International Journal of Maps in Mathematics
Volume:1, Issue:2, (2018), Pages:237-248
ISSN: 2636-7467 (Online)
www.journalmim.com

FIXED POINT THEOREMS OF HYBRID PAIRS OF SELF-MAPPINGS IN METRIC SPACE VIA NEW FUNCTIONS

MUZEYYEN SANGURLU SEZEN* AND ARSLAN HOJAT ANSARI

Abstract. In this article, we establish some fixed point theorems for new type generalized contractive mappings involving C-class functions in metric spaces. We provide an example in order to support the useability of our results. These results generalize some well-known results in the literature.

1. Introduction and Preliminaries

Banach [2] introduced a contraction principle which has been extended by many authors to more general contractive conditions in different spaces, for example (see [5–9]) . Kannan obtained the same conclusion as Banach's Theorem with different sufficient conditions (see [11,12]). The conclusion is called Kannan contraction: A mapping T on a metric space (X, d) and if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$\delta(Sx, Ty) \leq \alpha[d(fx, Sx) + d(gy, Ty)]$$

for all $x, y \in X$. Subrahmanyam [17] constructed to show that a metric space having the fixed point property for homeomorphisms need not be metrically topologically complete.

Received:2018-07-03 *Revised:2018-09-06* *Accepted:2018-09-15*

*2010 Mathematics Subject Classification.*47H10,54H25.

*Key words:*Strongly tangential, Hybrid pair, Fixed point

* Corresponding author

Also, he proved that: A metric space (X,d) is complete if and only if every Kannan contraction has a fixed point on X. On the other hand Markin [13] and Nadler [14] initiated the study of fixed points of set valued and multivalued mappings using the Hausdorff metric. Tomar et al. proved strict coincidence and common strict fixed point of strongly tangential hybrid pairs of self-mappings satisfying Kannan type contraction [18].

In this paper, we present new general results of strongly tangential hybrid pairs of self-mappings satisfying Kannan type contraction involving C-class functions. Also we establish coincidence and common fixed point using Hausdorff distance. The obtained results extend many recent results in the literature.

Let (X,d) be a metric space and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. Functions $\delta(A,B)$ and $D(A,B)$ are defined as: $\delta(A,B)$ and $D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$

for all $A,B \in CB(X)$. If $A = \{a\}$, then $\delta(A,B) = \delta(a,B)$. If $A = \{a\}$ and $B = \{b\}$, then $\delta(A,B) = d(a,b)$. It follows immediately from the definition of δ that

(a) $\delta(A,B) = \delta(B,A) > 0$,

(b) $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$,

(c) $\delta(A,B) = 0$ iff $A = B = \{a\}$,

(d) $\delta(A,A) = diamA$, for all $A,B,C \in CB(X)$.

Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$ for all $A \in CB(X)$. Also $H(A,B) = 0$ iff $A = B$.

Let (X,d) be a metric space and $h : X \to X$ be a single valued mapping and $T : X \to CB(X)$ be a multivalued mapping. Then (h,T) is called a hybrid pair of mapping. For a multivalued mapping $T : X \to CB(X)$, a point $u \in X$ is

(a) fixed point if $u \in Tu$,

(b) strict fixed point (or a stationary fixed point or absolute fixed point) if $Tu = \{u\}$. For a hybrid pair (h,T), a point $u \in X$ is

(c) coincidence point if $hu = Tu$,

(d) strict coincidence point if $Tu = \{hu\}$,

(e) common fixed point if $u = hu \in Tu$,

(f) common strict fixed point if $hu = Tu = \{u\}$.

Definition 1.1. *[10] Let (X, d) be a metric space. A hybrid pair of mappings (h, T) is weakly commuting if $hTx \in CB(X)$ and $\delta(Thx, hTx) \leq \max\{\delta(hx, Tx), diam(hTx)\}$ for all $x \in X$. Note that if T is a single valued mapping, then the set $\{hTx\}$ consists of a single point. Hence, $diamhTx = 0$ for all $x \in X$ and definition of weak commutativity of a hybrid pair of self mappings reduces to the weak commutativity of a single valued pair of self mappings given by Sessa [15], that is, $d(Thx, hTx) \leq d(hx, Tx)$ for all $x \in X$.*

Definition 1.2. *[16] Let (X, d) be a metric space. A pair of single valued self mappings (h, g) is tangential with respect to a pair of multivalued self mappings (S, T) if*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A \text{ for some } z \in X.$$

Definition 1.3. *[3] Let (X, d) be a metric space. A pair of single valued self mappings (h, g) is strongly tangential with respect to a pair of multivalued self mappings (S, T) if*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A \text{ for some } z \in hX \cap gX.$$

Definition 1.4. *[3] Let (X, d) be a metric space. A single valued self mapping h is strongly tangential with respect to multivalued self mapping T if*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} hy_n = z \in A \text{ for some } z \in hX.$$

Definition 1.5. *[4] Let $h : X \to X$ be a single valued mapping while $T : X \to CB(X)$ be a multivalued mapping. The mapping h is said to be coincidentally idempotent with respect to mapping T, if $hx \in Tx$ imply $hhx = hx$.*

Definition 1.6. *[1] A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following conditions:*

(1) $f(s,t) \leq s$,

(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$ and $f(0,0) = 0$.

We denote C-class functions as $\mathcal{C}$.

2. Main Results

Let Φ denote all functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy

(1) φ is continuous and non-decreasing,

(2) $\varphi(t) = 0$ and only if $t = 0$,

(3) $\varphi(t) < t$, for all $t \in (0,\infty)$ and Ψ denote all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy

(1) $\psi(0) \geq 0$, $\psi(t) > 0$ for all $t > 0$

(2) $\psi(t) \leq t$, for all $t \in (0,\infty)$.

Theorem 2.1. *Let (X,d) be a metric space and $h,g : X \to X$ be single valued and $S,T : X \to CB(X)$ be multi-valued mappings. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(\delta(Sx,Ty)) \leq f(\psi(d(hx,Sx) + d(gy,Ty)), \varphi(d(hx,Sx) + d(gy,Ty))) \tag{2.1}$$

for all $x,y \in X$ and pair of (h,g) is strongly tangential with respect to (S,T). Then pairs (h,S) and (g,T) have strict coincidence point. Morever, h,g,S and T have a unique common strict fixed point if hybrid pairs (h,S) and (g,T) are coincidentally idempotent.

Proof. Suppose that (h,g) is strongly tangential with respect to (S,T). We introduce $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to\infty} hx_n = \lim\limits_{n\to\infty} gy_n = z \in A = \lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Ty_n$, where $A \in CB(X)$ and $z \in hX \cap gX$. Hence, there exists $u,v \in X$ such that $hu = gv = z$. Now we claim that $z = hu \in Su$. We take $x = u$ and $y = y_n$ in (2.1),

$$\psi(\delta(Su,Ty_n)) \leq f(\psi(d(hu,Su) + d(gy_n,Ty_n)), \varphi(d(hu,Su) + d(gy_n,Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu,Su)) \leq \psi(\delta(Su,A)) \leq f(\psi(d(hu,Su)), \varphi(d(hu,Su))) \leq \psi(d(hu,Su)).$$

Then, we have

$$f(\psi(d(hu,Su)), \varphi(d(hu,Su))) = \psi(d(hu,Su)).$$

From the property of f, we get

$$\psi(d(hu, Su)) = 0 \text{ or } \varphi(d(hu, Su)) = 0 \text{ so } d(hu, Su) = 0.$$

Therefore, $hu \in Su$, that is, $\delta(Su, A) = 0$ and so we get $Su = \{hu\}$. Hence, h and S have a strict coincidence point. Now we show that $z = gv \in Tv$, using $x = x_n$ and $y = v$ in (2.1)

$$\psi(\delta(Sx_n, Tv)) \leq f(\psi(d(hx_n, Sx_n) + d(gv, Tv)), \varphi(d(hx_n, Sx_n) + d(gv, Tv)).$$

Taking limit as $n \to \infty$, from property of f, we get

$$\psi(\delta(A, Tv)) \leq f(\psi(d(gv, Tv)), \varphi(d(gv, Tv)) \leq \psi(d(gv, Tv))) \leq \psi(d(z, A)).$$

From $gv = z \in A$, we have

$$\psi(d(gv, Tv)) \leq \psi(\delta(A, Tv)) \leq f(\psi(d(gv, Tv)), \varphi(d(gv, Tv))) \leq \psi(d(gv, Tv)).$$

So,

$$\psi(d(gv, Tv)) = 0 \text{ or } \varphi(d(gv, Tv)) = 0 \text{ so } d(gv, Tv) = 0.$$

Therefore, $gv \in Tv$, that is, $\delta(Tu, A) = 0$ and so we have $Tv = \{gu\}$. Hence, g and T have a strict coincidence point. Thus, $z \in Su = Tv = \{z\}$. Now since (h, S) is coincidentally idempotent, $hu \in Su$ implies $hhu = hu \in Su$. Now we claim that $z = hz \in Sz$. We take $x = z$ and $y = y_n$ in (2.1),

$$\psi(\delta(Sz, Ty_n)) \leq f(\psi(d(hu, Su) + d(gy_n, Ty_n)), \varphi(d(hu, Su) + d(gy_n, Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu, Su)) \leq \psi(\delta(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

Since $hz = z \in A$, we get,

$$\psi(d(hz, Sz)) \leq \psi(\delta(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

So,

$$\psi(d(hz, Sz)) = 0 \text{ or } \varphi(d(hz, Sz)) = 0 \text{ so } d(hz, Sz) = 0.$$

Therefore, $hz \in Sz$, that is, $\delta(Sz, A) = 0$ and so we have $Sz = \{hz = z\}$. Similarly, (g, T) is coincidentally idempotent $gv \in Tv$ implies $ggv = gv \in Tv$.

Now we claim that $z = gz \in Tz$. We take $x = x_n$ and $y = z$ in (2.1),

$$\psi(\delta(Sx_n, Tz)) \leq f(\psi(d(hx_n, Sx_n) + d(gz, Tz)), \varphi(d(hx_n, Sx_n) + d(gz, Tz))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(\delta(A, Tz)) \leq f(\psi(d(z, A) + d(gz, Tz)), \varphi(d(z, A) + d(gz, Tz)))$$

or

$$\psi(\delta(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Since $gz = z \in A$, we get,

$$\psi(d(gv, Tz)) \leq \psi(\delta(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Then,

$$\psi(d(gz, Tz)) = 0 \text{ or } \varphi(d(gz, Tz)) = 0 \text{ so } d(gz, Tz) = 0.$$

Hence, $gz \in Tz$, that is, $\delta(Tz, A) = 0$ and so we get $Tz = \{gz = z\}$. Therefore z is a common strict fixed point of h, g, T and S.

Let z and w be two common strict fixed points such that $z \neq w$. Now from (2.1), we have,

$$\begin{aligned}\psi(\delta(Sz, Tw)) &\leq f(\psi(d(hz, Sz) + d(gw, Tw)), \varphi(d(hz, Sz) + d(gw, Tw))) \\ &\leq f(0, 0) \leq 0.\end{aligned}$$

$$\delta(Sz, Tw) \leq 0$$

but

$$\delta(Sz, Tw) > 0$$

which is a contradiction. Hence, z is a unique common strict fixed point of h, g, T and S.

Taking $h = g$ and $T = S$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let (X, d) be a metric space and $h : X \to X$ be single valued and $T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(\delta(Tx, Ty)) \leq f(\psi(d(hx, Tx) + d(hy, Ty)), \varphi(d(hx, Tx) + d(hy, Ty))) \tag{2.2}$$

for all $x, y \in X$ and pair of h is strongly tangential with respect to T. Then h and T have strict coincidence point. Morever, h and T have a unique common strict fixed point if hybrid pair (h, T) is coincidentally idempotent.

Theorem 2.2. *Let (X,d) be a metric space and $h,g : X \to X$ be single valued and $S,T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(H(Sx,Ty)) \leq f(\psi(d(hx,Sx)+d(gy,Ty)),\varphi(d(hx,Sx)+d(gy,Ty)) \tag{2.3}$$

for all $x,y \in X$ and pair of (h,g) is strongly tangential with respect to (S,T). Then pairs (h,S) and (g,T) have coincidence point. Morever, h,g,S and T have a common fixed point if hybrid pairs (h,S) and (g,T) are coincidentally idempotent.

Proof. Suppose that (h,g) is strongly tangential with respect to (S,T). We introduce $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to\infty} hx_n = \lim\limits_{n\to\infty} gy_n = z \in A = \lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Ty_n$, where $A \in CB(X)$ and $z \in hX \cap gX$. Hence, there exists $u,v \in X$ such that $fu = gv = z$. Now we claim that $z = hu \in Su$. We take $x = u$ and $y = y_n$ in (2.3),

$$\psi(H(Su,Ty_n)) \leq f(\psi(d(hu,Su)+d(gy_n,Ty_n)),\varphi(d(hu,Su)+d(gy_n,Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu,Su)) \leq \psi(H(Su,A)) \leq f(\psi(d(hu,Su)),\varphi(d(hu,Su))) \leq \psi(d(hu,Su)).$$

Then, we have

$$f(\psi(d(hu,Su)),\varphi(d(hu,Su))) = \psi(d(hu,Su)).$$

From the property of f, we get

$$\psi(d(hu,Su)) = 0 \text{ or } \varphi(d(hu,Su)) = 0 \text{ so } d(hu,Su) = 0.$$

Hence, $hu \in Su$. Hence, h and S have a coincidence point. Now we show that $z = gv \in Tv$, using $x = x_n$ and $y = v$ in (2.3)

$$\psi(H(Sx_n,Tv)) \leq f(\psi(d(hx_n,Sx_n)+d(gv,Tv)),\varphi(d(hx_n,Sx_n)+d(gv,Tv))).$$

Taking limit as $n \to \infty$, from property of f, we get

$$\psi(H(A,Tv)) \leq f(\psi(d(gv,Tv)),\varphi(d(gv,Tv))) \leq \psi(d(gv,Tv)) \leq \psi(d(z,A)).$$

From $gv = z \in A$, we have

$$\psi(d(gv,Tv)) \leq \psi(H(A,Tv)) \leq f(\psi(d(gv,Tv)),\varphi(d(gv,Tv))) \leq \psi(d(gv,Tv)).$$

So,

$$\psi(d(gv,Tv)) = 0 \text{ or } \varphi(d(gv,Tv)) = 0 \text{ so } d(gv,Tv) = 0.$$

Therefore, $gv \in Tv$. Hence, g and T have a coincidence point. Thus, $z \in Su = Tv = \{z\}$. Now since (h, S) is coincidentally idempotent, $hu \in Su$ implies $hhu = hu \in Su$. Now we claim that $z = hz \in Sz$. We take $x = z$ and $y = y_n$ in (2.3),

$$\psi(H(Sz, Ty_n)) \leq f(\psi(d(hu, Su) + d(gy_n, Ty_n)), \varphi(d(fu, Su) + d(gy_n, Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu, Su)) \leq \psi(H(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

Since $hz = z \in A$, we get,

$$\psi(d(hz, Sz)) \leq \psi(H(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

So,

$$\psi(d(hz, Sz)) = 0 \text{ or } \varphi(d(hz, Sz)) = 0 \text{ so } d(fz, Sz) = 0.$$

Hence, $hz \in Sz$. Similarly, (g, T) is coincidentally idempotent $gv \in Tv$ implies $ggv = gv \in Tv$.

Now we claim that $z = gz \in Tz$. We take $x = x_n$ and $y = z$ in (2.3),

$$\psi(H(Sx_n, Tz)) \leq f(\psi(d(hx_n, Sx_n) + d(gz, Tz)), \varphi(d(hx_n, Sx_n) + d(gz, Tz))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(H(A, Tz)) \leq f(\psi(d(z, A) + d(gz, Tz)), \varphi(d(z, A) + d(gz, Tz)))$$

or

$$\psi(H(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Since $gz = z \in A$, we get,

$$\psi(d(gv, Tz)) \leq \psi(H(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Then,

$$\psi(d(gz, Tz)) = 0 \text{ or } \varphi(d(gz, Tz)) = 0 \text{ so } d(gz, Tz) = 0.$$

Thus, $gz \in Tz$. Therefore z is a common fixed point of h, g, T and S.

Let z and w be two common fixed points such that $z \neq w$. Now from condition (1), we have,

$$\begin{aligned}\psi(H(Sz, Tw)) &\leq f(\psi(d(hz, Sz) + d(gw, Tw)), \varphi(d(hz, Sz) + d(gw, Tw))) \\ &\leq f(0, 0) \leq 0.\end{aligned}$$

$$H(Sz, Tw) \leq 0$$

but

$$H(Sz, Tw) > 0$$

which is a contradiction. Hence, z is a unique common fixed point of h, g, T and S.

Taking $h = g$ and $T = S$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.2. *Let (X, d) be a metric space and $h : X \to X$ be single valued and $T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(H(Tx, Ty)) \leq f(\psi(d(hx, Tx) + d(hy, Ty)), \varphi(d(hx, Tx) + d(hy, Ty)) \tag{2.4}$$

for all $x, y \in X$ and pair of h is strongly tangential with respect to T. Then h and T have coincidence point. Morever, h and T have a common fixed point if hybrid pair (h, T) is coincidentally idempotent.

Example 2.1. *Let $X = [0, 5]$, d be usual metric on X, Let a hybrid pair of mappings h,T : $X \to X$ by*

$$hx = \begin{cases} \frac{4-x}{2} & , \quad x \in [0,2] \\ 4 & , \quad x \in (2,5] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \{\frac{4}{3}\} & , \quad x \in [0,2] \\ 1 & , \quad x \in (2,5] \end{cases}.$$

Define $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{t}{5}, \varphi(t) = 3t$ and $F(s, t) = ks$ for $k \in (0, 1)$. Consider two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n = \frac{4}{3} - \frac{1}{n}$ and $y_n = \frac{4}{3}$ for all $n > 1$. Clearly $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} hy_n = \frac{4}{3} \in \{\frac{4}{3}\} = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n$ and $\frac{4}{3} \in hX$. Hence h is strongly tangential with respect to T. The point $z = \frac{4}{3}$ is a strict coincidence point and $hh\frac{4}{3} = h\frac{4}{3}$, that is, (h, T) is coincidentally idempotent.
For $x, y \in [0, 2]$, we have

$$\psi(\delta(Tx, Ty)) = 0 \leq k.\frac{2d(hx, Tx)}{5},$$

for $x \in [0, 2]$ and $y \in (2, 5]$, we have

$$\psi(\delta(Tx, Ty)) = \frac{1}{3} \leq k.\frac{d(hx, Tx) + d(hy, Ty)}{5},$$

for $x, y \in (2, 5]$, we have

$$\psi(\delta(Tx, Ty)) = 0 \leq k.\frac{2d(hx, Tx)}{5},$$

for $x \in (2, 5]$ and $y \in [0, 2]$, we have

$$\psi(\delta(Tx, Ty)) = \frac{1}{3} \leq k.\frac{d(hx, Tx) + d(hy, Ty)}{5}.$$

Thus h and T satisfy Corollary 2.1, for $k = \frac{1}{3} \in (0, 1)$. Also $T\frac{4}{3} = \{h\frac{4}{3}\} = \{\frac{4}{3}\}$, that is, $\frac{4}{3}$ is the unique common strict fixed point of h and T.

3. Application

In this section, we generalize the results of Theorem 4.1 given by Tomar et al. [18].

Let $B(W)$ be the set of all closed and bounded real-valued functions on W. For an arbitrary $p,k \in B(W)$ define $\|p\| = \sup_{x\in W}|p(x)|$, $\|k\| = \sup_{x\in W}|k(x)|$ and $\delta(p,k) = \sup_{x\in W}|p(x)-k(x)|$. Also, $(B(W),\|.\|)$ is a Banach space wherein convergence is uniform. Consider the operators $T_i, A_i : B(W) \to B(W)$ given by

$$\begin{cases} T_i p(x) = \sup_{y\in D}\{g(x,y) + G_i(x,y,p(\tau(x,y)))\}, & i=1,2, \\ A_i k(x) = \sup_{y\in W}\{g^{'}(x,y) + G_i^{'}(x,y,p(\tau(x,y)))\}, & i=1,2, \end{cases} \tag{3.1}$$

for $p,k \in B(W)$, where $\tau : W\times D \to W$, $g.g^{'} : W\times D \to \mathbb{R}$, $G_i.G_i^{'} : W\times D\times\mathbb{R} \to \mathbb{R}$ are given mappings, while $W \in U$ is a state space, $D \in V$ is a decision space and U, V are Banach spaces. These mappings are well-defined if the functions $g_i, g_i^{'}$,G_i and $G_i^{'}$ are bounded. Also, denote

$$\Theta(p,k) = f(\psi(d(A_1p,T_1p)+d(A_2k,T_2k)),\varphi(d(A_1p,T_1p)+d(A_2k,T_2k)))$$

for $p,k \in B(W)$.

Theorem 3.1. *Let $T_i, A_i : B(W) \to B(W)$ given by (3.1), for $i=1,2$. Suppose that the following conditions hold:*

(1) For all $x \in W, y \in D$, $|G_1(x,y,p(\tau(x,y)) - G_2(x,y,p(\tau(x,y))| \leq \Theta(p,k)$,

(2) For $i=1,2$, $g.g^{'} : W\times D \to \mathbb{R}$, $G_i.G_i^{'} : W\times D\times\mathbb{R} \to \mathbb{R}$ are bounded functions,

(3) There exists sequences $\{p_n\}$,$\{k_n\} \in B(W)$ and functions $p^* \in B(W)$ such that

$$\lim_{n\to\infty} T_1p_n = \lim_{n\to\infty} T_2k_n = A \in B(W)$$

and

$$\lim_{n\to\infty} A_1p_n = \lim_{n\to\infty} A_2k_n = p^* \in A \text{ and } p^* \in A_1\cap A_2,$$

(4) $A_1A_1p = A_1p$ whenever $A_1p \in T_1p$ and $A_2A_2k = A_2k$ whenever $A_2k \in T_2k$ for some $p,k \in B(W)$. Then, the equation system (5) has a bounded solution.

Proof. Let $\delta(h,k) = \sup_{x\in W}|h(x)-k(x)|$ for any $h,k \in B(W)$ and $\psi(t)=t$ for $t \in [0,+\infty)$. Let λ be an arbitrary positive number, $x \in W$. Then there exists $y_1,y_2 \in D$ such that

$$T_1h(x) < g(x,y_1) + G_1(x,y_1,h(\tau(x,y_1)) + \lambda \tag{3.2}$$

$$T_2k(x) < g(x, y_2) + G_2(x, y_2, k(\tau(x, y_2)) + \lambda. \tag{3.3}$$

From the definition, we have

$$T_1h(x) > g(x, y_2) + G_1(x, y_2, h(\tau(x, y_2)) + \lambda \tag{3.4}$$

$$T_2k(x) > g(x, y_1) + G_2(x, y_1, k(\tau(x, y_1)) + \lambda. \tag{3.5}$$

From (3.2) and (3.5), we get

$$T_2k(x) - T_1h(x) < \Theta(h, k) + \lambda. \tag{3.6}$$

Combining, we get

$$|T_1h(x) - T_2k(x)| < \Theta(h, k) + \lambda.$$

Implying thereby

$$\delta(T_1h(x), T_2k(x)) < \Theta(h, k) + \lambda. \tag{3.7}$$

Also, (3.7) does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily. Hence, we obtain

$$\delta(T_1h(x), T_2k(x)) < \Theta(h, k)$$

for each $t \in (0, \infty)$. From condition (3), (A_1, A_2) is strongly tangential with respect to (T_1, T_2). Thus, from condition (4) and taking $h = A_1, S = T_1, g = A_2, T = T_2$ all the conditions of Theorem 2.1 are satisfied. Hence, from Theorem 2.1, T_1,T_2, A_1 and A_2 have a unique common fixed point, the system of functional equations (3.1) has a unique bounded solution.

References

[1] Ansari AH. Note on ψ−ϕ-contractive type mappings and related fixed point, The 2nd Regional Conference on Mathematics And Applications, Payame Noor University 2014; 377-380.

[2] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 1922; 3; 133-181.

[3] Chauhan S, Imdad M, Karapinar E, Fisher B. An integral type fixed point theorem for multi-valued mappings employing strongly tangential property, J. Egyptian Math. Soc 2014; 22(2); 258-264.

[4] Imdad M, Ahmad A, Kumar S. On nonlinear nonself hybrid contractions, Radovi Matematicki 2001; 10; 233-244.

[5] Isik H, Turkoglu D. Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces, Fixed Point Theory Appl. 2013; 2013(51); 12 pages.

[6] Isik H, Turkoglu D. Coupled fixed point theorems for new contractive mixed monotone mappings and applications to integral equations, Filomat 2014; 28(6); 1253-1264.

[7] Isik H, Samet B, Vetro C. Cyclic admissible contraction and applications to functional equations in dynamic programming, Fixed Point Theory Appl. 2015; 2015(163); DOI 10.1186/s13663-015-0410-6.

[8] Isik H, Turkoglu D. Generalized weakly α-contractive mappings and applications to ordinary differential equations. Miskolc Mathematical Notes 2016; 17(1); 365-379.

[9] Isik H, Radenovic S. A new version of coupled fixed point results in ordered metric spaces with applications. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics 2017; 79(2), 131-138.

[10] Kaneko H. A common fixed point of weakly commuting multivalued mappings, Math. Japon. 1988; 33(5); 741-744.

[11] Kannan R. Some results on fixed points, Bull. Calcutta Math. Soc 1968; 60; 71-76.

[12] Kannan R. Some results on fixed points. II, American Mathematical Monthly 1969; 76(4); 405-408.

[13] Markin J. A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc 1968; 74; 639-640.

[14] Nadler SB. Multi-valued contraction mappings, Pacific J. Math 1969; 30(2); 475-486.

[15] Sessa S. On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math 1982; 32(46); 149-153.

[16] Sintunavarat W, Kumam P. Gregus-type common fixed point theorems for tangential multi-valued mappings of integral type in metric spaces, Int. J. Math. Math. Sci 2011; 12 pages, DOI:10.1155/2011/923458.

[17] Subrahmanyam V. Completeness and fixed-points, Monatsh. Math 1975; 80(4); 325-330.

[18] Tomar A, Upadhyay S, Sharma R. Strict Coincidence and Common Strict Fixed Point of Hybrid Pairs of Self-mappings with Application, Math. Sci. and Appl. E-Notes 2017; 5(2); 51-59.

Giresun University, Department of Mathematics, 28100, Giresun-Turkey

E-mail address: muzeyyen.sezen@giresun.edu.tr

Department of Mathematics, Faculty of Science, Iran

E-mail address: mathanalsisamir4@gmail.com