International Journal of Maps in Mathematics

Volume 8, Issue 2, 2025, Pages:769-790

E-ISSN: 2636-7467

www.simadp.com/journalmim

FAULHABER-TYPE FORMULAS FOR THE SUMS OF POWERS OF ARITHMETIC SEQUENCES

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Abstract. In this article, we derive explicit formulas for computing the sums of powers in arithmetic sequences. We begin with a historical odyssey, tracing the contributions of some of the world's most influential mathematicians whose work has shaped and inspired our approach. We then present two distinct Faulhaber-type formulas–one involving Bernoulli numbers and closely resembling the classical formula for sums of powers of integers. To establish these results, we employ two different techniques: the first is based on the principle of invariance, while the second uses the differencing operator applied to polynomials. Although the methods differ in form, we emphasize that they share the similar computational complexity, a point we demonstrate with illustrative examples at the end.

Keywords: Faulhaber-type formulas, Bernoulli numbers, Principle of invariance, Differencing operator.

2020 Mathematics Subject Classification: 11B25, 11B68, 11B75, 11Y35, 11Y55.

1. Introduction

According to legend, a precocious primary school student by the name of Carl Friedrich Gauss surprised his teacher by calculating

$$S = 1 + 2 + 3 + \cdots + 100,$$

Received: 2025.05.22 *Revised: 2025.08.20* *Accepted: 2025.08.26*

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almost without effort. His trick was to consider the sum twice, once adding in ascending order, and once in descending order, then adding them side-by-side. In modern notation,

$$2S = \sum_{n=1}^{100} n + \sum_{n=1}^{100} (101 - n) = \sum_{n=1}^{100} 101 = 100 \times 101 = 10,100.$$

Once Gauss had calculated twice the sum he wanted, the only thing left was to divide his result by 2, and voila! He had obtained $S = 5,050$. Extending his technique to compute the sum of the first N positive integers, the general formula

$$\sum_{n=1}^{N} n = 1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

is easily obtained. The elegance of Gauss's technique is made more evident when we consider that it can be used to compute the sum of *any* finite arithmetic sequence $\{a, a+r, \dots, a+Nr\}$. Indeed, we obtain

$$\sum_{n=0}^{N} (a + rn) = \frac{(2a + rN)(N+1)}{2}.$$

Despite the brilliance of young Gauss's tenacious tallying, the formulas above were known at least 2000 years before his birth! In fact, formulas for the sum of squares and the sum of cubes,

$$\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6},$$
$$\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4},$$

respectively, were known in antiquity [20, 27, 19, 2, 18, 17, 3]. The former was described by the legendary Greek polymath Archimedes in his work *On Conoids and Spheroids* [1, 6], and the latter is famously attributed to Nicomachus of Gerasa, another well-known Greek mathematician, who published the result in his *Introduction to Arithmetic* [2, 5]. Due to the nature of ancient Greek mathematics, these proofs are geometric in nature, and certainly valuable from both a mathematical and a historical standpoint. The interested reader can find more information in David M. Burton's historical treatise [2].

A formula for the sum of the fourth powers of the first N natural numbers, $\sum n^4$, is described by Pierre de Fermat in letters written in 1636 to Gilles Persone de Roberval and Marin Mersenne, all of whom were prominent mathematicians at the time [2]. It is possible that Fermat was under the impression that such a formula was unknown at the time [27]. Fermat's belief notwithstanding, formulas for the sums of powers of integers up to the 17th power, and even as high as the 23rd power, were known to the German mathematician Johann Faulhaber as late as 1631 [27, 8, 21]. Arguably, Faulhaber's greatest contribution was the discovery of the fact that, in the case of an odd exponent k, the sum $\sum_{n=1}^{N} n^k$ is a polynomial in terms of the variable $a = \frac{N(N+1)}{2}$ and, moreover, for odd $k > 1$ the polynomial is divisible by a^2. He also described a method to obtain a formula for the sum $\sum n^{2k}$ once the formula for the sum $\sum n^{2k+1}$ is known [27, 26, 8, 21, 23, 16]. It is not clear whether Faulhaber knew how to prove his assertions in generality; in his day, mathematical discoveries were usually kept secret, given as challenges to other mathematicians, or intentionally written in code! [27, 15, 21]

A proof of the general explicit formula for the sum $\sum n^k$, and its rigorous verification, would have to wait until 1834, with the publication of Carl Jacobi's paper *De usu legitimo formulae summatoriae Maclaurinianae* [9]. Jacobi's formula incorporates the Bernoulli numbers, which is the sequence of rational numbers $\{B_N\}_{N=0}^{\infty}$ given recursively by the formula [1]

$$B_0 = 1, \qquad \sum_{n=0}^{N} \binom{N+1}{n} B_n = 0, \quad N \geq 1. \tag{1.1}$$

The first few nonzero Bernoulli numbers are given in Table 1.1 below. Note that for all odd $N > 1$, $B_N = 0$.

TABLE 1.1. Bernoulli numbers.

N	0	1	2	4	6	8	10	12
B_N	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$

The numbers B_N can also be defined explicitly by the formula

$$B_N = \sum_{n=0}^{N} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^N}{n+1}, \quad \text{where } 0^0 \text{ and } \binom{0}{0} \text{ are both taken to be 1.}$$

Though this will not be necessary for our discussion, it is still an important observation, since any explicit formula for calculating sums of powers of integers which utilizes the Bernoulli numbers would be easily stymied by requiring the Bernoulli numbers to be calculated recursively. We will refer to the Bernoulli numbers again in Section 2.2.

The Bernoulli numbers were discovered independently by the Swiss mathematician Jakob Bernoulli, as well as the Japanese mathematician Seki Takakazu [2, 11, 10]. Specifically,

[1] There are two common conventions for defining the Bernoulli numbers, but only the sign of B_1 is affected by the choice. We use the convention in which $B_1 = -\frac{1}{2}$.

Bernoulli discovered the sequence while attempting to derive formulas for the sums of powers of integers. As an interesting side note, both Bernoulli's and Takakazu's discoveries were published posthumously, the former in 1713 and the latter a year earlier in 1712 [11, 10]. Armed with the Bernoulli numbers, Jacobi was able to rigorously verify Faulhaber's formula (often referred to as Bernoulli's formula) [9],

$$\sum_{n=1}^{N} n^k = \frac{N+1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \tag{1.2}$$

Our goal is to find a Faulhaber-type formula for sums of powers of finite arithmetic sequences, i.e., sums of the form

$$S_{N,k}[a,r] := \sum_{n=0}^{N} (a+rn)^k,$$

where a and r are arbitrary real numbers, and k and N are nonnegative integers. Using the binomial theorem and changing the order of summation yields

$$\begin{aligned} S_{N,k}[a,r] &= \sum_{n=0}^{N} (a+nr)^k \\ &= \sum_{n=0}^{N} \sum_{m=0}^{k} \binom{k}{m} a^{k-m} n^m r^m \\ &= \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m \sum_{n=0}^{N} n^m \\ &= \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m S_{N,m}[0,1]. \end{aligned} \tag{1.3}$$

The aforementioned formula provides a straightforward recursive approach for calculating $S_{N,k}[a,r]$, contingent upon having knowledge of the sums $S_{N,m}[0,1]$. If the values of these sums are unknown, this method can become quite laborious. As an example, let us use the formula above to find closed formulas for the first few sums $S_{N,k}[a,r]$. Note that

$$S_{N,0}[0,1] = N+1,$$

and that

$$\begin{aligned}
S_{N,1}[a,r] &= (N+1)a + \frac{N(N+1)}{2}r, \\
S_{N,2}[a,r] &= (N+1)a^2 + 2\frac{N(N+1)}{2}ar + \frac{N(N+1)(2N+1)}{6}r^2, \\
S_{N,3}[a,r] &= (N+1)a^3 + 3\frac{N(N+1)}{2}a^2r + 3\frac{N(N+1)(2N+1)}{6}ar^2 \\
&\qquad + \frac{N^2(N+1)^2}{4}r^3.
\end{aligned}$$

The next important development in our story comes courtesy of Blaise Pascal [27, 2]. In his famous work *Treatise on the Arithmetic Triangle* [12], the author outlines a formula for the sums of powers of terms in an arithmetic sequence. He illustrates his process via the example

$$P = 5^3 + 8^3 + 11^3 + 14^3 = 4712,$$

and claims that the process can be generalized. In fact, his approach is similar to the one we will begin with in Section 2, before using it to derive our explicit formula. First, Pascal showed (using techniques he had developed earlier in the paper) that

$$(t+3)^4 - t^4 = 12t^3 + 54t^2 + 108t + 81.$$

He then substituted $t = 5$, $t = 8$, $t = 11$, and $t = 14$ into this identity and added the results, noticing that the left side "telescopes." At this point, we are left with

$$\begin{aligned}
17^4 - 5^4 = {}& 12(5^3 + 8^3 + 11^3 + 14^3) + 54(5^2 + 8^2 + 11^2 + 14^2) \\
& + 108(5 + 8 + 11 + 14) + 81(1 + 1 + 1 + 1),
\end{aligned}$$

from which we compute

$$17^4 - 5^4 = 12P + 54(406) + 108(38) + 81(4).$$

Finally, solving for P gives $P = 4712$ as expected.

More generally, Pascal recognized that on one hand, the sum

$$\sum_{n=1}^{N} \left[(a + (n+1)r)^{k+1} - (a + nr)^{k+1} \right]$$

"telescopes" to the expression $[a + (N+1)r]^{k+1} - a^{k+1}$, while on the other hand, it can be expanded (using the entries from his eponymous triangle) into a combination of sums with exponents lower than $k+1$. At this point, he solved for $S_{N,k}[a,r]$ in terms of the sums of lower powers, i.e., the sums $S_{n,0}[a,r]$, $S_{n,1}[a,r], \ldots,$ and $S_{n,k-1}[a,r]$.

Observe that Pascal's method is recursive in nature, requiring the formulas for the sums of all lower powers in order to compute the desired sum. This makes the method easy to describe, but impractical to use directly for larger numbers of terms or for higher powers. However, this recursive approach is the first step to obtaining our Faulhaber-type formulas, which do not rely on recursion. These formulas exhibit a remarkable level of effectiveness, which is made apparent by their resilience and their ability to unify and generalize various formulas involving the summation of different powers of integers. This potent set of formulas demonstrates its versatility when applied to the computation of exceedingly high powers of arithmetic sums, as well as when dealing with a substantial number of terms within these sums of arithmetic sequences.

In Section 2, we derive an explicit formula to calculate $S_{N,k}[a, r]$ which is quite similar to Equation (1.2). We first obtain a recursive formula in the style of Pascal. Next, we develop a unique approach to proving Faulhaber's formula without induction. The crux of our proof relies on a clever argument in which we show that a particular expression, which we call $Q_{N,k}(j)$, is invariant with respect to j. It turns out that $\frac{N+1}{k+1}Q_{N,k}(0)$ and $\frac{N+1}{k+1}Q_{N,k}(k-1)$ are equal to the left- and right-hand sides of Equation (1.2), respectively, from which Faulhaber's formula follows immediately. We also provide a more traditional inductive proof of Equation (1.2). Finally, we combine Faulhaber's formula with our recursive formula, culminating in Theorem 2.3.

In Section 3, we use methods from the theory of finite differences to derive a somewhat different version of Faulhaber's formula which does not require any knowledge of the Bernoulli numbers. As in Section 2, we take inspiration from Pascal and consider a special telescoping sum; the notation of difference operators arises naturally as a result, and provides us with an alternate, yet equally robust, Faulhaber-type formula, stated explicitly in Theorem 3.1.

2. Faulhaber's Formula with Bernoulli Numbers

In what follows, N and k are always nonnegative integers, a and r are real numbers, and we use the conventions that $0^0 = 1$ and $\binom{0}{0} = 1$.

2.1. **Recursive Formulas.**

Recall that in Section 1 we introduced the notation

$$S_{N,k}[a, r] = \sum_{n=0}^{N} (a + nr)^k = a^k + (a + r)^k + \cdots + (a + Nr)^k.$$

Our first theorem is a recursive formula for $S_{N,k}[a, r]$ in terms of the sums $S_{N,m}[a, r]$ with lower exponents $m = 0, 1, \ldots, k - 1$. This is an important first step in finding an explicit formula for $S_{N,k}[a,r]$, and a generalization of Pascal's method, which we discussed in Section 1. Immediately thereafter, we provide as a corollary a recursive formula for the sums $S_{N,k}[0,1]$ in terms of the sums $S_{N,m}[0,1]$. We make extensive use of this corollary in our deductive proof of Faulhaber's formula for $S_{N,k}[0,1]$, as well as the Faulhaber-type formula for $S_{N,k}[a,r]$.

Theorem 2.1. *The sums $S_{N,k}[a,r]$ are given recursively in k by the formula*

$$S_{N,k}[a,r] = (N+1)a^k + \sum_{m=1}^{k} \binom{k}{m} \frac{r^m}{m+1} \left[(N+1)^{m+1}a^{k-m} - S_{N,k-m}[a,r]\right].$$

Proof. The result is trivial when $r=0$, so we prove it for nonzero r. We begin by considering the telescoping sum

$$T(N) := \sum_{n=0}^{N} \left([a+(n+1)r]^{k+1} - [a+nr]^{k+1}\right).$$

On one hand, if we first telescope $T(N)$, then use the binomial theorem to expand the expression $[a+(N+1)r]^{k+1}$, we obtain

$$\begin{aligned}
T(N) &= [a+(N+1)r]^{k+1} - a^{k+1} \\
&= \sum_{m=0}^{k+1} \left[\binom{k+1}{m}(N+1)^m r^m a^{k+1-m}\right] - a^{k+1} \\
&= \sum_{m=1}^{k+1} \binom{k+1}{m}(N+1)^m r^m a^{k+1-m} \\
&= \sum_{m=0}^{k} \binom{k+1}{m+1}(N+1)^{m+1} r^{m+1} a^{k-m} \\
&= (k+1)r \sum_{m=0}^{k} \binom{k}{m} \frac{(N+1)^{m+1} r^m}{m+1} a^{k-m}.
\end{aligned}$$

On the other hand, if we first distribute the sum in n, then use the binomial theorem to expand the expression $[a+(n+1)r]^{k+1} = [(a+nr)+r]^{k+1}$, we are left with

$$\begin{aligned}
T(N) &= \sum_{n=0}^{N}\left[\sum_{m=0}^{k+1}\binom{k+1}{m} r^m (a+nr)^{k+1-m}\right] - S_{N,k+1}[a,r] \\
&= \sum_{m=0}^{k+1}\left[r^m\binom{k+1}{m}\sum_{n=0}^{N}(a+nr)^{k+1-m}\right] - S_{N,k+1}[a,r] \\
&= \sum_{m=0}^{k+1}\left[\binom{k+1}{m} r^m S_{N,k+1-m}[a,r]\right] - S_{N,k+1}[a,r] \\
&= \sum_{m=1}^{k+1}\binom{k+1}{m} r^m S_{N,k+1-m}[a,r] \\
&= \sum_{m=0}^{k}\binom{k+1}{m+1} r^{m+1} S_{N,k-m}[a,r] \\
&= (k+1)r\sum_{m=0}^{k}\binom{k}{m}\frac{r^m}{m+1} S_{N,k-m}[a,r].
\end{aligned}$$

Equating the two expressions above and dividing by $(k+1)r$ gives us

$$\sum_{m=0}^{k}\binom{k}{m}\frac{(N+1)^{m+1}r^m}{m+1}a^{k-m} = \sum_{m=0}^{k}\binom{k}{m}\frac{r^m}{m+1}S_{N,k-m}[a,r],$$

from which we deduce

$$\begin{aligned}
S_{N,k}[a,r] &= \sum_{m=0}^{k}\binom{k}{m}\frac{(N+1)^{m+1}r^m}{m+1}a^{k-m} - \sum_{m=1}^{k}\binom{k}{m}\frac{r^m}{m+1}S_{N,k-m}[a,r] \\
&= (N+1)a^k + \sum_{m=1}^{k}\binom{k}{m}\frac{r^m}{m+1}\left((N+1)^{m+1}a^{k-m} - S_{N,k-m}[a,r]\right),
\end{aligned}$$

which is the desired result. □

Corollary 2.1. *The sums $S_{N,k}[0,1]$ are given recursively in k by the formula*

$$S_{N,k}[0,1] = \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1}\sum_{m=0}^{k-1}\binom{k+1}{m}S_{N,m}[0,1].$$

Proof. First, consider the mth term

$$\tau_m := \binom{k}{m}\frac{r^m}{m+1}(N+1)^{m+1}a^{k-m}$$

from the sum in Theorem 2.1. Observe that when $a=0$ and $r=1$,

$$\tau_m = \frac{(N+1)^{k+1}}{k+1}\,\delta_k(m),$$

where $\delta_k(m)$ is the Kroenecker delta. If we substitute the revised τ_m back into the sum from Theorem 2.1, then channel our "inner Gauss" and rewrite the sum in *descending* order, we conclude

$$\begin{aligned}
S_{N,k}[0,1] &= \frac{(N+1)^{k+1}}{k+1} - \sum_{m=1}^{k} \binom{k}{m} \frac{1}{m+1} S_{N,k-m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \sum_{m=0}^{k-1} \binom{k}{k-m} \frac{1}{k-m+1} S_{N,m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1} \sum_{m=0}^{k-1} \binom{k}{m} \frac{k+1}{k+1-m} S_{N,m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1} \sum_{m=0}^{k-1} \binom{k+1}{m} S_{N,m}[0,1].
\end{aligned}$$

This is precisely the desired result. □

Before moving on to the next section and demonstrating one of our main results, it is worth noting that Theorem 2.1 remains a labor-intensive method for directly computing the sums $S_{N,k}[a,r]$, yet it yields identical outcomes to those previously observed.

$$\begin{aligned}
S_{N,1}[a,r] &= (N+1)a + \frac{r}{2}((N+1)^2 - S_{N,0}[a,r]) \\
&= (N+1)a + \frac{r}{2}((N+1)^2 - (N+1)) \\
&= (N+1)a + \frac{N(N+1)}{2} r, \\
S_{N,2}[a,r] &= (N+1)a^2 + r((N+1)^2 a - S_{N,1}[a,r]) + \frac{r^2}{3}((N+1)^3 - S_{N,0}[a,r]) \\
&= (N+1)a^2 + r\left((N+1)^2 a - (N+1)a - \frac{N(N+1)}{2} r\right) \\
&\quad + \frac{r^2}{3}((N+1)^3 - (N+1)) \\
&= (N+1)a^2 + 2\frac{N(N+1)}{2} ar + \frac{N(N+1)(2N+1)}{6} r^2,
\end{aligned}$$

and, after much simplification which we leave to the reader,

$$\begin{aligned}
S_{N,3}[a,r] &= (N+1)a^3 + \frac{3r}{2}((N+1)^2 a^2 - S_{N,2}[a,r]) \\
&\quad + r^2((N+1)^3 a - S_{N,1}[a,r]) + \frac{r^3}{4}((N+1)^4 - S_{N,0}[a,r]) \\
&= (N+1)a^3 + 3\frac{N(N+1)}{2} a^2 r + 3\frac{N(N+1)(2N+1)}{6} ar^2 \\
&\quad + \frac{N^2(N+1)^2}{4} r^3.
\end{aligned}$$

As we can see, even for small k, direct use of Theorem 2.1 to calculate a formula for $S_{N,k}[a,r]$ is quite time-consuming. However, without an efficient algorithm for generating formulas for the sums $S_{N,k}[0,1]$, it remains our only option. This is precisely the issue we address in the next section, and unsurprisingly, it is Theorem 2.1 (or rather, its corollary) which we rely on most heavily to obtain our results.

2.2. **Connections with Bernoulli numbers.** In Section 1, we introduced the Bernoulli numbers $\{B_j\}$, which are a recursively defined sequence of rational numbers. For our purposes, we shall utilize the following reformulation of Equation (1.1):

$$B_0 = 1, \quad B_{j+1} = -\frac{1}{j+2}\sum_{n=0}^{j}\binom{j+2}{n}B_n, \quad j \geq 0. \tag{2.4}$$

In order to prove the main results for this section, we first establish two intermediate lemmas. The first of these can be regarded as an alternate recursive definition of the Bernoulli numbers. The second lemma invokes the first to demonstrate the invariance under j of a particular quantity, which we refer to as $Q_{N,k}(j)$, via repeated use of Corollary 2.1. Before embarking on this quest of quantification, we introduce a bit of notation to clean up future calculations. Let j, k, and m be integers, with $0 \leq j, m < k$, and define the sums

$$\Theta_{j,k}(m) := \sum_{n=0}^{j} B_n \binom{k+1}{n}\binom{k+1-n}{m},$$

and

$$Q_{N,k}(j) := \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m - \frac{1}{N+1}\sum_{m=0}^{k-j-1}\Theta_{j,k}(m)S_{N,m}[0,1].$$

Armed with this new notation, we are in position to establish the lemmas.

Lemma 2.1. *For any integers j and k satisfying $0 \leq j < k$, we have*

$$\Theta_{j,k}(k-j-1) = -(k-j)\binom{k+1}{j+1}B_{j+1}.$$

Proof. We use the well-known identities

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{k}\binom{n-k}{\ell-k} = \binom{n}{\ell}\binom{\ell}{k}, \quad \text{and} \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

along with the recursive definition of the Bernoulli numbers expressed in Equation (2.4), to establish the equality

$$\begin{aligned}\Theta_{j,k}(k-j-1) &= \sum_{n=0}^{j}\binom{k+1}{n}\binom{k+1-n}{j+2-n}B_n \\ &= \binom{k+1}{j+2}\sum_{n=0}^{j}\binom{j+2}{n}B_n \\ &= \frac{k-j}{j+2}\binom{k+1}{j+1}\sum_{n=0}^{j}\binom{j+2}{n}B_n \\ &= -(k-j)\binom{k+1}{j+1}B_{j+1},\end{aligned}$$

which is precisely the desired result. □

Lemma 2.2. *For all integers j, $0 \le j < k$,*

$$Q_{N,k}(j) = Q_{N,k}(j+1).$$

Proof. Before beginning, in order to prevent heads from unnecessarily spinning, we briefly outline the process. We first suitably partition $Q_{N,k}(j)$ into two terms $A(j)$ and $B(j)$. Following this, with surgical precision we will introduce an auxiliary term $C(j)$ which in turn is split into two more terms $D(j)$ and $E(j)$ so that $A(j) + D(j) = A(j+1)$ and $E(j) + B(j) - C(j) = B(j+1)$. In the end, we will have

$$\begin{aligned}Q_{N,k}(j) &= A(j) + B(j) \\ &= A(j) + C(j) + B(j) - C(j) \\ &= [A(j) + D(j)] + [E(j) + B(j) - C(j)] \\ &= A(j+1) + B(j+1) \\ &= Q_{N,k}(j+1).\end{aligned} \tag{2.5}$$

Now for the gory details. Let

$$\begin{aligned}A(j) &:= \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m, \\ B(j) &:= -\frac{1}{N+1}\sum_{m=0}^{k-j-1}\Theta_{j,k}(m)S_{N,m}[0,1],\end{aligned}$$

so that $Q_{N,k}(j) = A(j) + B(j)$. Next, we define $C(j)$ to be the last term in $B(j)$, which corresponds to $m = k - j - 1$. In other words,

$$C(j) := -\frac{1}{N+1}\Theta_{j,k}(k-j-1)S_{N,k-j-1}[0,1].$$

Using our result from Lemma 2.1, we reformulate $C(j)$ to be

$$C(j) = \frac{k-j}{N+1}\binom{k+1}{j+1}S_{N,k-j-1}[0,1]\, B_{j+1}. \tag{2.6}$$

Subsequently, we apply the recursive formula from Corollary 2.1 to the sum $S_{N,k-j-1}[0,1]$ and obtain

$$S_{N,k-j-1}[0,1] = \frac{1}{k-j}\left[(N+1)^{k-j} - \sum_{m=0}^{k-j-2}\binom{k-j}{m}S_{N,m}[0,1]\right]. \tag{2.7}$$

Inserting (2.7) into (2.6), we are left with the following decomposition of $C(j)$:

$$\begin{aligned} C(j) &=: D(j) + E(j) \\ &= \binom{k+1}{j+1}(N+1)^{k-(j+1)}B_{j+1} + \frac{-B_{j+1}}{N+1}\sum_{m=0}^{k-j-2}\binom{k-j}{m}\binom{k+1}{j+1}S_{N,m}[0,1]. \end{aligned}$$

Notice that adding $D(j)$ to $A(j)$ gives $A(j+1)$. Explicitly, we have

$$\begin{aligned} A(j) + D(j) &= \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m + \binom{k+1}{j+1}(N+1)^{k-(j+1)}B_{j+1} \\ &= \sum_{m=0}^{j+1}\binom{k+1}{m}(N+1)^{k-m}B_m \\ &= A(j+1). \end{aligned}$$

Furthermore, adding $E(j)$ to $B(j) - C(j)$ gives $B(j+1)$. Indeed,

$$\begin{aligned} E(j) + (B(j) - C(j)) &= \frac{-1}{N+1}B_{j+1}\sum_{m=0}^{k-j-2}\binom{k-j}{m}\binom{k+1}{j+1}S_{N,m}[0,1] + \\ &\quad + \frac{-1}{N+1}\sum_{m=0}^{k-j-2}\Theta_{j,k}(m)S_{N,m}[0,1] \\ &= \frac{-1}{N+1}\sum_{m=0}^{k-j-2}\left[\binom{k+1}{j+1}\binom{k+1-(j+1)}{m}B_{j+1} + \Theta_{j,k}(m)\right]S_{N,m}[0,1] \\ &= \frac{-1}{N+1}\sum_{m=0}^{k-(j+1)-1}\Theta_{j+1,k}(m)S_{N,m}[0,1] \\ &= B(j+1). \end{aligned}$$

This completes the proof as previously outlined in Equation (2.5). □

We are now ready to present two different proofs of Faulhaber's eponymous Formula. The first is a direct proof which, as far as the authors are aware, is a novel approach to the problem. Thanks to the work put into Lemmas 2.1 and 2.2, it is both short and elegant. The second proof invokes the principle of strong induction on k, and relies on the recursion formula of Corollary 2.1. Faulhaber's Formula is well-known, yet is instrumental in proving the first of our two main results, a Faulhaber-type formula for $S_{N,k}[a,r]$ which utilizes the Bernoulli numbers.

Theorem 2.2 (Faulhaber's Formula)**.** *For any $k \geq 0$,*

$$S_{N,k}[0,1] = \sum_{n=0}^{N} n^k = \frac{N+1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \tag{2.8}$$

Remark 2.1. *It is worth nothing that it is common to use N instead of $N+1$ in* (2.8) *when B_1 is taken to be $1/2$ instead of $-1/2$. That is, for any $k \geq 0$, if $B_1 = 1/2$, then*

$$S_{N,k}[0,1] = \sum_{n=0}^{N} n^k = \frac{1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} N^{k-m+1} B_m. \tag{2.9}$$

Direct Proof. On the one hand, the recursion formula of Corollary 2.1 is equivalent to the equation

$$Q_{N,k}(0) = \frac{k+1}{N+1} S_{N,k}[0,1].$$

On the other hand, using the recursive definition of the Bernoulli numbers from Equation (2.4), we observe that

$$\begin{aligned}
\Theta_{k-1,k}(0) &= \sum_{n=0}^{k-1} B_n \binom{k+1}{n} \binom{k+1-n}{0} \\
&= \sum_{n=0}^{k-1} B_n \binom{k+1}{n} \\
&= -(k+1)B_k \\
&= -\binom{k+1}{k} (N+1)^{k-k} B_k.
\end{aligned}$$

Consequently, after recalling that $S_{N,0}[0,1] = N+1$, we arrive at

$$\begin{aligned} Q_{N,k}(k-1) &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m - \frac{1}{N+1} \sum_{m=0}^{k-(k-1)-1} \Theta_{k-1,k}(m) S_{N,m}[0,1] \\ &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m - \frac{1}{N+1} \Theta_{k-1,k}(0)\, S_{N,0}[0,1] \\ &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m + \binom{k+1}{k} (N+1)^{k-k} B_k \\ &= \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \end{aligned}$$

Finally, by the invariance of $Q_{N,k}(j)$ under j, which we painstakingly established in Lemma 2.2, we conclude

$$Q_{N,k}(0) = Q_{N,k}(k-1),$$

from which the desired result follows immediately. □

And now we present our strong induction proof of Theorem 2.2.

Strong Induction Proof. In the base case where $k = 0$, we know $S_{N,0}[0,1] = (N+1)$. Keeping in mind that $B_0 = 1$, the right-hand side of Equation (2.8) is

$$\frac{N+1}{0+1} \binom{0+1}{0} (N+1)^{0-0} B_0 = N+1,$$

hence the base case is proved. Next, suppose that for some $K > 0$ Equation (2.8) holds whenever $0 \leq k \leq K-1$. We will show that the result also holds when $k = K$. Indeed, by Corollary 2.1, we have

$$S_{N,K}[0,1] = \frac{(N+1)^{K+1}}{K+1} - \frac{1}{K+1} \sum_{m=0}^{K-1} \binom{K+1}{m} S_{N,m}[0,1]. \tag{2.10}$$

We now use the induction hypothesis, reverse the order of summation a la Gauss, and utilize the identity

$$\frac{1}{m+\ell+1} \binom{K+1}{m+\ell} \binom{m+\ell+1}{m} = \frac{1}{\ell+1} \binom{K+1}{\ell} \binom{K-\ell+1}{m},$$

to compute

$$\begin{aligned}\sum_{m=0}^{K-1}\binom{K+1}{m}S_{N,m}[0,1] &= \sum_{m=0}^{K-1}\binom{K+1}{m}\left[\frac{N+1}{m+1}\sum_{\ell=0}^{m}\binom{m+1}{\ell}(N+1)^{m-\ell}B_\ell\right]\\ &= \sum_{m=0}^{K-1}\binom{K+1}{m}\left[\frac{N+1}{m+1}\sum_{\ell=0}^{m}\binom{m+1}{m-\ell}(N+1)^{\ell}B_{m-\ell}\right]\\ &= \sum_{\ell=0}^{K-1}(N+1)^{\ell+1}\sum_{m=\ell}^{K-1}\frac{1}{m+1}\binom{K+1}{m}\binom{m+1}{m-\ell}B_{m-\ell}\\ &= \sum_{\ell=0}^{K-1}(N+1)^{\ell+1}\sum_{m=0}^{K-\ell-1}\frac{1}{m+\ell+1}\binom{K+1}{m+\ell}\binom{m+\ell+1}{m}B_m\\ &= \sum_{\ell=0}^{K-1}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}\sum_{m=0}^{K-\ell-1}\binom{K-\ell+1}{m}B_m.\end{aligned}$$

At this stage, we use the definition of the Bernoulli numbers from Equation (2.4) to write

$$\sum_{m=0}^{K-1}\binom{K+1}{m}S_{N,m}[0,1] = -\sum_{\ell=0}^{K-1}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}(K-\ell+1)B_{K-\ell}.$$

Additionally, multiplication by a clever choice of 1 yields

$$(N+1)^{K+1} = \binom{K+1}{K}\frac{(N+1)^{K+1}}{K+1}(K-K+1)B_{K-K},$$

which, when substituted into Equation (2.10) above, enables us to conclude

$$\begin{aligned}S_{N,K}[0,1] &= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}(K-\ell+1)B_{K-\ell}\\ &= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{K-\ell}\frac{(N+1)^{K-\ell+1}}{K-\ell+1}(\ell+1)B_\ell\\ &= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{\ell}(N+1)^{K-\ell+1}B_\ell\\ &= \frac{N+1}{K+1}\sum_{m=0}^{K}\binom{K+1}{m}(N+1)^{K-m}B_m.\end{aligned}$$

This completes the induction proof. □

Our main result for this paper is a byproduct of all we have done thus far, but especially Equation (1.3) and Theorem 2.2.

Theorem 2.3 (Faulhaber's Formula for $S_{N,k}[a,r]$)**.** *For all $a, r \in \mathbb{R}$, we have*

$$S_{N,k}[a,r] = \sum_{m=0}^{k}\binom{k}{m}\frac{a^{k-m}r^m}{m+1}\left[\sum_{j=0}^{m}\binom{m+1}{j}(N+1)^{m-j+1}B_j\right].$$

Proof. The result follows from Equation (1.3), which states

$$S_{N,k}[a,r] = \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m S_{N,m}[0,1],$$

along with a straightforward use of Theorem 2.2 to replace the expression $S_{N,m}[0,1]$. □

There are a number of alternative formulations and proofs of Faulhaber's formula. They are proven using tools which include, but are not limited to, generating functions, matrix techniques, Bernoulli polynomials, the Stirling numbers of the first and second kinds, and finite discrete convolutions [21, 14, 23]. In this section, we have explored Faulhaber's Formula through the use of the Bernoulli numbers. In what follows, we turn our attention to the versatile theory of finite differences, which is applied extensively in fields as far-flung as statistics, combinatorics, numerical analysis, differential equations, and even particle physics. In particular, we make use of the finite discrete difference operator, which we introduce in the next section.

3. Faulhaber's Formula via Differencing

We have already seen that the sums of powers of integers can be calculated using a telescoping sum. We formalize this notion by introducing the (forward, finite) difference operator Δ whose action on a functions f is defined by [4]

$$\Delta[f](x) = f(x+1) - f(x).$$

Further, for m a nonnegative integer, the mth order difference operator Δ^m is given recursively by

$$\Delta^0[f] = f, \qquad \Delta^{m+1}[f] = \Delta[\Delta^m[f]], \quad m \geq 1.$$

For example,

$$\begin{aligned}
\Delta[f](x) &= f(x+1) - f(x) \\
\Delta^2[f](x) &= f(x+2) - 2f(x+1) + f(x) \\
\Delta^3[f](x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x).
\end{aligned}$$

In general, we can express $\Delta^m[f]$ in the following manner.

Lemma 3.1. *For any function f and integer $m \geq 0$,*

$$\Delta^m[f](x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+j).$$

Proof. We use induction on m. When $m = 0$, we have $\Delta^0[f](x) = f(x)$ by definition, and

$$\sum_{j=0}^{0}(-1)^{0-j}\binom{0}{j}f(x+j) = (-1)^0\binom{0}{0}f(x+0) = f(x),$$

thus the result holds in the base case. Now, suppose the result holds for some natural number m. Recalling Pascal's identity,

$$\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j},$$

we compute

$$\begin{aligned}
\Delta^{m+1}[f](x) &= \Delta\left[\Delta^m[f]\right](x) \\
&= \Delta\left[\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(\cdot+j)\right](x) \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+1+j) - \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{m+1-j)}\binom{m}{j-1}f(x+j) + \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m}{j}f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{(m+1)-j)}\left[\binom{m}{j-1} + \binom{m}{j}\right]f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}f(x+j).
\end{aligned}$$

This completes the induction proof. □

Since it will be relevant later in the discussion, we also present a special case of Lemma 3.1 as a corollary.

Corollary 3.1. *If $P(x) = (a + rx)^k$, then*

$$\Delta^m[P](0) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(a+jr)^k. \tag{3.11}$$

From here on, we focus our attention to the case in which Δ is acting on a polynomial P, despite the richness of the more general theory [4]. It is not difficult to see that the operator Δ reduces the degree of a polynomial P by 1. That is to say, if P is a polynomial of degree $\deg(P)$, then $\Delta[P]$ is a polynomial of degree $\deg(P)-1$. This follows easily from the binomial theorem. Indeed,

$$(x+1)^d - x^d = \sum_{j=0}^{d-1}\binom{d}{j}x^j.$$

Furthermore, for any integer $m > \deg(P)$, we observe that $\Delta^m[P] = 0$.

One important advantage of using difference operators is it gives us the ability to write polynomials evaluated at integer inputs, say $n \geq 0$, as

$$P(n) = \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j}. \tag{3.12}$$

This special case of a far more general result (called the Gregory-Newton interpolation formula, first published by Newton in 1687 [13]) can be easily verified by a robust and interesting use of induction on n. Indeed, in the base case of $n = 0$, both sides of Equation (3.12) are equal to $P(0)$. Now assume Equation (3.12) holds true for some $n \geq 0$. Then, noticing that $P(n+1) = P(n) + \Delta[P](n)$, and appealing to Pascal's identity and the fact that $\Delta^{\deg(P)+1}[P] = 0$, we can conclude

$$\begin{aligned} P(n+1) &= P(n) + \Delta[P](n) \\ &= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} + \sum_{j=0}^{\deg(P)} \Delta^{j+1}[P](0) \binom{n}{j} \\ &= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} + \sum_{j=1}^{\deg(P)} \Delta^j[P](0) \binom{n}{j-1} \\ &= P(0) + \sum_{j=1}^{\deg(P)} \Delta^j[P](0) \left[\binom{n}{j} + \binom{n}{j-1} \right] \\ &= P(0) + \sum_{j=1}^{\deg(P)} \Delta^j[P](0) \binom{n+1}{j} \\ &= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n+1}{j}. \end{aligned}$$

Now that Equation (3.12) has been verified, we are ready to take the plunge back into our original problem. Recall that we are seeking a way to reduce the number of computations required to evaluate the sums $S_{N,k}[a,r]$, which are merely sums of the form $\sum_{n=0}^{N} P(n)$ for a particular polynomial P. Armed with Equation (3.12), and the aptly named "hockey stick identity," that is,

$$\sum_{n=0}^{N} \binom{n}{j} = \sum_{n=j}^{N} \binom{n}{j} = \binom{N+1}{j+1}, \tag{3.13}$$

we can convert a sum $\sum_{n=0}^{N} P(n)$ with $N+1$ terms into a sum with only $\deg(P)+1$ terms. This can be quite advantageous if N is significantly greater than $\deg(P)$. Indeed, interchanging the finite sums in n and j allows us to write [4]

$$\sum_{n=0}^{N} P(n) = \sum_{n=0}^{N} \left[\sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} \right] = \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{N+1}{j+1}. \tag{3.14}$$

Let us consider a simple polynomial: the monic monomial $P(n) = n^k$ of degree $k \geq 1$. In light of Corollary 3.1, it is not difficult to see that we can write

$$\Delta^j[n^k](0) = j! \left\{ {k \atop j} \right\},$$

where $\left\{ {k \atop j} \right\}$ denotes the Stirling number of the second kind given by

$$\left\{ {k \atop j} \right\} = \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} \ell^k, \qquad j \geq 0.$$

In view of Equation (3.14) and the fact $\left\{ {k \atop 0} \right\} = 0$, we obtain the formula

$$\sum_{n=1}^{N} n^k = \sum_{j=1}^{k} j! \left\{ {k \atop j} \right\} \binom{N+1}{j+1}. \tag{3.15}$$

We now illustrate the utility of the formulas discussed above by way of a concrete example.

Example 3.1. *Let $P(n) = n^2$. We have*

$$\Delta[n^2](0) = 1! \left\{ {2 \atop 1} \right\} = 1, \quad \textit{and} \quad \Delta^2[n^2](0) = 2! \left\{ {2 \atop 2} \right\} = 2,$$

and hence, by (3.12) *we are able to write n^2 as the sum*

$$n^2 = \Delta P(0) \binom{n}{1} + \Delta^2 P(0) \binom{n}{2} = \binom{n}{1} + 2\binom{n}{2}.$$

In light of Equation (3.13)*, we have recovered the well-known formula*

$$\sum_{n=0}^{N} n^2 = \sum_{j=1}^{2} j! \left\{ {2 \atop j} \right\} \binom{N+1}{j+1} = \binom{N+1}{2} + 2\binom{N+1}{3} = \frac{N(N+1)(2N+1)}{6}.$$

We are now ready to present yet another Faulhaber-type formula, one which does not make use of the Bernoulli numbers, but makes use of the terms of the original sum $S_{N,k}[a,r]$, rather than being written in powers of a, r, and $N+1$.

Theorem 3.1 (Faulhaber-type Formula for $S_{N,k}[a,r]$)**.** *For all $a, r \in \mathbb{R}$,*

$$S_{N,k}[a,r] = \sum_{m=0}^{k} \left[\binom{N+1}{m+1} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (a+jr)^k \right]. \tag{3.16}$$

Proof. Letting $P(n) = (a + rn)^k$ in Corollary 3.1, and substituting into Equation 3.14, we get

$$S_{N,k}[a,r] = \sum_{n=0}^{N}(a+nr)^k = \sum_{n=0}^{N} P(n) = \sum_{m=0}^{k} \Delta^m[P](0)\binom{N+1}{m+1}$$
$$= \sum_{m=0}^{k}\left[\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(a+jr)^k\right)\binom{N+1}{m+1}\right]$$

from which we get the formula in Equation 3.16, completing the proof. □

As a parting gift, we illustrate the utility of Theorems 2.3 and 3.1 with a worked example. One common aspect of the calculations which should be noted is the number of terms required to evaluate the sum. In fact, a careful inspection of the two formulas reveals that they both require that $(k + 1)(k + 2)/2$ terms be taken into account. In the example presented below, we have $k = 2$, and there are indeed $(3)(4)/2 = 6$ terms in both calculations. This is far better than the 101 terms it would take in order to calculate the sum directly.

Example 3.2. *Let us compute*

$$S_{100,2}[5,3] = \sum_{n=0}^{100}(5+3n)^2 = 5^2 + 8^2 + 11^2 + \cdots + 305^2.$$

Recalling the values $B_0 = 1$, $B_1 = -1/2$, *and* $B_2 = 1/6$, *Theorem 2.3 states*

$$S_{100,2}[5,3] = \sum_{m=0}^{2}\binom{2}{m}\frac{5^{2-m}3^m}{m+1}\left[\sum_{j=0}^{m}\binom{m+1}{j}101^{m+1-j}B_j\right]$$
$$= \binom{2}{0}\frac{5^2 3^0}{1}\left[\sum_{j=0}^{0}\binom{1}{j}101^{1-j}B_j\right] + \binom{2}{1}\frac{5^1 3^1}{2}\left[\sum_{j=0}^{1}\binom{2}{j}101^{2-j}B_j\right]$$
$$+ \binom{2}{2}\frac{5^0 3^2}{3}\left[\sum_{j=0}^{2}\binom{3}{j}101^{3-j}B_j\right]$$
$$= 2525B_0 + 15\left[\binom{2}{0}101^2B_0 + \binom{2}{1}101^1B_1\right]$$
$$+ 3\left[\binom{3}{0}101^3B_0 + \binom{3}{1}101^2B_1 + \binom{3}{2}101^1B_2\right]$$
$$= 3(101^3) - \frac{3}{2}(101^2) - \frac{1}{2}(101)$$
$$= 3,199,175,$$

while the formula from Theorem 3.1 yields

$$
\begin{aligned}
S_{100,2}[5,3] &= \sum_{m=0}^{2}\left[\binom{101}{m+1}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(5+3j)^2\right] \\
&= \binom{101}{1}(-1)^0\binom{0}{0}5^2 + \binom{101}{2}\left((-1)^1\binom{1}{0}5^2 + (-1)^0\binom{1}{0}8^2\right) \\
&\quad + \binom{101}{3}\left((-1)^2\binom{2}{0}5^2 + (-1)^1\binom{2}{1}8^2 + (-1)^0\binom{2}{2}11^2\right) \\
&= 3,199,175.
\end{aligned}
$$

As expected, the two results above are in agreement. It is also worth noting that the second calculation from Theorem 3.1 is a more robust and practical choice, as it avoids the use of Bernoulli numbers.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper. The second author thanks Caterpillar Inc. for their financial support through fellowship grant CAT2511136.

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