



A NOTE ON CONNECTED POWER DOMINATION OF CERTAIN CLASSES OF GRAPHS

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Abstract. The efficiency of monitoring the electrical power systems can be optimised with the use of Phasor Measurement Unit (PMU). A complete monitoring of the network can be effectively implemented through the power domination problem, downsizing the count of PMUs. In this paper, we present a detailed study on the Connected Power Domination (CPD) of a graph, a variant of power domination with an additional constraint of being connected. Tight upper bounds on the Connected Power Domination Number of a graph G , denoted $\gamma_{p,c}(G)$ are established and also the nature of the CPD set of unicyclic graphs is explored. Further, this study gives the exact $\gamma_{p,c}$ -set for certain classes of graphs including, Grid graph, Join of graphs, Kragujevac tree and Honeycomb network.

Keywords: Connected Power Domination, Connected component, Unicyclic graph, Pendant path, Grid graph, Join of graphs, Kragujevac tree, Honeycomb network

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1. INTRODUCTION

The need for real-time monitoring and having control over their vast network led the electrical power companies to opt for the use of PMU placements in their networks. These units give a greater understanding of the network performance, which includes the voltage measure and phase angle measure. Taking into account the excessive maintenance cost of the PMUs, the aim is to downsize the quantity of PMUs used. The optimal PMU placement (OPP) problem first appeared in the literature in the early 90's and was widely explored in [1, 5, 6, 15, 16]. This problem was then parametrized into a graph theoretical problem by Haynes in 2002 [13], by associating the vertices of the graph with electrical nodes and the transmission cables with edges. Here, the aim is to find the minimum number of PMU locations to monitor the entire network.

The above discussed problem was formally defined as follows: Let G be the graph model of the given electrical network and let S denote the PMU locations. In line with Ohm's law and Kirchhoff's current law, the system operates with simple monitoring rules as outlined below.

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- (1) The set S is monitored along with its neighbors and incident edges (Ohm's law)
- (2) If a vertex has exactly one unmonitored neighbor, then that neighbor is also monitored (Kirchhoff's current law)

The OPP problem is closely associated with two widely recognised problems, the domination problem and the vertex colouring problem.

Furthermore, in order to improve the smooth operation, response time and efficiency of the system, the electrical power companies might consider including the constraint of positioning the PMU units in a confined and interconnected environment. This added constraint allows the benefit of reduction in the number of processing hubs and data collection infrastructures required. This inspired the problem of Connected Power Domination (CPD), a variation of power domination, where the minimal Power Domination Set (PDS) is required to induce a connected subgraph. Moreover, the CPD of a graph cannot be computed in polynomial time as it is NP-complete to find the PDS of a general graph [4]. Nevertheless, it is polynomial-time solvable for restricted classes such as chain graphs and threshold graphs [10]. Furthermore, the CPD problem has been studied for certain product graphs [8].

2. PRELIMINARIES

The study includes only connected, finite, undirected and simple graphs. For a simple connected graph $G = (V, E)$ with $|V(G)| = n$, the set V comprises of all the vertices and the set E includes the edges of the graph. The adjacency of the vertices (neighbors) is justified by the existence of the edge between them and is denoted by $v \sim w$. Given any arbitrary vertex u , the *open neighborhood* of u , $N(u)$ consists all the adjacent vertices of u , whereas u along with $N(u)$ makes up its *closed neighborhood* $N[u]$. The open neighborhood of the set S is the collection $\{N(w)|w \in S\}$ and $\{N[w]|w \in S\}$ is its closed neighborhood. The *degree* of u , denoted by $d(u)$, refers to the cardinality of $N(u)$, with degree one vertex being referred to as *pendant (leaf)* and the vertex with $|N(u)| = n - 1$ as *universal vertex*. The *maximum degree* $\Delta(G)$ of a graph is obtained by $\max\{d(u)|u \in V(G)\}$ and the *minimum degree* $\delta(G)$ by $\min\{d(u)|u \in V(G)\}$. The distance between two vertices $u, v \in V(G)$, denoted $d(u, v)$, is the length of the shortest $u - v$ path in G . The diameter of G is given by $\text{diam}(G) = \max\{d(u, v)|u, v \in V(G)\}$. The isomorphism between two graphs G_i and G_j is symbolised by $G_i \cong G_j$. $\omega(G)$ denotes the number of connected components of G . Considering any subset $S \subset V(G)$, $G[S]$ represents the *S-induced subgraph* of G , constructed with the vertex set S and the edges that have endpoints in S . A graph having precisely one cycle is termed as an *unicyclic graph*. A spider is a tree in which only one vertex, called the branch vertex is of degree greater than 2. Let $K_n, K_{m,n}, W_{1,n}, S_{1,n}$ and $B_{m,n}$ denote the complete graph, complete bipartite graph, wheel graph, star graph and bistar graph, respectively. Helm graph H_n , is obtained by adjoining a pendant vertex to all the vertices in $W_{1,n}$ except the central vertex. Kayak paddle $KP(k, m, l)$ is obtained by adjoining two cycles C_k and C_m by a path of length l [7].

Given any subset $D \subset V(G)$, D is the dominating set of G provided that $N[D]$ is equivalent to $V(G)$. The domination number of G , denoted by $\gamma(G)$, is the cardinality of the smallest dominating set of G [12]. In addition, if $G[D]$ is connected, then D is called the connected dominating set (CDS), where $\gamma_c(G)$ denotes the cardinality of the minimal CDS of G [17].

In 2002, Haynes et al. [13] formulated a new variation of domination, called the power domination. The set S is called the power dominating set of G if all the vertices are monitored by S , adhering to the rules provided below.

- (1) *Domination*: Initially $N[S]$ is monitored
- (2) *Propagation*: For any monitored vertex u , if $N[u] - w$ is monitored, with $w \in N(u)$ then w is also monitored

The power domination number symbolised by $\gamma_p(G)$ is the minimal cardinality of all such PDS. In this article, we discuss in detail a variant of power domination called the Connected Power Domination (CPD).

Definition 2.1. [4] For a connected graph $G = (V, E)$, a set $S_c \subset V(G)$ is a connected power dominating set (CPDS) of G if S_c is a PDS and $G[S_c]$ is connected. $\gamma_{p,c}(G)$ denotes the connected power domination number of G , which is the order of the smallest CPDS by cardinality.

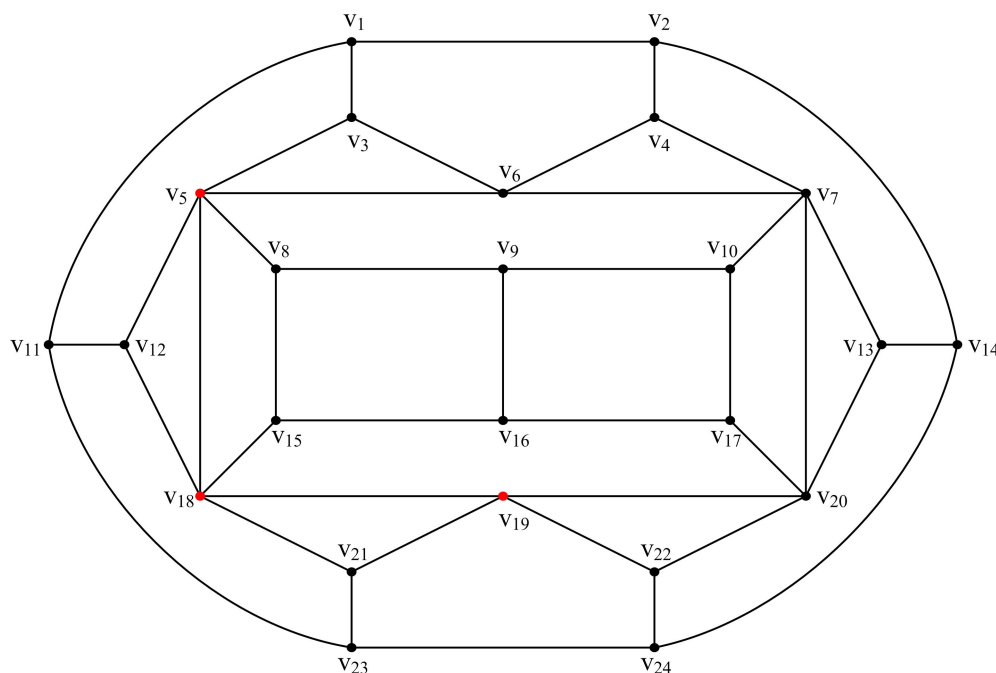


FIGURE 1. A graph G with connected power dominating set

Figure 1 gives a clear demonstration of the CPDS of the graph G . Here, the entire graph can be power dominated by the set $S = \{v_5, v_{19}\}$. However, to incorporate the constraint of connectedness, it is necessary to extend the set S by including the vertex v_{18} , resulting in the minimum CPDS being $S_c = \{v_5, v_{18}, v_{19}\}$. We further observe that, the minimum dominating set is $D = \{v_2, v_5, v_9, v_{18}, v_{20}, v_{23}\}$ and the minimum CDS is $D_c = \{v_2, v_4, v_5, v_6, v_8, v_9, v_{18}, v_{19}, v_{20}, v_{21}, v_{23}\}$. Thus, $\gamma_p(G) = 2$, $\gamma_{p,c}(G) = 3$, $\gamma(G) = 6$ and $\gamma_c(G) = 11$.

For any connected graph $G \not\cong P_n$, consider a vertex $u \in V(G)$ of degree greater than 2. A *pendant path incident to a vertex u* , is a maximal set P of $V(G)$ such that $G[P]$ is a connected

component of $G - u$, which is a path. One of the end vertices of P is adjacent to u . The vertex in $P \cap N(u)$ is the *base* of P . Let $p(u)$ denote the number of pendant paths incident to u [3]. Further, we define

$$\begin{aligned}\mathcal{P}_1(G) &= \{u \in V(G) \mid p(u) = 1 \text{ and } \omega(G - u) = 2\} \\ \mathcal{P}_2(G) &= \{u \in V(G) \mid p(u) = 0 \text{ and } \omega(G - u) = 2\} \\ \mathcal{P}_3(G) &= \{u \in V(G) \mid \omega(G - u) > 2\} \\ \mathcal{P}_4(G) &= \{u \in V(G) \mid d(u) = 1\} \\ \mathcal{M}(G) &= \mathcal{P}_2(G) \cup \mathcal{P}_3(G)\end{aligned}$$

3. BOUNDS ON THE CONNECTED POWER DOMINATION NUMBER OF A GRAPH

In this section, we establish bounds on the connected power domination number for general graphs. Furthermore, we study its behaviour for certain unicyclic graphs.

Proposition 3.1. [17] *For any connected graph G , $\gamma(G) \leq \gamma_c(G)$.*

Proposition 3.2. [13] *For any graph G , $1 \leq \gamma_p(G) \leq \gamma(G)$.*

Lemma 3.1. [4] *Let $G = (V, E)$ be a connected graph different from a path and R be an arbitrary connected power dominating set of G . Then $\mathcal{M}(G) \subset R$.*

Lemma 3.2. [4] *Let G be a graph different from a path. Then, no minimum connected power dominating set of G contains a leaf of G .*

Theorem 3.1. [4] *Let T be a tree. Then, $\gamma_{p,c}(T) = \max\{1, |\mathcal{M}(T)|\}$.*

Theorem 3.2. *For any graph G , $\gamma_p(G) \leq \gamma_{p,c}(G) \leq \gamma_c(G)$.*

Proof. Let G be a graph with $n \geq 2$ vertices. Let D, D_c, S and S_c denote the minimal of dominating set, connected dominating set, power dominating set and connected power dominating set of G respectively. By the definition of connected power domination, we have $S_c = S \cup X$, where $X \subset V(G)$ such that $G[S \cup X]$ is connected. If $X = \emptyset$, then $S_c = S$ and if $X \neq \emptyset$, then $|S_c| > |S|$. Thus, $\gamma_p(G) \leq \gamma_{p,c}(G)$. Note that, S and X are mutually disjoint sets, thus $|S_c| = |S| + |X|$. Similarly, $D_c = D \cup Y$, where $Y \subset V(G)$ such that $G[D \cup Y]$ is connected. Thus, $|D_c| = |D| + |Y|$. From Proposition 3.2, we know that $|S| \leq |D|$. That is,

$$|S_c| - |X| \leq |D_c| - |Y| \quad (3.1)$$

To claim that, $|S_c| \leq |D_c|$, we consider the following cases.

Case 1: If $X = Y = \emptyset$, then eqn. 3.1 implies $|S_c| \leq |D_c|$.

Case 2: Let $X = \emptyset$ and $Y \neq \emptyset$. From eqn. 3.1, we have $|S_c| \leq |D_c| - |Y|$. Note that $|Y| > 0$, thus $|S_c| < |D_c|$.

Case 3: When $X \neq \emptyset$ and $Y = \emptyset$, we have $|S_c| - |X| \leq |D_c|$. Suppose, $|S_c| > |D_c|$. Then, we have

$$\begin{aligned}|S_c| - |X| &> |D_c| - |X| \\ \implies |S| &> |D_c| - |X|\end{aligned}$$

a contradiction, since $|X| > 0$ and $|S| \leq |D| \leq |D_c|$. Therefore, $|S_c| \leq |D_c|$.

Case 4: Let $X \neq \emptyset$ and $Y \neq \emptyset$. Suppose, $|X| \leq |Y|$, then,

$$\begin{aligned} |S_c| - |X| &\leq |D_c| - |Y| \\ \implies |S| + |Y| &\leq |D_c| \\ \implies |S| + |X| + k &\leq |D_c| \quad (\text{since } |Y| = |X| + k, \text{ for some } k) \\ \text{Therefore, } |S_c| &\leq |D_c| \end{aligned}$$

Suppose, $|X| > |Y|$, then it is easy to claim that, $|S_c| \leq |D_c|$. Hence, $\gamma_{p,c}(G) \leq \gamma_c(G)$. $\square$

Theorem 3.3. *Let G be a graph of order $n \geq 3$ with no universal vertex. Then $\gamma_{p,c}(G) \leq n - \Delta(G) - 1$.*

Proof. Let t be a vertex of maximum degree $\Delta(G)$ in G . Construct a spanning tree T of G such that every neighbor of t in G is also a neighbor of t in T . Let $N(t)$ denote the neighbors of t in T .

Case 1 : Suppose all the vertices in the set $V(T) - \{t\}$ is of degree one or two. Define $V(T) - \{t\} = X \cup Y$, where $X = \{v \in V(T) - \{t\} \mid d(v) = 1\}$ and $Y = \{v \in V(T) - \{t\} \mid d(v) = 2\}$. If $Y = \emptyset$, then $\Delta(G) = n - 1$, a contradiction. Therefore, Y is a non-empty subset of $V(T) - \{t\}$. In order to construct a connected power domination set, consider the vertex $t \in T$. Note that this vertex dominates all the vertices of $N(t)$ and the remaining vertices of $V(T) - N[t]$ are power dominated by t , as they all belong to a pendant path with base as one of the vertices of $N(t)$. Thus, $\gamma_{p,c}(G) = 1 \leq n - \Delta(G) - 1$.

Case 2 : Suppose $V(T) - \{t\}$ has at least one vertex of degree greater than 2. Define $V(T) - \{t\} = X \cup Y \cup Z$, where $X = \{v \in V(T) - \{t\} \mid d(v) = 1\}$, $Y = \{v \in V(T) - \{t\} \mid d(v) = 2\}$ and $Z = \{v \in V(T) - \{t\} \mid d(v) > 2\}$ with $|Z| \geq 1$. Now, T has at least $\Delta(G) + |Z|$ leaves. Thus, to monitor all the vertices, we need at most $n - (\Delta(G) + |Z|)$ vertices. Since $|Z| \geq 1$, we have $\gamma_{p,c}(G) \leq n - (\Delta(G) + 1)$. $\square$

Theorem 3.4. *For any tree $T \not\cong P_n$, $\gamma_{p,c}(T) = 1$ if and only if T is a spider.*

Proof. If T is a spider and let v be the branch vertex of T , then clearly $\{v\}$ is a connected power dominating set of T . Conversely, if T is not a spider, then there is a vertex w , such that $v \neq w$ with degree greater than 2. Thus, $\gamma_{p,c}(T) \geq 2$. $\square$

Theorem 3.5. *For any connected graph G of order $n \leq 5$, $\gamma_p(G) = \gamma_{p,c}(G)$.*

Proof. Let G be a connected graph with $n \leq 5$ vertices. Then, $\Delta(G) \leq 4$. Suppose the graph G is of order $n \leq 4$, then G is either P_n or C_n or contains $S_{1,n-1}$. Thus, $\gamma_{p,c}(G) = 1$. When $n = 5$, G either contains $S_{1,4}$ or $\Delta(G) \leq 3 = n - 2$. In either case $\gamma_{p,c}(G) = 1$, with the latter condition being justified by Theorem 3.3. $\square$

Theorem 3.6. *Let T be a tree of order $n \geq 2$ with diameter at most 3. Then, $\gamma_p(T) = \gamma_{p,c}(T)$.*

Proof. Let T be a tree of diameter at most 3. Clearly, T has no more than 2 vertices of degree greater than 2. If there is only one such vertex, then T is a spider, thus by Theorem 3.4,

$\gamma_p(G) = \gamma_{p,c}(G)$. In the case where there are 2 vertices, u and v of degree greater than 2, $S = \{u, v\}$ is a power dominating set of T , since every other vertex in T is power dominated by the vertices in S . Suppose u and v are not adjacent, there is a uv -path P_n ($n \geq 3$) in T , which implies that the neighbor of u and the neighbor of v not in the uv -path are at a distance exceeding 3, a contradiction to the fact that $\text{diam}(T) \leq 3$. Thus, $\gamma_p(T) = \gamma_{p,c}(T)$. $\square$

Theorem 3.7. *Let T be a tree and S_c be the minimum connected power dominating set of T . Then, every vertex $v \in V - S_c$ is either a pendant vertex or a vertex of degree two with $p(v) = 1$.*

Proof. For a path, the result clearly holds. Hence, let us assume that $T \not\cong P_n$ and S_c be the connected power dominating set of T . The vertices in T can be partitioned into four disjoint sets as $\mathcal{P}_1(T)$, $\mathcal{P}_2(T)$, $\mathcal{P}_3(T)$ and $\mathcal{P}_4(T)$. From Theorem 3.1, it is clear that $S_c = \mathcal{M}(T)$. Thus, the vertices in $V - S_c$ are the pendant vertices and the degree 2 vertices which belong to a pendant path. $\square$

In the following results we consider unicyclic graphs G with cycle C such that $V(C) = \{v \in V(G) \mid d(v) \geq 2 \text{ and } p(v) = 0\}$.

Theorem 3.8. *Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. If $\langle S \rangle$ is connected, then $\gamma_{p,c}(G) = |\mathcal{M}(G)|$.*

Proof. Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. Then, the only connected components of $G - C$ are trees different from a path. We claim that $\mathcal{M}(G)$ is the minimum connected power dominating set of G . The vertices in G can be partitioned into two distinct subsets as $V(G) = X \cup Y$, where $X = \{v \in V(G) \mid \omega(G - v) \geq 2\}$ and $Y = \{v \in V(G) \mid \omega(G - v) = 1\}$. Note that $X = \mathcal{P}_1(G) \cup \mathcal{P}_2(G) \cup \mathcal{P}_3(G)$. Clearly, $Y = V(C - S) \cup \mathcal{P}_4(G)$. Now, deleting the set $\mathcal{P}_1(G) \cup \mathcal{P}_4(G) \cup V(C - S)$ does not disconnect the graph G , as S is connected. Thus, $\mathcal{M}(G)$ is a connected set. Also, $S \subset \mathcal{M}(G)$ initially dominates all its neighbors including the vertices in $N(S) \cap C$. These vertices, being in $V(C - S)$ and having degree exactly two, initiate the propagation phase, which in turn monitors all the vertices in $V(C - S)$. Further, let $u \in \mathcal{P}_3(G) \subset \mathcal{M}(G)$ be a vertex to which a pendant path is attached. Note that u dominates initially all its neighbors which includes the base of each pendant paths and power dominates all the vertices in the path through the propagation step. Also, the vertices in $\mathcal{P}_4(G)$ are either adjacent to $\mathcal{P}_3(G)$ or belong to a pendant path. In both cases, they are dominated and hence power dominated by $\mathcal{M}(G)$. Thus, $\mathcal{M}(G)$ is a connected power dominating set of G . Therefore, $\gamma_{p,c}(G) \leq |\mathcal{M}(G)|$. By Lemma 3.1, we have $|\mathcal{M}(G)| \leq \gamma_{p,c}(G)$ and hence $\gamma_{p,c}(G) = |\mathcal{M}(G)|$. $\square$

Theorem 3.9. *Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. If $\{T_i\}$ is the family of connected components of $G - C$, with $\langle S \rangle$ being connected, then $\gamma_{p,c}(G) = \sum_i \gamma_{p,c}(T_i) + |S| + |\mathcal{F}(G)|$, where $\mathcal{F}(G) = \{v \in V(G) \mid d(v) = 2, p(v) = 0 \text{ in } G \text{ and } p(v) = 1 \text{ in } T_i\}$.*

Proof. If G is an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$, then each connected component T_i of $G - C$ is a tree different from path. From Theorem 3.1, we have $\mathcal{M}(T_i)$ to be the minimum connected power dominating set of T_i . Now if there

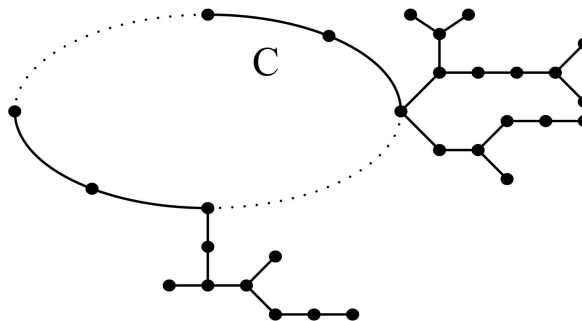


FIGURE 2. An unicyclic graph with cycle C

is a vertex in each $\mathcal{M}(T_i)$ that is adjacent to any vertex in S , then $(\bigcup_i \mathcal{M}(T_i)) \cup S$ is a connected power dominating set of G . Suppose, there is a connected component T_j such that, there is no vertex in $\mathcal{M}(T_j)$ adjacent to any vertex in S , then $(\bigcup_i \mathcal{M}(T_i)) \cup S$ is not connected. That is, there is a vertex u of degree two in T_j such that it belongs to a pendant path in T_j but not in G . Thus, $(\bigcup_i \mathcal{M}(T_i)) \cup S \cup \mathcal{F}(G)$ where $\mathcal{F}(G) = \{v \in V(G) | d(v) = 2, p(v) = 0 \text{ in } G \text{ and } p(v) = 1 \text{ in } T_i\}$ is a connected power dominating set of G . Clearly, $\bigcup_i \mathcal{M}(T_i)$, S and $\mathcal{F}(G)$ are mutually disjoint. Therefore we have, $|\mathcal{M}(G)| \leq \sum_i |\mathcal{M}(T_i)| + |S| + |\mathcal{F}(G)|$. Furthermore, for every i we have $\mathcal{M}(T_i) \subset \mathcal{M}(G)$. By definition, $S \subset \mathcal{M}(G)$ and $\mathcal{F}(G) \subset \mathcal{P}_2(G) \subset \mathcal{M}(G)$. Thus, $\sum_i |\mathcal{M}(T_i)| + |S| + |\mathcal{F}(G)| \leq |\mathcal{M}(G)|$. Therefore, $\gamma_{p,c}(G) = \sum_i \gamma_{p,c}(T_i) + |S| + |\mathcal{F}(G)|$. $\square$

Theorem 3.10. Let G be an unicyclic graph of order n with cycle C , $\mathcal{P}$ be the set of all vertices belonging to the pendant paths of G and $S = \{v \in V(C) | d(v) \geq 3 \text{ and } p(v) = 0\}$. If $|S| > 1$ is not connected, then $\gamma_{p,c}(G) = n - |\mathcal{P}| - \max\{d(u_i, u_{i+1}) - 1 | u_i \in S, i \in \{1, \dots, |S| - 1\}\}$.

Proof. Let G be an unicyclic graph of order n with cycle C and $\mathcal{P}$ be the set of all vertices belonging to the pendant paths of G . Since $S = \{v \in V(C) | d(v) \geq 3 \text{ and } p(v) = 0\}$, the only connected component of $G - C$ are trees not isomorphic to P_n . Therefore, the vertex set of G can be expressed as

$$V(G) = V(C - S) \cup \mathcal{M}(G) \cup \mathcal{P} \quad (3.2)$$

Note that, the sets $V(C - S)$, $\mathcal{M}(G)$ and $\mathcal{P}$ are mutually disjoint. Given that, S is not connected, there is a vertex, say u in S , such that the shortest path between u and $S - \{u\}$ has at least one vertex of the set $V(C - S)$. Now, label the vertices in S as $u_1, u_2, \dots, u_{|S|}$ in a clockwise manner. Then, $d(u_i, u_{i+1}) - 1$ denotes the number of vertices between u_i and u_{i+1} in C . Let Q denote the set of all vertices belonging to the largest path in $C - S$. We claim that $V(G) - (Q \cup \mathcal{P})$ is the minimum connected power dominating set for G . Note that, removing Q from G does not disconnect G , since Q is contained in C . Also, the vertices in Q having degree two are power dominated either by u_i or u_{i+1} . Similarly, the vertices in $V(C - Q)$ are power dominated by S . Further, removing the vertices in the pendant path does not disconnect the graph. Moreover, they are monitored through the propagation step initiated by the vertex in $\mathcal{P}_3(G)$, which is a neighbor of the base of the pendant path. Now, it is clear that the $\mathcal{M}(G)$ is enough to power dominate the entire graph G . Since $S \subset \mathcal{M}(G)$ and S is not connected, $\mathcal{M}(G)$ is also not connected. In order to

achieve the connectivity, we must include the vertices in the set $V(C - (S \cup Q))$ to $\mathcal{M}(G)$. From eqn. 3.2 and $Q \subset V(C - S)$, we have $\mathcal{M}(G) \cup V(C - (S \cup Q)) = V(G) - (Q \cup \mathcal{P})$. Thus, $V(G) - (Q \cup \mathcal{P})$ is a minimum connected power dominating set of G . Consequently, $\gamma_{p,c}(G) = n - |\mathcal{P}| - \max\{d(u_i, u_{i+1}) - 1 | u_i \in S, i \in \{1, \dots, |S| - 1\}\}$. $\square$

4. CONNECTED POWER DOMINATION OF SOME CLASSES OF GRAPHS

In this section, we determine the connected power domination number for certain classes of graphs, including Grid graph, Join of graphs, Kragujevac tree and Honeycomb network.

Definition 4.1. [14] *The Grid graph $G_{m,n}$ is obtained by the cartesian product of the paths P_m and P_n .*

Theorem 4.1. *Let G be a Grid graph $G_{m,n}$, then*

$$\gamma_{p,c}(G_{m,n}) = \begin{cases} \lceil \frac{n}{3} \rceil, & n < 6, n \leq m \\ n - 4, & 6 \leq n \leq m \end{cases}$$

Proof. Let us denote the vertices of $G_{m,n}$ as $v_{i,j}$ where $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. It is clear that the vertex set can be partitioned into layers, say, $L_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n}\}$. To construct the CPDS of $G_{m,n}$, we consider the following cases.

Case 1: Let $n \in \{2, 3\}$ and $m \geq 2$. We consider the set $\{v_{1,2}\}$ to be the CPDS of $G_{m,n}$. Initially, the vertex $v_{1,2}$ dominates the entire L_1 layer and the vertex $v_{2,2}$. Now, all the remaining vertices are dominated through the propagation phase initiated by the vertices in L_1 . Thus, $\gamma_{p,c}(G_{m,n}) = 1 = \lceil \frac{n}{3} \rceil$.

Case 2: Let $n \in \{4, 5\}$; $m \geq n$. We consider the $\gamma_{p,c}$ -set as $S_c = \{v_{2,2}, v_{2,3}\}$. Initially, the vertices in $\gamma_{p,c}$ -set dominates the vertices $\{v_{1,2}, v_{1,3}, v_{2,1}, v_{2,4}, v_{3,2}, v_{3,3}\}$. Now, $v_{1,1}$ and $v_{1,4}$ being the only unmonitored neighbors of $v_{1,2}$ and $v_{1,3}$ respectively, they are also monitored when $n = 4$. Whereas, when $n = 5$, we see that the vertex $v_{1,4}$ power dominates $v_{1,5}$, since $N(v_{1,4}) - \{v_{1,5}\} \in N(S_c)$. Thus, in both cases, the entire L_1 layer is monitored, which in turn power dominates the entire graph. Note that $\langle S_c \rangle \cong P_2$, and hence it is connected. Thus, $\gamma_{p,c}(G_{m,n}) = \lceil \frac{n}{3} \rceil$.

Case 3: When $n \geq 6$, $m \geq n$. Let us consider the set $S_c = \{v_{2,j} | 3 \leq j \leq n - 2\}$ to be the CPDS of $G_{m,n}$. Clearly, $\langle S_c \rangle$ is connected. The set S_c initially dominates $\{v_{2,2}, v_{2,n-1}, v_{1,j}, v_{3,j} | 3 \leq j \leq n - 2\}$. Now, the vertices $v_{1,2}$ and $v_{1,n-1}$ being the only unmonitored neighbors of $N[S_c] \cap L_1$, are monitored in the first step of the propagation phase, which then in turn dominates $v_{1,1}$ and $v_{1,n}$ respectively. As a result, the entire L_1 layer ends up being monitored. This ultimately leads to the entire graph getting monitored. Thus, $\gamma_{p,c}(G_{m,n}) = n - 4$. $\square$

Definition 4.2. [2] *The graph join $G \vee H$ of two graphs G and H is a graph on the vertex set $V(G) \cup V(H)$ with edge connecting every vertex of G to every vertex of H , along with the edges in G and H .*

Theorem 4.2. *Let G and H be any two connected graphs, then*

$$\gamma_{p,c}(G \vee H) = \begin{cases} 1, & \text{if } \min\{\gamma_{p,c}(G), \gamma_{p,c}(H)\} = 1 \\ 2, & \text{otherwise} \end{cases}$$

Proof. Let $G \vee H$ be the join of two graphs G with order $n \geq 2$ and graph H with order $m \geq 2$. Let the vertex set of G be $V(G) = \{x_i | 1 \leq i \leq n\}$ with $d(x_l) = \Delta(G)$ and the vertex set of H be $V(H) = \{y_j | 1 \leq j \leq m\}$ with $d(y_k) = \Delta(H)$. By the definition of $G \vee H$, we have $V(G \vee H) = \{x_i, y_j | 1 \leq i \leq n; 1 \leq j \leq m\}$. Clearly, $S_c = \{x_l, y_k\}$ is a connected power domination set of $G \vee H$, since x_l is adjacent to every vertex of H and y_k is adjacent to every vertex of G . Also, note that the subgraph induced by S_c is connected. Therefore, $\gamma_{p,c}(G \vee H) \leq 2$. For the case $\min\{\gamma_{p,c}(G), \gamma_{p,c}(H)\} = 1$, we claim that $\gamma_{p,c}(G \vee H) = 1$. Without loss of generality, let $\gamma_{p,c}(G) = 1$, then all the vertices in $G \vee H$ are dominated by x_l except the vertices $V(G) - N[x_l]$. These remaining vertices are then power dominated by $V(G) \cap N(x_l)$. Thus, in this case $\gamma_{p,c}(G \vee H) = 1$ $\square$

Definition 4.3. [11] Let P_3 be the 3-vertex tree, rooted at one of its terminal vertices. For $k = 2, 3, \dots$, construct the rooted tree B_k by identifying the roots of k copies of P_3 . The vertex obtained by identifying the roots of P_3 -trees is the root of B_k .

Definition 4.4. [11] Let $t \geq 2$ be an integer. Let $\beta_1, \beta_2, \dots, \beta_t$ be the rooted trees, i.e., $\beta_1, \beta_2, \dots, \beta_t \in \{B_2, B_3, \dots\}$. A Kragujevac tree $Kg_{t,k}$ is a tree possessing a central vertex of degree t , adjacent to the roots of $\beta_1, \beta_2, \dots, \beta_t$.

Theorem 4.3. Let G be a Kragujevac tree $Kg_{t,k}$, then $\gamma_{p,c}(Kg_{t,k}) = t + 1$.

Proof. Let $Kg_{t,k}$ be a Kragujevac tree with a central vertex v of degree $t \geq 2$ and branches B_k . Since each $B_k \cong$ spider, from Theorem 3.4 it is clear that the CPDS of B_k is the root vertex of B_k . Thus, $N(v)$ is enough to power dominate the whole tree. But in order to satisfy the additional constraint of connectedness, we construct the CPDS of $Kg_{t,k}$ to be $N[v]$. Hence, $\gamma_{p,c}(Kg_{t,k}) = |N[v]| = t + 1$ $\square$

Definition 4.5. [9] A n -dimensional Honeycomb network is represented by $HC(n)$, where the parameter n denotes the number of hexagons between the middle to the boundary hexagon.

Theorem 4.4. Let G be a Honeycomb network $HC(n)$, then

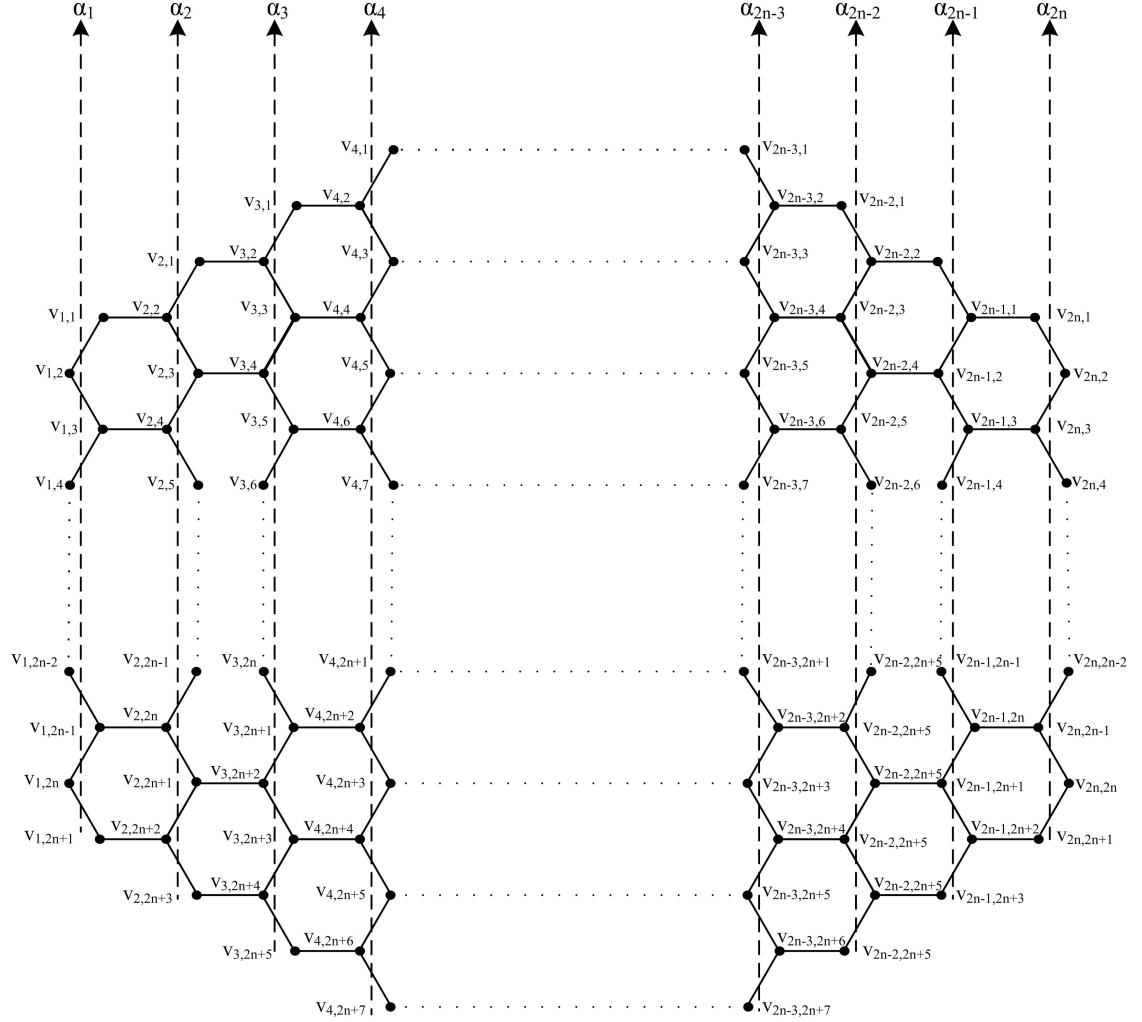
$$\gamma_{p,c}(HC(n)) = \begin{cases} 1, & n = 1 \\ 2(n - 1), & n > 1 \end{cases}$$

Proof. Let $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ represent the α -lines (vertical lines) of the Honeycomb network $HC(n)$, given from left to right. The vertices on each α_i are named from top to bottom as $v_{i,j}$ where $1 \leq i \leq 2n$ and $j = \begin{cases} 1, 2, \dots, 2(n + i) - 1; & \text{for } i = 1, 2, \dots, n \\ 1, 2, \dots, 2(3n - i) + 1; & \text{for } i = n + 1, n + 2, \dots, 2n \end{cases}$

See Figure 3. To construct the CPDS of $HC(n)$, we consider the following cases:

Case 1: When $n = 1$, $HC(n) \cong C_6$. Thus, $\gamma_{p,c}(HC(n)) = 1$.

Case 2: For $n = 2$, it is clear that a single vertex is not enough to power dominate the entire graph $HC(2)$, since each vertex is either of degree 3 or a vertex of degree 2. Let the $\gamma_{p,c}$ -set for the case $n = 2$, be $S_c = \{v_{2,2}, v_{2,3}\}$. Initially, S_c dominates $\{v_{1,1}, v_{2,1}, v_{2,4}, v_{3,3}\}$. Then the vertices $v_{1,1}$ and $v_{2,1}$ being degree 2 vertices initiates the propagation phase, following the monitoring pattern: $v_{1,1} \rightarrow v_{1,2} \rightarrow v_{1,3}$ and $v_{2,1} \rightarrow v_{3,1} \rightarrow v_{3,2}$. Now, each of the degree 3

FIGURE 3. $HC(n)$ with α -lines

vertices in the set $\{v_{1,3}, v_{2,4}, v_{3,2}, v_{3,3}\}$ having only one unmonitored neighbor further proceeds with the propagation phase. Proceeding in a similar way, the entire graph is monitored by the set S_c . Thus, $\gamma_{p,c}(HC(n)) = 2 = 2(n - 1)$.

Case 3: When $n > 2$, the $\gamma_{p,c}$ -set is $S_c = \{v_{i,j} \mid 2 \leq i \leq n; j \in \{2(i-1), 2i-1\}\}$. This set, initially dominates the vertex set $\{v_{1,1}, v_{n+1,2n-1}, v_{i,j} \mid 2 \leq i \leq n; j \in \{2i-3, 2i\}\}$. Similarly, following the propagation phase, the whole graph is monitored. Thus, $\gamma_{p,c}(HC(n)) = 2(n - 1)$. $\square$

Since the proofs of the subsequent theorems are evident, it is left for the understanding of the readers.

Theorem 4.5. *If $G \in \{K_n, K_{m,n}, W_{1,n}, S_{1,n}, B_{m,n}\}$ be a connected graph with $m, n \geq 2$ vertices, then $\gamma(G) = \gamma_p(G) = \gamma_{p,c}(G)$.*

Theorem 4.6. *If $G \in \{P_n, C_n, H_n, P_2 \times P_n\}$ be a connected graph with $n \geq 2$ vertices, then $\gamma_{p,c}(G) < \gamma(G)$.*

In the following table, we present the power domination number and connected power domination number for certain graphs, demonstrating that the connected power domination number is always greater than or equal to the power domination number.

TABLE 4.1. List of Graphs with Power Domination and Connected Power Domination Number

Family of Graphs	$\gamma_p(G)$	$\gamma_{p,c}(G)$
Petersen graph	2	3
Complete Binary Tree	$2^{(n-1)}$	$2^n - 1$
Kayak paddle ($KP(k, m, l)$)	2	$l + 1$

5. CONCLUSION

In this paper, we conducted a comprehensive study of connected power domination in graphs and established bounds on the connected power domination number. Furthermore, the structure and behavior of connected power dominating sets in unicyclic graphs were studied. Moreover, exact values of the connected power domination number were obtained for specific graph classes such as Grid graph, Join of graphs, Kragujevac tree and Honeycomb network. The results obtained in this study provide a foundation for further exploration in this area.

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