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ANALYTICAL EXPLORATION OF WEYL-CONFORMAL CURVATURE TENSOR IN LORENTZIAN β-KENMOTSU MANIFOLDS ENDOWED WITH GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. This paper investigates the conformal curvature properties of *Lorentzian β-Kenmotsu ($L\beta K$) manifolds* admitting a *generalized Tanaka-Webster (g-TW) connection*. We begin by establishing the fundamental preliminaries of $L\beta K$ manifolds and exploring their curvature properties under the influence of g-TW connection. The study then focuses on specific curvature conditions, namely $\widetilde{R}\cdot\widetilde{S} = 0$, $\widetilde{S}\cdot\widetilde{R} = 0$, conformal flatness, ζ-conformal flatness, and pseudo-conformal flatness, to examine their geometric and structural implications. Additionally, we construct an explicit example of a *3-dimensional $L\beta K$ manifold* that admits a g-TW connection, providing concrete validation of our theoretical results. The findings contribute to the broader understanding of curvature behaviors in almost contact pseudo-Riemannian geometry and extend the study of non-Riemannian connections in Lorentzian manifolds.

Keywords: Lorentzian β-Kenmotsu manifolds, Generalized Tanaka-Webster connection, Weyl-conformal curvature tensor, Generalized η-Einstein manifolds.

2020 Mathematics Subject Classification: Primary: 53C05, 53C15, 53C25, Secondary: 53D10.

1. Introduction

The *Tanaka–Webster connection* was introduced by *Tanno* [16] as a generalization of the well-known connection formulated in the late 1970s by *Tanaka* [15] and independently by

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Webster [19]. This connection coincides with the classical *Tanaka–Webster connection* when the associated *CR-structure* is integrable. It is defined as the canonical affine connection on a *non-degenerate, pseudo-Hermitian CR-manifold.*

For a *real hypersurface* in a *Kähler manifold* endowed with an *almost contact structure* (ϕ, ζ, η, g), *Cho* [3, 4] adapted *Tanno's g-Tanaka–Webster connection* for a nonzero real constant k. Utilizing this connection, several researchers have explored various geometric properties of *real hypersurfaces in complex space forms* [17].

A *Riemannian manifold* is termed *semisymmetric* if its curvature tensor satisfies

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot R = 0, \tag{1.1}$$

where $R(\mathcal{H}_1, \mathcal{H}_2)$ is regarded as a field of linear operators acting on R. It is well established that the class of *semisymmetric manifolds* properly contains *locally symmetric manifolds* (where $\nabla R = 0$). The concept of *semisymmetry* in Riemannian geometry was first investigated by *E. Cartan, A. Lichnerowicz, R. S. Couty, and N. S. Sinjukov.*

A Riemannian manifold is called *Ricci semisymmetric* if its curvature tensor satisfies

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot S = 0, \tag{1.2}$$

where S denotes the *Ricci tensor* of type $(0, 2)$. The class of *Ricci semisymmetric manifolds* contains *Ricci symmetric manifolds* (where $\nabla S = 0$) as a proper subset. Several researchers have studied these manifolds extensively. It is known that every *semisymmetric manifold* is *Ricci semisymmetric*, but the converse does not always hold. However, under certain additional conditions, the equations

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot R = 0 \quad \text{and} \quad R(\mathcal{H}_1, \mathcal{H}_2) \cdot S = 0$$

become equivalent. *Szabó* classified *semisymmetric manifolds* locally in [14], while fundamental studies in this area were carried out by *Szabó* [14], *Boeckx et al.* [2], and *Kowalski* [6].

One notable example of a curvature condition related to *semisymmetry* is

$$Q \cdot R = 0, \tag{1.3}$$

where Q is the *Ricci operator* defined by

$$S(\mathcal{H}_1, \mathcal{H}_2) = g(Q\mathcal{H}_1, \mathcal{H}_2).$$

Such curvature conditions naturally extend to *pseudosymmetry-type* conditions. The condition $Q \cdot R = 0$ was extensively studied by *Verstraelen et al.* in [18].

Several properties on $\mathcal{M}_\beta$ and the g-TW connection have also been researched by numerous geometers, such as ([1, 7, 8, 9, 10, 11, 12, 13]). Inspired by these foundational works, the present paper aims to *characterize Lorentzian β-Kenmotsu manifolds admitting the generalized Tanaka–Webster connection.*

The arrangement of this paper is structured as follows: Section 2 presents the fundamental definitions and preliminary results related to Lorentzian β-Kenmotsu (LβK) manifolds. We introduce the structure equations and discuss essential properties that will be used in subsequent sections. In section 3, we explore the curvature properties of a $L\beta K$ manifold admitting the generalized Tanaka-Webster (g-TW) connection. We derive explicit expressions for the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ with respect to g-TW connection and establish some interesting geometric properties. Section 4 investigates the condition $\widetilde{R} \cdot \widetilde{S} = 0$ in a $L\beta K$ manifold equipped with g-TW connection. We demonstrate that under this condition, the manifold becomes a generalized η-Einstein manifold with respect to the g-TW connection. In section 5, we analyze the condition $\widetilde{S} \cdot \widetilde{R} = 0$ and establish that the $L\beta K$ manifold satisfying this curvature restriction is also a generalized η-Einstein manifold with respect to g-TW connection. Section 6 is devoted to the study of conformally flat $L\beta K$ manifolds under the influence of g-TW connection. We prove that such manifolds naturally admit a generalized η-Einstein structure with respect to g-TW connection. In section 7, we focus on ζ-conformally flat $L\beta K$ manifolds and derive certain interesting curvature properties arising from this condition. Section 8 examines the notion of pseudo-conformal flatness in the framework of $L\beta K$ manifolds. Finally, in section 9, we construct an explicit example of a 3-dimensional $L\beta K$ manifold admitting g-TW connection and verify that it satisfies the curvature conditions discussed in the previous sections. This structured approach ensures a coherent development of our results, highlighting the interplay between various curvature conditions and the geometry of Lorentzian β-Kenmotsu manifolds.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold is termed as $L\beta K$ manifold $(\mathcal{M}_\beta)$, if it possesses a $(1,1)$-tensor field ϕ, a contravariant vector field ζ, a covariant vector field η and a Lorentzian metric g satisfying

$$\phi^2\mathcal{H}_1 = \mathcal{H}_1 + \eta(\mathcal{H}_1)\zeta, \quad g(\mathcal{H}_1, \zeta) = \eta(\mathcal{H}_1), \tag{2.4}$$

$$\eta(\zeta) = -1, \quad \phi(\zeta) = 0, \quad \eta(\phi\mathcal{H}_1) = 0, \tag{2.5}$$

$$g(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{2.6}$$

$$g(\phi\mathcal{H}_1, \mathcal{H}_2) = g(\mathcal{H}_1, \phi\mathcal{H}_2), \tag{2.7}$$

for all vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}_\beta$. Furthermore, $\mathcal{M}_\beta$ satisfies

$$\nabla_{\mathcal{H}_1}\zeta = \beta[\mathcal{H}_1 - \eta(\mathcal{H}_1)\zeta], \tag{2.8}$$

$$(\nabla_{\mathcal{H}_1}\eta)(\mathcal{H}_2) = \beta[g(\mathcal{H}_1, \mathcal{H}_2) - \eta(\mathcal{H}_1)\eta(\mathcal{H}_2)], \tag{2.9}$$

$$(\nabla_{\mathcal{H}_1}\phi)(\mathcal{H}_2) = \beta[g(\phi\mathcal{H}_1, \mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\phi\mathcal{H}_1], \tag{2.10}$$

where ∇ represents the covariant differentiation operator with respect to the Lorentzian metric g. Moreover, on $\mathcal{M}_\beta$, the following relations hold

$$\eta(R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3) = \beta^2[g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2) - g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1)], \tag{2.11}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \beta^2[g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 - g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1], \tag{2.12}$$

$$R(\zeta, \mathcal{H}_1)\mathcal{H}_2 = \beta^2[\eta(\mathcal{H}_2)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_2)\zeta], \tag{2.13}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\zeta = \beta^2[\eta(\mathcal{H}_1)\mathcal{H}_2 - \eta(\mathcal{H}_2)\mathcal{H}_1], \tag{2.14}$$

$$S(\mathcal{H}_1, \zeta) = -2n\beta^2\eta(\mathcal{H}_1), \tag{2.15}$$

$$Q\mathcal{H}_1 = -2n\beta^2\mathcal{H}_1, \quad Q\zeta = -2n\beta^2\zeta, \tag{2.16}$$

$$S(\zeta, \zeta) = 2n\beta^2, \tag{2.17}$$

$$g(Q\mathcal{H}_1, \mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2) = -2n\beta^2 g(\mathcal{H}_1, \mathcal{H}_2), \tag{2.18}$$

$$S(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2) - 2n\beta^2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{2.19}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$, where R, S and Q stand for the curvature tensor, the Ricci tensor and the Ricci operator on $\mathcal{M}_\beta$, respectively.

Let $\{e_1, e_2, e_3, \dots, e_n = \zeta\}$ be an orthonormal basis for the tangent space at any point on the manifold $\mathcal{M}_\beta$. The Ricci tensor S and the scalar curvature r of the manifold are given by the following expression

$$S(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, \mathcal{H}_1)\mathcal{H}_2, e_i), \tag{2.20}$$

where ε_i are the signs corresponding to the metric signature.

On $L\beta K$-manifolds, the scalar curvature r is given by

$$r = \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i), \tag{2.21}$$

where ε_i are the signs corresponding to the metric signature. Additionally, we have

$$g(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{2n+1} \varepsilon_i g(\mathcal{H}_1, e_i) g(\mathcal{H}_2, e_i), \tag{2.22}$$

where $\mathcal{H}_1, \mathcal{H}_2 \in \chi(\mathcal{M}_\beta)$ and $\varepsilon_i = g(e_i, e_i) = \pm 1$.

Definition 2.1 *A $L\beta K$-manifold $\mathcal{M}_\beta$ is referred to as a generalized η-Einstein manifold if its Ricci tensor S takes the form*

$$S(\mathcal{H}_1, \mathcal{H}_2) = \nu_1 g(\mathcal{H}_1, \mathcal{H}_2) + \nu_2 \eta(\mathcal{H}_1)\eta(\mathcal{H}_2) + \nu_3 \Phi(\mathcal{H}_1, \mathcal{H}_2), \tag{2.23}$$

where $\Phi(\mathcal{H}_1, \mathcal{H}_2) = g(\phi\mathcal{H}_1, \mathcal{H}_2)$ is the fundamental 2-form of the manifold $\mathcal{M}_\beta$ and ν_1, ν_2, ν_3 are smooth functions on $\mathcal{M}_\beta$.

If $\nu_3 = 0$, then $\mathcal{M}_\beta$ is said to be an η-Einstein manifold.

If $\nu_2 = 0, \nu_3 = 0$, then $\mathcal{M}_\beta$ is said to be an Einstein manifold.

Definition 2.2 *In a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold, the Weyl-conformal curvature tensor $\mathcal{C}$ (also known as conformal curvature tensor) with respect to the Levi-Civita connection is defined as follows (see* [20]*):*

$$\begin{aligned}\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - \frac{1}{(2n-1)}\Big[S(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - S(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 + g(\mathcal{H}_2, \mathcal{H}_3)Q\mathcal{H}_1 \\ - g(\mathcal{H}_1, \mathcal{H}_3)Q\mathcal{H}_2\Big] + \frac{r}{2n(2n-1)}\Big[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2\Big].\end{aligned} \tag{2.24}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$, R and r represent the curvature tensor and the scalar curvature with respect to the Levi-Civita connection, respectively.

Definition 2.3 *The sectional curvature $\kappa(\mathcal{H}_1, \mathcal{H}_2)$ of a manifold is given by*

$$\kappa(\mathcal{H}_1, \mathcal{H}_2) = -\frac{R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)}{g(\mathcal{H}_1, \mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_2) - g(\mathcal{H}_1, \mathcal{H}_2)^2}, \tag{2.25}$$

where $R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)$ represents the associated curvature tensor.

3. Curvature properties of a $L\beta K$ manifold admitting g-TW connection

The g-TW connection $\widetilde{\nabla}$, associated with the Levi-Civita connection ∇, is defined by [16, 5]

$$\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_2 = \nabla_{\mathcal{H}_1}\mathcal{H}_2 + (\nabla_{\mathcal{H}_1}\eta)(\mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\nabla_{\mathcal{H}_1}\zeta - \eta(\mathcal{H}_1)\phi\mathcal{H}_2, \tag{3.26}$$

for any vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}_\beta$. Using (2.8) and (2.9) in (3.26), we obtain

$$\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_2 = \nabla_{\mathcal{H}_1}\mathcal{H}_2 + \beta g(\mathcal{H}_1, \mathcal{H}_2)\zeta - \beta\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\phi\mathcal{H}_2, \tag{3.27}$$

for all smooth vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}_\beta$.
Substituting $\mathcal{H}_2 = \zeta$ in (3.27), we have

$$\widetilde{\nabla}_{\mathcal{H}_1}\zeta = 2\beta\mathcal{H}_1. \tag{3.28}$$

Let $\widetilde{R}$ and R denote the curvature tensors of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇, respectively. The curvature tensor of a $(2n+1)$-dimensional $L\beta K$ manifold with respect to the g-TW connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \widetilde{\nabla}_{\mathcal{H}_1}\widetilde{\nabla}_{\mathcal{H}_2}\mathcal{H}_3 - \widetilde{\nabla}_{\mathcal{H}_2}\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_3 - \widetilde{\nabla}_{[\mathcal{H}_1,\mathcal{H}_2]}\mathcal{H}_3. \tag{3.29}$$

By virtue of (3.27) in (3.29), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = {} & R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \rho\eta(\mathcal{H}_1)[g(\mathcal{H}_2, \mathcal{H}_3)\zeta - \eta(\mathcal{H}_3)\mathcal{H}_2] \\ & - \rho\eta(\mathcal{H}_2)[g(\mathcal{H}_1, \mathcal{H}_3)\zeta - \eta(\mathcal{H}_3)\mathcal{H}_1] + 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2] \\ & - 2\beta[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3)\zeta - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)\zeta],\end{aligned} \tag{3.30}$$

where $\rho = \zeta\beta$ and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are any vector fields on $\mathcal{M}_\beta$.
By taking the inner product of (3.30) with the vector field $\mathcal{H}_4$, we have

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = {} & R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) + \rho\eta(\mathcal{H}_1)[\eta(\mathcal{H}_4)g(\mathcal{H}_2, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ & - \rho\eta(\mathcal{H}_2)[\eta(\mathcal{H}_4)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4)] \\ & + 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ & - 2\beta\eta(\mathcal{H}_4)[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)],\end{aligned} \tag{3.31}$$

where $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = g(\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3, \mathcal{H}_4)$ is the curvature tensor associated with $\widetilde{\nabla}$.

Let $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_1 = \mathcal{H}_4 = e_i$ in (3.31) and summing over i for $1 \leq i \leq (2n+1)$, we obtain

$$\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = S(\mathcal{H}_2, \mathcal{H}_3) + (6n\beta^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3) + (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) - 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{3.32}$$

for all vector fields $\mathcal{H}_2, \mathcal{H}_3$ on $\mathcal{M}_\beta$, where $\widetilde{S}$ and S denote the Ricci tensor of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇ respectively.

Using (3.32) , the Ricci operator $\widetilde{Q}$ with respect to the connection $\widetilde{\nabla}$ is determined by

$$\widetilde{Q}\mathcal{H}_2 = Q\mathcal{H}_2 + (6n\beta^2 - \rho)\mathcal{H}_2 + (2n-1)\rho\eta(\mathcal{H}_2)\zeta - 2\beta\phi\mathcal{H}_2. \tag{3.33}$$

Let $\widetilde{r}$ and r denote the scalar curvature of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇, respectively. Let $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_2 = \mathcal{H}_3 = e_i$ in (3.32) and summing over i for $1 \leq i \leq (2n+1)$, we obtain

$$\widetilde{r} = r + 6n(2n+1)\beta^2 - 4n\rho - 2\beta\psi, \tag{3.34}$$

where $\psi = trace(\phi)$.

From above discussion, we state the following:

Theorem 3.1 *In a (2n+1)-dimensional LβK manifold admitting g-TW connection $\widetilde{\nabla}$, the following holds:*

(i) The curvature tensor $\widetilde{R}$, Ricci tensor $\widetilde{S}$, Ricci operator $\widetilde{Q}$, and scalar curvature $\widetilde{r}$ with respect to $\widetilde{\nabla}$ are given by (3.30), (3.32), (3.33), and (3.34) respectively,

(ii) $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{H}_2, \mathcal{H}_1)\mathcal{H}_3 = 0$,

(iii) $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 + \widetilde{R}(\mathcal{H}_3, \mathcal{H}_1)\mathcal{H}_2 = 0$,

(iv) The Ricci tensor $\widetilde{S}(\mathcal{H}_1, \mathcal{H}_2)$ is symmetric in nature.

Now, let $\mathcal{M}_\beta$ be a Ricci flat with respect to the g-TW connection $\widetilde{\nabla}$. Then from (3.32), we lead to

$$S(\mathcal{H}_2, \mathcal{H}_3) = -(6n\beta^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3) - (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) + 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{3.35}$$

where $\rho = \zeta\beta$ and $\Phi(\mathcal{H}_2, \mathcal{H}_3) = g(\mathcal{H}_2, \phi\mathcal{H}_3)$.

This leads to the following result:

Theorem 3.2 *A $L\beta K$ manifold $\mathcal{M}_\beta$ is Ricci flat with respect to the g-TW connection $\widetilde{\widetilde{\nabla}}$ if and only if it is a generalized η-Einstein manifold with respect to the Levi-Civita connection ∇.*

Now, if $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0$, then by virtue of (3.31), we have

$$\begin{aligned} R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = &-\rho\eta(\mathcal{H}_1)[\eta(\mathcal{H}_4)g(\mathcal{H}_2, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ &+ \rho\eta(\mathcal{H}_2)[\eta(\mathcal{H}_4)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4)] \\ &- 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ &+ 2\beta\eta(\mathcal{H}_4)[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)], \end{aligned} \tag{3.36}$$

Let $\zeta^\perp = \{\mathcal{H}_1 : g(\mathcal{H}_1, \zeta) = 0, \forall \mathcal{H}_1 \in \chi(\mathcal{M}_\beta)\}$ denotes a $(2n+1)$-dimensional distribution orthogonal to ζ, then for any $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \in \zeta^\perp$, (3.36) takes the form

$$R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = -3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)]. \tag{3.37}$$

Thus, we can state the following:

Theorem 3.3 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$. The curvature tensor of $\mathcal{M}_\beta$ determined by $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \in \zeta^\perp$ with respect to $\widetilde{\widetilde{\nabla}}$ vanishes if and only if $\mathcal{M}_\beta$ with respect to the Levi-Civita connection ∇ is isomorphic to the hyperbolic space $H^{2n+1}(-3\beta^2)$.*

Replacing $\mathcal{H}_3$ by $\mathcal{H}_1$ and $\mathcal{H}_4$ by $\mathcal{H}_2$ in (3.37), we have

$$\kappa(\mathcal{H}_1, \mathcal{H}_2) = -\frac{R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)}{g(\mathcal{H}_1, \mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_2) - g(\mathcal{H}_1, \mathcal{H}_2)^2} = -3\beta^2. \tag{3.38}$$

Hence, we obtain the following result:

Corollary 3.1 *If $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0$ in a $L\beta K$ manifold, then the sectional curvature of the plane section determined by $\mathcal{H}_1, \mathcal{H}_2 \in \zeta^\perp$ is $-3\beta^2$.*

Furthermore, we obtain the following results:

Lemma 3.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$, then we have the following*

(i) $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\zeta = (2\beta^2 - \rho)[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2]$,

(ii) $\widetilde{R}(\zeta, \mathcal{H}_1)\mathcal{H}_2 = (2\beta^2 - \rho)[g(\mathcal{H}_1, \mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\mathcal{H}_1] - 2\beta\Phi(\mathcal{H}_1, \mathcal{H}_2)\zeta$,

(iii) $\widetilde{R}(\mathcal{H}_1, \zeta)\mathcal{H}_2 = -(2\beta^2 - \rho)[g(\mathcal{H}_1, \mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\mathcal{H}_1] + 2\beta\Phi(\mathcal{H}_1, \mathcal{H}_2)\zeta$,

(iv) $\widetilde{R}(\zeta, \mathcal{H}_1)\zeta = (2\beta^2 - \rho)\phi^2\mathcal{H}_1$,

(v) $\widetilde{S}(\mathcal{H}_1, \zeta) = 2n(2\beta^2 - \rho)\eta(\mathcal{H}_1)$,

(vi) $\widetilde{Q}\zeta = 2n(2\beta^2 - \rho)\zeta$,

(vii) $\eta(\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3) = (2\beta^2 - \rho)[g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1) - g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2)] + 2\beta[\eta(\mathcal{H}_2)\Phi(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_1)\Phi(\mathcal{H}_2, \mathcal{H}_3)]$,

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

Now, we define conformal curvature tensor with respect to g-TW connection $\widetilde{\nabla}$.

Definition 3.1 *The conformal curvature tensor $\widetilde{\mathcal{C}}$ for a (2n+1)-dimensional $L\beta K$ manifold $\mathcal{M}_\beta$ admitting g-TW connection is defined as*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 + g(\mathcal{H}_2, \mathcal{H}_3)\widetilde{Q}\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\widetilde{Q}\mathcal{H}_2\Big] + \frac{\widetilde{r}}{2n(2n-1)}\Big[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2\Big]. \tag{3.39}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$. Here $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor and the scalar curvature with respect to the connection $\widetilde{\nabla}$, respectively on $\mathcal{M}_\beta$.

Also, we can state the following:

Lemma 3.2 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$. Let $\widetilde{\mathcal{C}}$ be the conformal curvature tensor with respect to $\widetilde{\nabla}$. Then, we have the following*

(i) $\widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{\mathcal{C}}(\mathcal{H}_2, \mathcal{H}_1)\mathcal{H}_3 = 0$,

(ii) $\widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{\mathcal{C}}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 + \widetilde{\mathcal{C}}(\mathcal{H}_3, \mathcal{H}_1)\mathcal{H}_2 = 0$,

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

4. Lorentzian β-Kenmotsu manifold admitting g-TW connection satisfying $\widetilde{R} \cdot \widetilde{S} = 0$ condition

Let us consider a $L\beta K$ manifold admitting g-TW connection satisfying the condition

$$\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2) \cdot \widetilde{S} = 0, \tag{4.40}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}_\beta$.

From (4.40), we infer

$$(\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2) \cdot \widetilde{S})(\mathcal{F}_1, \mathcal{F}_2) = \widetilde{S}(\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{F}_1, \mathcal{F}_2) + \widetilde{S}(\mathcal{F}_1, \widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{F}_2) = 0, \tag{4.41}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{M}_\beta$.

Substituting $\mathcal{H}_1 = \zeta$ in (4.41), we have

$$\widetilde{S}(\widetilde{R}(\zeta, \mathcal{H}_2)\mathcal{F}_1, \mathcal{F}_2) + \widetilde{S}(\mathcal{F}_1, \widetilde{R}(\zeta, \mathcal{H}_2)\mathcal{F}_2) = 0, \tag{4.42}$$

By virtue of (3.30), we have

$$\widetilde{S}(\widetilde{R}(\zeta, \mathcal{H}_2)\mathcal{F}_1, \mathcal{F}_2) = (2\beta^2 - \rho)[g(\mathcal{H}_2, \mathcal{F}_1)\widetilde{S}(\zeta, \mathcal{F}_2) - \eta(\mathcal{F}_1)\widetilde{S}(\mathcal{H}_2, \mathcal{F}_2)] - 2\beta\Phi(\mathcal{H}_2, \mathcal{F}_1)\widetilde{S}(\zeta, \mathcal{F}_2), \tag{4.43}$$

and

$$\widetilde{S}(\mathcal{F}_1, \widetilde{R}(\zeta, \mathcal{H}_2)\mathcal{F}_2) = (2\beta^2 - \rho)[g(\mathcal{H}_2, \mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \zeta) - \eta(\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \mathcal{H}_2)] - 2\beta\Phi(\mathcal{H}_2, \mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \zeta), \tag{4.44}$$

where $\Phi(\mathcal{H}_2, \mathcal{F}_1) = g(\mathcal{H}_2, \phi\mathcal{F}_1)$ and $\Phi(\mathcal{H}_2, \mathcal{F}_2) = g(\mathcal{H}_2, \phi\mathcal{F}_2)$.

Substituting (4.43) and (4.44) in (4.42), we obtain

$$\begin{aligned}(2\beta^2 - \rho)[g(\mathcal{H}_2, \mathcal{F}_1)\widetilde{S}(\zeta, \mathcal{F}_2) - \eta(\mathcal{F}_1)\widetilde{S}(\mathcal{H}_2, \mathcal{F}_2) + g(\mathcal{H}_2, \mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \zeta) - \eta(\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \mathcal{H}_2)] \\ -2\beta[\Phi(\mathcal{H}_2, \mathcal{F}_1)\widetilde{S}(\zeta, \mathcal{F}_2) + \Phi(\mathcal{H}_2, \mathcal{F}_2)\widetilde{S}(\mathcal{F}_1, \zeta)] = 0.\end{aligned} \tag{4.45}$$

Setting $\mathcal{F}_1 = \zeta$ in (4.45) and on further simplification, we have

$$\widetilde{S}(\mathcal{H}_2, \mathcal{F}_2) = 2n(2\beta^2 - \rho)g(\mathcal{H}_2, \mathcal{F}_2) - 4n\beta\Phi(\mathcal{H}_2, \mathcal{F}_2). \tag{4.46}$$

Contracting above, we have

$$\widetilde{r} = 2n(2n+1)(2\beta^2 - \rho) - 4n\beta\psi, \tag{4.47}$$

where $\psi = trace(\phi)$.

By virtue of (3.32) in (4.46), we obtain

$$S(\mathcal{H}_2, \mathcal{F}_2) = -[2n\beta^2 + (2n-1)\rho]g(\mathcal{H}_2, \mathcal{F}_2) - (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{F}_2) - 2(2n-1)\beta\Phi(\mathcal{H}_2, \mathcal{F}_2). \tag{4.48}$$

Contracting above, we have

$$r = -2n(2n+1)\beta^2 - 2(2n-1)[n\rho + \beta\psi]. \tag{4.49}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 4.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying $\widetilde{R} \cdot \widetilde{S} = 0$ condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (4.46) and having scalar curvature $\widetilde{r}$ of the form (4.47), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (4.48) and having scalar curvature r of the form (4.49).

5. Lorentzian β-Kenmotsu manifold admitting g-TW connection satisfying $\widetilde{S} \cdot \widetilde{R} = 0$ condition

Let us consider a $L\beta K$ manifold admitting g-TW connection satisfying the condition

$$(\widetilde{S}(\mathcal{H}_1, \mathcal{H}_2) \cdot \widetilde{R})(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 = 0, \tag{5.50}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{M}_\beta$.

From(5.50), we infer that

$$(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}((\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{F}_1, (\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{H}_3 = 0, \tag{5.51}$$

where the endomorphism $\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2$ is defined by

$$(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{H}_3 = \widetilde{S}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2. \tag{5.52}$$

Substituting $\mathcal{H}_2 = \zeta$ in (5.51) and on further simplification, we obtain

$$\begin{aligned}\widetilde{S}(\zeta, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\zeta + \widetilde{S}(\zeta, \mathcal{F}_1)\widetilde{R}(\mathcal{H}_1, \mathcal{F}_2)\mathcal{H}_3 \\ -\widetilde{S}(\mathcal{H}_1, \mathcal{F}_1)\widetilde{R}(\zeta, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{S}(\zeta, \mathcal{F}_2)\widetilde{R}(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_3 - \widetilde{S}(\mathcal{H}_1, \mathcal{F}_2)\widetilde{R}(\mathcal{F}_1, \zeta)\mathcal{H}_3 \\ +\widetilde{S}(\zeta, \mathcal{H}_3)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\zeta = 0.\end{aligned} \tag{5.53}$$

Taking inner product of (5.53) with ζ, we have

$$\begin{aligned}\widetilde{S}(\zeta, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\eta(\mathcal{H}_1) + \widetilde{S}(\mathcal{H}_1, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3) + \widetilde{S}(\zeta, \mathcal{F}_1)\eta(\widetilde{R}(\mathcal{H}_1, \mathcal{F}_2)\mathcal{H}_3) \\ -\widetilde{S}(\mathcal{H}_1, \mathcal{F}_1)\eta(\widetilde{R}(\zeta, \mathcal{F}_2)\mathcal{H}_3) + \widetilde{S}(\zeta, \mathcal{F}_2)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_3) - \widetilde{S}(\mathcal{H}_1, \mathcal{F}_2)\eta(\widetilde{R}(\mathcal{F}_1, \zeta)\mathcal{H}_3) \\ +\widetilde{S}(\zeta, \mathcal{H}_3)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_1) - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\zeta) = 0.\end{aligned} \tag{5.54}$$

Setting $\mathcal{F}_1 = \mathcal{H}_3 = \zeta$ in (5.54) and on simplification, we have

$$\begin{aligned}(2\beta^2 - \rho)[\widetilde{S}(\mathcal{H}_1, \mathcal{F}_2) + \eta(\mathcal{F}_2)\widetilde{S}(\mathcal{H}_1, \zeta)] + 2n(2\beta^2 - \rho)^2[g(\mathcal{H}_1, \mathcal{F}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{F}_2)] \\ -4n\beta(2\beta^2 - \rho)\Phi(\mathcal{H}_1, \mathcal{F}_2) = 0.\end{aligned} \tag{5.55}$$

From (3.32), we have

$$\widetilde{S}(\mathcal{H}_1, \zeta) = 2n(2\beta^2 - \rho)\eta(\mathcal{H}_1). \tag{5.56}$$

Using (5.56) in (5.55), we obtain

$$\widetilde{S}(\mathcal{H}_1, \mathcal{F}_2) = -2n(2\beta^2 - \rho)g(\mathcal{H}_1, \mathcal{F}_2) - 4n(2\beta^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{F}_2) + 4n\beta\Phi(\mathcal{H}_1, \mathcal{F}_2). \tag{5.57}$$

Contracting above, we have

$$\widetilde{r} = -2n(2n-1)(2\beta^2 - \rho) + 4n\beta\psi, \tag{5.58}$$

where $\psi = trace(\phi)$.

Furthermore, using (3.32) in (5.57), we obtain

$$\begin{aligned}S(\mathcal{H}_1, \mathcal{F}_2) = [(2n+1)\rho - 10n\beta^2]g(\mathcal{H}_1, \mathcal{F}_2) + [(2n+1)\rho - 8n\beta^2]\eta(\mathcal{H}_1)\eta(\mathcal{F}_2) \\ + 2(2n+1)\beta\Phi(\mathcal{H}_1, \mathcal{F}_2).\end{aligned} \tag{5.59}$$

Contracting above, we have

$$r = 2n(2n+1)\rho - 2n(10n+1)\beta^2 + 2(2n+1)\beta\psi. \tag{5.60}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 5.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$ satisfying $\widetilde{S} \cdot \widetilde{R} = 0$ condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\widetilde{\nabla}}$ whose Ricci tensor is of the form (5.57) and having scalar curvature $\widetilde{r}$ of the form (5.58), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (5.59) and having scalar curvature r of the form (5.60).

6. Conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$.

Definition 6.1 *A $L\beta K$ manifold is said to be conformally flat with respect to g-TW connection $\widetilde{\widetilde{\nabla}}$ if it satisfies*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0, \tag{6.61}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

By virtue of (6.61) in (3.39), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 + g(\mathcal{H}_2, \mathcal{H}_3)\widetilde{Q}\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\widetilde{Q}\mathcal{H}_2\Big] \\ - \frac{\widetilde{r}}{2n(2n-1)}\Big[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2\Big].\end{aligned} \tag{6.62}$$

Taking inner product of (6.62) with ζ and on further simplification, we have

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \zeta) = \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1) - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2)\Big] \\ + \left[\frac{4n^2(2\beta^2 - \rho) - \widetilde{r}}{(2n-1)}\right][g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1) - g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2)].\end{aligned} \tag{6.63}$$

Further, on substituting $\mathcal{H}_4 = \zeta$ in (3.31) and using (2.12), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \zeta) = (2\beta^2 - \rho)[g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1) - g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2)] \\ + 2\beta[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)].\end{aligned} \tag{6.64}$$

Using (6.64) in (6.63), we infer

$$\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-\widetilde{S}(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)=\left[\frac{\widetilde{r}-2n(2\beta^2-\rho)}{2n}\right][g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)] + 2(2n-1)\beta\left[\eta(\mathcal{H}_2)g(\mathcal{H}_1,\phi\mathcal{H}_3)-\eta(\mathcal{H}_1)g(\mathcal{H}_2,\phi\mathcal{H}_3)\right]. \tag{6.65}$$

Assuming $\mathcal{H}_1=\zeta$ in (6.65) and on further simplification, we have

$$\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)=\left[\frac{\widetilde{r}-2n(2\beta^2-\rho)}{2n}\right]g(\mathcal{H}_2,\mathcal{H}_3)+\left[\frac{\widetilde{r}-2n(2n+1)(2\beta^2-\rho)}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) - 2(2n-1)\beta\Phi(\mathcal{H}_2,\mathcal{H}_3), \tag{6.66}$$

where $\Phi(\mathcal{H}_2,\mathcal{H}_3)=g(\mathcal{H}_2,\phi\mathcal{H}_3)$. Using (3.32) in (6.66), we obtain

$$S(\mathcal{H}_2,\mathcal{H}_3)=\left[\frac{r+2n\beta^2-2\beta\psi}{2n}\right]g(\mathcal{H}_2,\mathcal{H}_3)+\left[\frac{r+2n(2n+1)\beta^2-2\beta\psi}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) - 4(n-1)\beta\Phi(\mathcal{H}_2,\mathcal{H}_3). \tag{6.67}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 6.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying conformally flat condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (6.66), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (6.67).

7. ζ-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine ζ-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\nabla}$.

Definition 7.1 *A $L\beta K$ manifold is said to be ζ-conformally flat with respect to g-TW connection $\widetilde{\nabla}$ if it satisfies*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\zeta=0, \tag{7.68}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}_\beta$.

Setting $\mathcal{H}_3 = \zeta$ in (3.39) and using (7.68), we obtain

$$\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\zeta = \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2, \zeta)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \zeta)\mathcal{H}_2 + \eta(\mathcal{H}_2)\widetilde{Q}\mathcal{H}_1 - \eta(\mathcal{H}_1)\widetilde{Q}\mathcal{H}_2\Big] - \frac{\widetilde{r}}{2n(2n-1)}\Big[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2\Big]. \tag{7.69}$$

On further simplification, we have

$$\eta(\mathcal{H}_2)\widetilde{Q}\mathcal{H}_1 - \eta(\mathcal{H}_1)\widetilde{Q}\mathcal{H}_2 = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right]\Big[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2\Big]. \tag{7.70}$$

Taking inner product of (7.70) with $\mathcal{H}_3$, we have

$$\eta(\mathcal{H}_2)\widetilde{S}(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_1)\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right]\Big[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_3)\Big]. \tag{7.71}$$

Substituting $\mathcal{H}_1 = \zeta$ in (7.71), we obtain

$$\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right] g(\mathcal{H}_2, \mathcal{H}_3) + \left[\frac{\widetilde{r} - 2n(2n+1)(2\beta^2 - \rho)}{2n}\right] \eta(\mathcal{H}_2)\eta(\mathcal{H}_3). \tag{7.72}$$

Using (3.32) in (7.72), we have

$$S(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{r + 2n\beta^2 - 2\beta\psi}{2n}\right] g(\mathcal{H}_2, \mathcal{H}_3) + \left[\frac{r + 2n(2n+1)\beta^2 - 2\beta\psi}{2n}\right] \eta(\mathcal{H}_2)\eta(\mathcal{H}_3) + 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{7.73}$$

where $\Phi(\mathcal{H}_2, \mathcal{H}_3) = g(\mathcal{H}_2, \phi\mathcal{H}_3)$.

Thus, based on the discussion above, we can present the following theorem:

Theorem 7.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying ζ-conformally flat condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is an η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (7.72), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (7.73).

8. Pseudo-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine pseudo-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$.

Definition 8.1 *A $L\beta K$ manifold is said to be pseudo-conformally flat with respect to g-TW connection $\widetilde{\widetilde{\nabla}}$ if it satisfies*

$$g(\widetilde{\mathcal{C}}(\phi\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3,\phi\mathcal{H}_4)=0, \tag{8.74}$$

for any vector fields $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$ and $\mathcal{H}_4$ on $\mathcal{M}_\beta$.

By virtue of (3.39) and (8.74), we have

$$\begin{aligned}\widetilde{R}(\phi\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\phi\mathcal{H}_4)=&\frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-\widetilde{S}(\phi\mathcal{H}_1,\mathcal{H}_3)g(\mathcal{H}_2,\phi\mathcal{H}_4)\\&+g(\mathcal{H}_2,\mathcal{H}_3)\widetilde{S}(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-g(\phi\mathcal{H}_1,\mathcal{H}_3)\widetilde{S}(\mathcal{H}_2,\phi\mathcal{H}_4)\Big]\\&-\frac{\widetilde{r}}{2n(2n-1)}\left[g(\mathcal{H}_2,\mathcal{H}_3)g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-g(\phi\mathcal{H}_1,\mathcal{H}_3)g(\mathcal{H}_2,\phi\mathcal{H}_4)\right].\end{aligned} \tag{8.75}$$

Let $\{e_1,e_2,e_3,\ldots,e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_2=\mathcal{H}_3=e_i$ in (8.75) and summing over i for $1\leq i\leq(2n+1)$, we obtain

$$(2n+1)\widetilde{r}g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)=0. \tag{8.76}$$

Since $(2n+1)\neq 0$, therefore

$$\widetilde{r}g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)=0. \tag{8.77}$$

By virtue of (2.6), we have

$$\widetilde{r}\left[g(\mathcal{H}_1,\mathcal{H}_4)+\eta(\mathcal{H}_1)\eta(\mathcal{H}_4)\right]=0. \tag{8.78}$$

Replacing $\mathcal{H}_1$ by $\widetilde{Q}\mathcal{H}_1$ in (8.78), we have

$$\widetilde{r}\left[\widetilde{S}(\mathcal{H}_1,\mathcal{H}_4)+2n(2\beta^2-\rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4)\right]=0. \tag{8.79}$$

From above, we infer following cases:

Case I: If $\widetilde{r} = 0$. Then from (3.34), we obtain

$$r = -6n(2n+1)\beta^2 + 4n\rho + 2\beta\psi, \tag{8.80}$$

where $\psi = trace(\phi)$.

Case II: If $\widetilde{r} \neq 0$. Then from (8.79), we have

$$\widetilde{S}(\mathcal{H}_1, \mathcal{H}_4) = -2n(2\beta^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4). \tag{8.81}$$

Contracting above, we infer

$$\widetilde{r} = 2n(2\beta^2 - \rho). \tag{8.82}$$

Using (3.32) in (8.81), we obtain

$$S(\mathcal{H}_1, \mathcal{H}_4) = -(6n\beta^2 - \rho)g(\mathcal{H}_1, \mathcal{H}_4) - (4n\beta^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4) + 2\beta\Phi(\mathcal{H}_1, \mathcal{H}_4), \tag{8.83}$$

where $\Phi(\mathcal{H}_1, \mathcal{H}_4) = g(\mathcal{H}_1, \phi\mathcal{H}_4)$.
Contracting above, we have

$$r = -2n(6n+1)\beta^2 + 2n\rho + 2\beta\psi. \tag{8.84}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 8.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional LβK manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$ satisfying pseudo-conformally flat condition. Then we have the following:*
(i) The scalar curvature $\widetilde{r}$ with respect to $\widetilde{\widetilde{\nabla}}$ vanishes. Moreover, the scalar curvature r with respect to Levi-Civita connection ∇ is of the form (8.80), or
(ii) $\mathcal{M}_\beta$ is an η-Einstein manifold with respect to $\widetilde{\widetilde{\nabla}}$ whose Ricci tensor is of the form (8.81) and having scalar curvature $\widetilde{r}$ of the form (8.82). Moreover, $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (8.83) and having scalar curvature of the form (8.84).

9. Example of a three-dimensional Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we illustrate an example of a three-dimensional Lorentzian β-Kenmotsu manifold. Consider the three-dimensional manifold

$$\mathcal{M}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We define the vector fields

$$\vartheta_1 = e^{-z}\frac{\partial}{\partial x}, \quad \vartheta_2 = e^{-z}\frac{\partial}{\partial y}, \quad \vartheta_3 = e^{-z}\frac{\partial}{\partial z} = \zeta,$$

which remain linearly independent at each point in M.

The Lorentzian metric g is given by

$$g(\vartheta_1, \vartheta_1) = 1, \quad g(\vartheta_2, \vartheta_2) = 1, \quad g(\vartheta_3, \vartheta_3) = -1,$$

$$g(\vartheta_1, \vartheta_2) = g(\vartheta_2, \vartheta_3) = g(\vartheta_3, \vartheta_1) = 0,$$

which can be expressed as

$$g = e^{2z}(dx \otimes dx + dy \otimes dy - dz \otimes dz).$$

Let the 1-form η satisfy

$$\eta(\mathcal{H}_1) = g(\mathcal{H}_1, \vartheta_3)$$

The $(1,1)$-tensor field ϕ is defined as

$$\phi(\vartheta_1) = -\vartheta_2, \quad \phi(\vartheta_2) = -\vartheta_1, \quad \phi(\vartheta_3) = 0.$$

For any vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}^3$, the following conditions hold:

$$\phi^2(\mathcal{H}_1) = \mathcal{H}_1 + \eta(\mathcal{H}_1)\vartheta_3,$$

$$g(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2).$$

Thus, the structure $\mathcal{M}^3(\phi, \zeta, \eta, g)$ forms an almost contact metric structure on $\mathcal{M}^3$, where we set $\vartheta_3 = \zeta$.

The Lie brackets of the vector fields are computed as follows:

$$[\vartheta_1, \vartheta_3] = e^{-z}\vartheta_1, \quad [\vartheta_1, \vartheta_2] = 0, \quad [\vartheta_2, \vartheta_3] = e^{-z}\vartheta_2.$$

Using Koszul's formula, the Levi-Civita connection ∇ is obtained as

$$\begin{cases} \nabla_{\vartheta_1}\vartheta_1 = e^{-z}\vartheta_3, \quad \nabla_{\vartheta_2}\vartheta_1 = 0, \quad \nabla_{\vartheta_3}\vartheta_1 = 0, \\ \nabla_{\vartheta_1}\vartheta_2 = 0, \quad \nabla_{\vartheta_2}\vartheta_2 = e^{-z}\vartheta_3, \quad \nabla_{\vartheta_3}\vartheta_2 = 0, \\ \nabla_{\vartheta_1}\vartheta_3 = 0, \quad \nabla_{\vartheta_2}\vartheta_3 = 0, \quad \nabla_{\vartheta_3}\vartheta_3 = 0. \end{cases} \tag{9.85}$$

From the above results, setting $\beta = e^{-z}$, we conclude that $\mathcal{M}^3(\phi, \zeta, \eta, g)$ defines a $\mathcal{M}_\beta$ structure in dimension three. From (3.27) and (9.85), we obtain

$$\begin{cases} \widetilde{\nabla}_{\vartheta_1}\vartheta_1 = 2e^{-z}\vartheta_3, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_1 = 0, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_1 = -\vartheta_2, \\ \widetilde{\nabla}_{\vartheta_1}\vartheta_2 = 0, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_2 = 2e^{-z}\vartheta_3, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_2 = -\vartheta_1, \\ \widetilde{\nabla}_{\vartheta_1}\vartheta_3 = e^{-z}\vartheta_1, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_3 = e^{-z}\vartheta_2, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_3 = 0. \end{cases} \tag{9.86}$$

The components of the curvature tensor with respect to the Levi-Civita connection ∇ are given by:

$$\begin{cases} R(\vartheta_1,\vartheta_2)\vartheta_1 = e^{-2z}\vartheta_2, \quad R(\vartheta_2,\vartheta_3)\vartheta_1 = 0, \quad R(\vartheta_1,\vartheta_3)\vartheta_1 = e^{-2z}\vartheta_3, \\ R(\vartheta_1,\vartheta_2)\vartheta_2 = -e^{-2z}\vartheta_1, \quad R(\vartheta_2,\vartheta_3)\vartheta_2 = e^{-2z}\vartheta_3, \quad R(\vartheta_1,\vartheta_3)\vartheta_2 = 0, \\ R(\vartheta_1,\vartheta_2)\vartheta_3 = 0, \quad R(\vartheta_2,\vartheta_3)\vartheta_3 = e^{-2z}\vartheta_2, \quad R(\vartheta_1,\vartheta_3)\vartheta_3 = e^{-2z}\vartheta_1. \end{cases} \tag{9.87}$$

The components of the curvature tensor with respect to the g-TW connection $\widetilde{\nabla}$ are given by:

$$\begin{cases} \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_1 &= -2e^{-2z}\vartheta_2, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_1 = -2e^{-z}\vartheta_3, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_1 = -2e^{-2z}\vartheta_3 + \rho\vartheta_3, \\ \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_2 &= 2e^{-2z}\vartheta_1, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_2 = -2e^{-2z}\vartheta_3 + \rho\vartheta_3, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_2 = -2e^{-z}\vartheta_3, \\ \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_3 &= 0, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_3 = -2e^{-2z}\vartheta_2 + \rho\vartheta_2, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_3 = -2e^{-2z}\vartheta_1 + \rho\vartheta_1. \end{cases} \tag{9.88}$$

From (9.87), the non-vanishing components of Ricci tensor with respect to Levi-Civita connection ∇ is as follows

$$S(\vartheta_1,\vartheta_1) = -2e^{-2z}, \quad S(\vartheta_2,\vartheta_2) = -2e^{-2z}, \quad S(\vartheta_3,\vartheta_3) = 2e^{-2z}, \tag{9.89}$$

which implies that the scalar curvature r with respect to ∇ can be evaluated by

$$r = \sum_{i=1}^{3} \varepsilon_i S(e_i, e_i) = -6e^{-2z}. \tag{9.90}$$

Furthermore, from (9.88), the non-vanishing components of Ricci tensor with respect to the g-TW connection $\widetilde{\nabla}$ are given as

$$\widetilde{S}(\vartheta_1,\vartheta_1) = 4e^{-2z} - \rho, \quad \widetilde{S}(\vartheta_2,\vartheta_2) = 4e^{-2z} - \rho, \quad \widetilde{S}(\vartheta_3,\vartheta_3) = -4e^{-2z} + 2\rho, \tag{9.91}$$

which implies that the scalar curvature $\widetilde{r}$ with respect to $\widetilde{\nabla}$ can be evaluated by

$$\widetilde{r} = \sum_{i=1}^{3} \varepsilon_i \widetilde{S}(e_i, e_i) = 12e^{-2z} - 4\rho. \tag{9.92}$$

which can also be verified from (3.34) where ψ can be evaluated as

$$\psi = trace(\phi) = \sum_{i=1}^{3} \varepsilon_i \Phi(e_i, e_i) = 0. \tag{9.93}$$

10. Conclusion

In this paper, we conducted a comprehensive study of Lorentzian β-Kenmotsu ($L\beta K$) manifolds equipped with the generalized Tanaka-Webster (g-TW) connection. Beginning with fundamental definitions and preliminary results, we established the essential structure equations and derived explicit expressions for the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ in this setting. Our analysis revealed several significant geometric properties, including the conditions under which an $L\beta K$ manifold admitting the g-TW connection becomes a generalized η-Einstein manifold.

We demonstrated that a $L\beta K$ manifold satisfies crucial curvature identities, such as the symmetry and skew-symmetry of the curvature tensor, and explored conditions like $\widetilde{R} \cdot \widetilde{S} = 0$ and $\widetilde{S} \cdot \widetilde{R} = 0$, under which the manifold naturally admits a generalized η-Einstein structure. Further, we investigated the geometric implications of conformally flat and ζ-conformally flat conditions, showing that such manifolds inherently exhibit the generalized η-Einstein property with respect to the g-TW connection. Additionally, we examined the notion of pseudo-conformal flatness in $L\beta K$ manifolds, establishing key results regarding scalar curvature and the structure of the Ricci tensor.

To solidify our theoretical findings, we provided an explicit example of a three-dimensional $L\beta K$ manifold equipped with the g-TW connection and verified that it satisfies the curvature conditions discussed throughout the paper. This study offers new insights into the geometric nature of Lorentzian β-Kenmotsu manifolds and their curvature properties under different structural constraints. The results presented here open pathways for further research, including extensions to higher-dimensional cases, the study of additional curvature conditions, and potential applications in mathematical physics and relativity.

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