla^{\sharp}_{\mho_1}\mho_2 - \nabla^{\sharp}_{\mho_2}\mho_1 - [\mho_1,\mho_2].$$

We know that $\mathsf{g}(\mho_1,\zeta) = \eta(\mho_1)$. In view of (3.10) as

$$\nabla^{\sharp}_{\mho_1}\mho_2 = \nabla_{\mho_1}\mho_2 + \mathcal{T}(\mho_1,\mho_2). \tag{3.12}$$

Then

$$\mathcal{T}(\mho_1,\mho_2) = -\eta(\mho_1)\mho_2 - \mathsf{g}(\mho_1,\mho_2)\zeta. \tag{3.13}$$

Let us explain

$$\mathcal{T}(\mho_1,\mho_2,\mho_3) = \mathsf{g}(\mathcal{T}(\mho_1,\mho_2),\mho_3). \tag{3.14}$$

In light of (3.13) and (3.14) we obtain

$$(\nabla^{\sharp}{}_{\mho_1}\eta) = (\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_2) + \mathsf{g}(\mho_1,\mho_2)\eta(\zeta). \tag{3.15}$$

4. H*su*-unified structure manifold with a SSNMs-connection

Analogous to the definition of curvature tensor of $\mathcal{H}^n$ with respect to the Levi-Civita connection ∇, we express the curvature tensor of $\mathcal{H}^n$ with a new SSNMs-connection $\nabla^{\sharp}$, is given as

$$\mathcal{R}^{\sharp}(\mho_1,\mho_2,\mho 3) = \nabla^{\sharp}_{\mho_1}\nabla^{\sharp}_{\mho_2}\mho_3 - \nabla^{\sharp}_{\mho_2}\nabla^{\sharp}_{\mho_1}\mho_3 - \nabla^{\sharp}_{[\mho_1,\mho_2]}\mho 3. \tag{4.16}$$

Therefore, adopting (2.7), (3.15) and 3.14), the curvature tensor $\mathcal{R}^{\sharp}$ with respect to the SSNMs-connection $\nabla^{\sharp}$ is given by

$$\begin{aligned}\mathcal{R}^{\sharp}(\mho_1,\mho_2,\mho 3) = \mathcal{R}(\mho_1,\mho_2,\mho 3) &- [(\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_3) + \mathsf{g}(\mho_1,\mho_3)\eta(\zeta)]\mho_2 \\ &+[(\nabla_{\mho_2}\eta)\mho_3 + \eta(\mho_2)\eta(\mho_3) + \mathsf{g}(\mho_2,\mho_3)\eta(\zeta)]\mho_1 \\ &-[\nabla_{\mho_1}\zeta - \eta(\mho_1)\zeta]\mathsf{g}(\mho_2,\mho_3) + +\mathsf{g}(\mho_1,\mho_3)[\nabla_{\mho_2}\zeta - \eta(\mho_2)\zeta)],\end{aligned} \tag{4.17}$$

where $\mathcal{R}$ is curvature tensor of $\mathcal{H}^n$ with respect to the Riemannian connection ∇.

Let $\mathcal{H}^n$ be an n-dimensional $\mathcal{HUS}$-manifolds. Then, with regard to the SSNMs-connection $\nabla^{\sharp}$, the scalar curvature R_{scal} and Ricci tensor $\mathcal{S}_{ric}$ of the manifold $\mathcal{H}^n$ are given as

$$\mathcal{S}^{\sharp}_{ric}(\mho_1,\mho_2) = \sum_{i=1}^{n}\varepsilon_i\mathsf{g}(\mathcal{R}^{\sharp}(I_i,\mho_1,\mho_2),I_i) \tag{4.18}$$

and

$$R^{\sharp}_{scal} = \sum_{i=1}^{n}\varepsilon_i\mathcal{S}^{\sharp}_{ric}(I_i,I_i), \tag{4.19}$$

where $\{I_1,I_2\cdots,I_n\}$ is an orthonormal frame with $\mathsf{g}(I_i,I_i)=\varepsilon_i$.

Now, we gain the next result:

Theorem 4.1. *If* $(\mathcal{H}^n, \varphi, \mathsf{g}, r, \alpha)$ *be a* n*-dimensional* $\mathcal{HUS}$*-manifolds, the Ricci tensor* $\mathcal{S}^{\sharp}_{ric}$ *and the scalar curvature* $R^{\sharp}_{scal}$ *endowed with a SSNMs-connection* $\nabla^{\sharp}$ *are given by*

$$\mathcal{S}^{\sharp}_{ric}(\mho_1, \mho_2) = \mathcal{S}_{ric}(\mho_1, \mho_2) + (n-1)B(\mho_1, \mho_2) + \alpha^r \mathsf{g}(\mho_1, \mho_2) + \mathsf{g}(\varphi\mho_1, \mho_2) - \eta(\mho_1)\eta(\mho_2), \tag{4.20}$$

where $B(\mho_1, \mho_2) = (\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_2) + \mathsf{g}(\mho_1, \mho_2)\eta(\zeta)$.

$$R^{\sharp}_{scal} = R_{scal} + \alpha^r (n-1)(n+2). \tag{4.21}$$

where $\mathcal{S}_{ric}$ *and* R_{scal} *denotes the Ricci tensor and scalar curvature with respect to the Levi-Civita connection* ∇*, respectively.*

Proof. In the light of (4.17), (4.18) and (4.19), easily we obtain (4.20) and (4.21). □

5. Wintgen inequalities for submanifolds in Hsu-unified structure manifold with a $SSNMs$-connection

The generalized Wintgen inequalities for submanifolds in a $\mathcal{HUS}$-manifolds with a $SSNMs$-connection are derived in this section.

Let $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional $\mathcal{HUS}$-manifolds with a $SSNMs$-connection and induced metric g. Let $\nabla^{\sharp}$ and $\nabla^{\perp}$ represent the induced connections on the tangent bundle $T\mathcal{H}'$ and $T\mathcal{H}'^{\perp}$ of $\mathcal{H}'$, respectively and indicate the second fundamental form of $\mathcal{H}'$ by $\hbar$, for all $\mho_1, \mho_2 \in \Gamma(T\mathcal{H}')$ and $\mathbb{N} \in \Gamma(T^{\perp}\mathcal{H}')$.

Recall the Gauss and Weingarten formulas by

$$\nabla^{\sharp}_{\mho_1}\mho_2 = \nabla_{\mho_1}\mho_2 + \hbar(\mho_1, \mho_2), \tag{5.22}$$

and

$$\nabla^{\sharp}_{\mho_1}\mathbb{N} = -\Lambda_{\mathbb{N}}\mho_1 + \nabla^{\perp}_{\mho_1}\mathbb{N}, \tag{5.23}$$

where the shape operator of $\mathcal{H}'$ with respect to $\mathcal{N}$ is notated using $\Lambda_{\mathbb{N}}$. The equation that follows is widely recognized.

$$\mathsf{g}(\Lambda_{\mathbb{N}}\mho_1, \mho_2) = \mathsf{g}(\hbar(\mho_1, \mho_2), \mathbb{N}), \qquad \forall\ \mho_1, \mho_2 \in \Gamma(T\mathcal{H}'), \qquad \mathbb{N} \in \Gamma(T^{\perp}\mathcal{H}'). \tag{5.24}$$

Now, let the induced SSNMs-connection $\nabla^{\sharp}$ on $\mathcal{H}'$ indicated by $\nabla^{\sharp^{\mathcal{H}'}}$ and the induced Levi-Civita connection ∇ on $\mathcal{H}$ identified by $\nabla^{\mathcal{H}}$. Next, by the equation of Gauss

$$\mathcal{R}^{\sharp}(\mho_1, \mho_2, \mho_3, \mho_4) = \mathcal{R}(\mho_1, \mho_2, \mho_3, \mho_4) - \mathsf{g}(\hbar(\mho_1, \mho_4), \hbar(\mho_3, \mho_3)) + \mathsf{g}(\hbar(\mho_1, \mho_3), \hbar(\mho_2, \mho_4)) \tag{5.25}$$

for all $\mho_1, \mho_2, \mho_3, \mho_4 \in T\mathcal{H}'$ and $\hbar$ is the second fundamental form of $\mathcal{H}'$ in $\mathcal{H}$ with respect to the $\nabla^{\sharp}$.

Let $\{I_1, \ldots, I_m\}$ and $\{I_{m+1}, \ldots, I_n\}$ indicated a local orthonormal tangent frame of the tangent bundle $T\mathcal{H}'$ of $\mathcal{H}'$ and a local orthonormal normal frame of the normal bundle $T^{\perp}\mathcal{H}'$ of $\mathcal{H}'$ in $\mathcal{H}$.

The mean curvature vector Π of $\mathcal{H}'$ should be defined through

$$\Pi = \sum_{i=1}^{m} \frac{1}{m} \hbar(I_i, I_i) \tag{5.26}$$

as well as the squared norm of the second fundamental form by

$$||\hbar||^2 = \sum_{i,j=1}^{m} \mathtt{g}\big(\hbar(I_i, I_j), \hbar(I_i, I_j)\big)^2. \tag{5.27}$$

At $p \in \mathcal{H}'$, we express the scalar curvature R_{scal} as

$$R_{scal} = \sum_{1 \le i < j \le m} \mathcal{R}(I_i, I_j, I_j, I_i) \tag{5.28}$$

and specify the normalized scalar curvature Ω of $\mathcal{H}'$ by

$$\Omega = \frac{2R_{scal}}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1 \le i < j \le m} \mathbb{K}(I_i \wedge I_j) \tag{5.29}$$

where $\mathbb{K}$ is the sectional curvature function on $\mathcal{H}'$.

The following expression defines the scalar normal curvature $\mathbb{K}_{\mathbb{N}}$ in terms of the second fundamental form's components.[21]

$$\mathbb{K}_{\mathbb{N}} = \sum_{1 \le i < j \le m} \sum_{1 \le r < s \le n} (\sum_{k=1}^{m} \hbar^r_{jk} \hbar^s_{ik} - \hbar^r_{ik} \hbar^s_{jk})^2. \tag{5.30}$$

Additionally, the normalized scalar normal curvature has the following relationship.

$$\Omega_{Nor} = \frac{2}{m(m-1)} \sqrt{\mathbb{K}_{\mathbb{N}}}. \tag{5.31}$$

Now, we demonstrate the generalized Wintgen inequality for submanifolds of Hsu-unified structure manifold which possess an SSNMs connection.

Theorem 5.1. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \le \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m}. \tag{5.32}$$

Proof. Assume that $\{I_1, \ldots, I_m\}$ and $\{I_{m+1}, \ldots, I_n\}$ denotes the local orthonormal tangent frame and local orthonormal normal frame on $\mathcal{H}'$ respectively. Then, from (5.25) and Gauss equation, we have

$$\begin{aligned} \sum_{1 \le i < j \le m} \mathcal{R}^\sharp(I_i, I_j, I_j, I_i) \quad &= \quad R_{scal} + \alpha^r(m-1)(m+2) \\ &+ \sum_{r=m+1}^{n} \sum_{1 \le i < j \le m} \Big[h^r_{ii} h^r_{jj} - (h^r_{ij})^2 \Big]. \end{aligned} \tag{5.33}$$

Also

$$2\mathcal{R}^\sharp_{scal} = \sum_{1 \le i < j \le m} \mathcal{R}^\sharp(I_i, I_j, I_j, I_i). \tag{5.34}$$

Using (5.33) and (5.34), we obtain

$$2\mathcal{R}^{\sharp}_{scal} = R_{scal} + \alpha^{r}(m-1)(m+2) + \sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}\Big[h^{r}_{ii}h^{r}_{jj} - (h^{r}_{ij})^{2}\Big]. \quad (5.35)$$

We also note that

$$m^{2}||\mathbb{H}||^{2} = \sum_{r=m+1}^{n}\Big(\sum_{i=1}^{m}h^{r}_{ii}\Big)^{2} = \frac{1}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(h^{r}_{ii}-h^{r}_{jj})^{2} + \frac{2m}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}h^{r}_{ii}h^{r}_{jj}. \quad (5.36)$$

In [20], Lu proved the Normal Scalar Curvature Conjecture and the Böttcher-Wenzel Conjecture. He also develops a new Bochner formula and uses the results to establish new pinching theorems for minimal submanifolds in spheres.
Therefore, from (page 6, Lemma 1, [20]) it is known

$$\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(\hbar^{r}_{ii}-\hbar^{r}_{jj})^{2} + 2m\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(\hbar^{r}_{ij})^{2} \geq 2m\Big[\sum_{m+1\leq r<s\leq n}\sum_{1\leq i<j\leq m}\big(\sum_{k=1}^{m}(\hbar^{r}_{jk}\hbar^{s}_{ik}-\hbar^{r}_{ik}\hbar^{s}_{jk})\big)^{2}\Big]^{\frac{1}{2}} \quad (5.37)$$

Thanks to (5.36), (5.37) and (5.30), we have

$$m^{2}||\mathbb{H}||^{2} - m^{2}\Omega_{Nor} \geq \frac{2m}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}[\hbar^{r}_{ii}\hbar^{r}_{jj} - (\hbar^{r}_{ij})^{2}]. \quad (5.38)$$

Hence, taking view of (5.31), (5.35) and (5.38), we find

$$\Omega_{Nor} - ||\mathbb{H}||^{2} \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^{r}(m+2)}{m} - \Omega$$

whereby proving the inequality (5.32). □

Theorem 5.2. *If $\mathcal{H}'$ be m-dimensional submanifold of a n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection. The equality holds if and only if the shape operators of $\mathcal{H}'$ in $\mathcal{H}$ take the following forms with the suitable orthonormal frames*

$$\{b_{1}\cdots b_{m}\} \quad \textit{and} \quad \{b_{m+1}\cdots b_{n}, b_{n+1}=\zeta\}$$

$$\Lambda_{b_{m}+1} = \begin{pmatrix} \theta_{1} & \delta & 0 & \dots & 0 \\ \delta & \theta_{1} & 0 & \dots & 0 \\ 0 & 0 & \theta_{1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_{1} \end{pmatrix}, \quad (5.39)$$

$$\Lambda_{b_m+2} = \begin{pmatrix} \theta_2+\delta & 0 & 0 & \dots & 0 \\ 0 & \theta_2-\delta & 0 & \dots & 0 \\ 0 & 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_2 \end{pmatrix}, \tag{5.40}$$

$$\Lambda_{b_m+3} = \begin{pmatrix} \theta_3 & 0 & 0 & \dots & 0 \\ 0 & \theta_3 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_3 \end{pmatrix}, \quad \Lambda_{b_m+4} = \dots = \Lambda_{b_{2m}} = \Lambda_{b_{2m+1}} = 0, \tag{5.41}$$

where $\theta_1, \theta_2, \theta_3$, and δ are real numbers.

Example 5.1. [6] *Totally umbilical submanifolds of real space forms, trivially realize the equality in (1.2). For a classification of totally umbilical submanifolds in real space forms.*

Example 5.2. *If M^2 is a surface in a real space form $N^4(c)$ with an ellipse of curvature a circle, then the equality is realized in (1.2)* [8].

Example 5.3. *A special case of the surfaces in Example 5.2 are the superminimal (i.e. minimal and ellipse of curvature a circle) surfaces in R^4. Also a cylinder on a superminimal surface in R^4 satisfies equality in (1.2)* [8].

As a consequence of Theorem 5.1 and the above table, we give the following results:

Corollary 5.1. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost complex manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} - \frac{(m+2)}{m}. \tag{5.42}$$

Or

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)i}{m}. \tag{5.43}$$

Corollary 5.2. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost tangent manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)}. \tag{5.44}$$

Corollary 5.3. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost product manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m}. \tag{5.45}$$

Corollary 5.4. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional GF-manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^2(m+2)}{m}. \tag{5.46}$$

Remark 5.1. *In fact for particular values of r and α we can turn up the similar equality condition for almost complex manifold, almost tangent manifold, almost product manifold and GF-manifold and π-structure manifolds, respectively.*

Again, in view of Theorem 5.1, we can turn up the following inequalities for the totally umbilical submanifolds as well.

Theorem 5.3. *If $\mathcal{H}'$ be m-dimensional totally umbilical submanifold of n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m}. \tag{5.47}$$

Moreover, we gain the same inequality for the various manifolds in the following table:

Value of r and α	Manifolds type	Type of Inequality
$r = \pm 1$, $\alpha = -1$ or $\alpha = \pm i$	almost complex manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)i}{m}$
$r \neq 0, \alpha = 0,$	almost tangent manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)}$
$r = 0$, $\alpha = +1$	almost product manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m}$
$r = 2$, $\alpha \neq 0$	GF-manifold [27] and π-structure	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^2(m+2)}{m}$

6. Some Implications

In this section, we use universal lower bounds of the norms of specific functions involving scalar curvature, normalized scalar curvature, and mean curvature to establish topological obstructions for be m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection. The Betti numbers are used to express these barriers.

By extending Wintgen's inequality in the previously given form, Guadalupe and Rodriguez [12] created an inequality for compact surfaces in a simple way. This inequality establishes a connection between the integral of $c + \mathbb{H}^2 - |\Omega_{Nor}|$ and the Euler-Poincar´e characteristic of the surface.

In (Theorem 1, [26]), Onti et al. used universal lower limits on the $L^{m/2}$-norms of various functions involving Ω, $\mathbb{H}$, and $\Omega^{\perp}$ to deduce several topological obstacles for compact m-dimensional submanifolds. The Betti numbers are used to describe these barriers. $\mathcal{H}^{m+n}(c)$ for compact submanifolds into space, where $c \geq 0$. It is assumed that every manifold examined in this work is oriented, connected without limits, and has an i-th Betti number $B_i(\mathcal{H}^m; \mathbb{F})$ over an arbitrary coefficient field $\mathbb{F}$.

Onti et al. [26], apply for any $1 \leq k \leq n-1$, but it generally fails for $k = n$, where the involved norm vanishes precisely for Wintgen ideal submanifolds. They demonstrated this by providing a method of constructing new compact 3-dimensional minimal Wintgen ideal submanifolds in even-dimensional spheres. Specifically, they proved that such submanifolds exist in $\mathbb{S}^6$ with arbitrarily large first Betti number.

Remark 6.1. *Let m -diemnsional Riemannian manifold M^m denote by $\lambda_1 \leq \cdots \leq \lambda_m$ the eigenvalues of the normalized Ricci tensor at each point. In view of above facts Onti et al.*

[26] *proved an inequality for given integers* $m \geq 3$ *and* $n \geq 1$, *there exists for every* $\lambda \in [0, 1)$ *a positive constant* $\varepsilon_\lambda(m, n)$, *depending only on* m *and* n, *such that if* M^m *is a compact subamnifolds of a Riemannian manifold* $M^{m+n}(c)$, *then*

$$\int_{M^m} (\mathbb{H}^2 + c - \lambda\Omega_{Nor} - \Omega)^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n) \sum_{i=1}^{m-1} B_i(M^m; \mathbb{F}) \tag{6.48}$$

for any coefficient field $\mathbb{F}$.

Moreover, Let M be homeomorphic to the n-sphere $\mathbb{S}^n$. If $f : M \longrightarrow \mathbb{S}^n$ is a minimal immersion with trivial normal bundle, then f is totally geodesic.

Proof. The following outcome is obtained by combining (5.32), (6.48), Theorem 5.1, and the aforementioned Remark 6.1. □

Theorem 6.1. *Given integers* $m \geq 3$ *and* $n \geq 1$, *there exists for every* $\lambda \in [0, 1)$ *a positive constant* $\varepsilon_\lambda(m, n)$, *depending only on* m *and* n, *such that if* m*-dimensional submanifold* $\mathcal{H}'$ *of an* n*-dimensional* $\mathcal{HUS}$*-manifolds* $\mathcal{H}$ *with a* $SSNMs$*-connection. Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n) \sum_{i=1}^{m-1} B_i(M^m; \mathbb{F}). \tag{6.49}$$

In particular, $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or or it is an Eells-Kuiper manifold if

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n). \tag{6.50}$$

Proof. Using (6.48) and (5.32), we easily get the (6.50). □

In particular, tangent manifolds $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold

Corollary 6.1. *Given integers* $m \geq 3$ *and* $n \geq 1$, *there exists for every* $\lambda \in [0, 1)$ *a positive constant* $\varepsilon_\lambda(m, n)$, *depending only on* m *and* n, *such that if* m*-dimensional submanifold* $\mathcal{H}'$ *of an* n*-dimensional almost complex manifold* $\mathcal{H}$ *with a* $SSNMs$*-connection . Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n) \sum_{i=1}^{m-1} B_i(M^m; \mathbb{F}). \tag{6.51}$$

In particular, almost complex manifold $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or or it is an Eells-Kuiper manifold, then we have

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n). \tag{6.52}$$

Corollary 6.2. *Given integers* $m \geq 3$ *and* $n \geq 1$, *there exists for every* $\lambda \in [0, 1)$ *a positive constant* $\varepsilon_\lambda(m, n)$, *depending only on* m *and* n, *such that if* m*-dimensional submanifold* $\mathcal{H}'$ *of an* n*-dimensional almost tangent manifold* $\mathcal{H}$ *with a* $SSNMs$*-connection . Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m, n) \sum_{i=1}^{m-1} B_i(M^m; \mathbb{F}). \tag{6.53}$$

If an almost tangent manifold $\mathcal{H}$ with a $SSNMs$-connection s homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.54}$$

Corollary 6.3. *Given integers $m\geq 3$ and $n\geq 1$, there exists for every $\lambda\in[0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional almost product manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n)\sum_{i=1}^{m-1}B_i(M^m;\mathbb{F}). \tag{6.55}$$

If almost product manifold $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.56}$$

Corollary 6.4. *Given integers $m\geq 3$ and $n\geq 1$, there exists for every $\lambda\in[0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional GF-manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{\alpha^2(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n)\sum_{i=1}^{m-1}B_i(M^m;\mathbb{F}). \tag{6.57}$$

In particular, GF-manifold $\mathcal{H}^m$ is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then we have

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}\frac{\alpha^2(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.58}$$

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