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CONFORMAL SOLITONS IN RELATIVISTIC MAGNETO-FLUID SPACETIMES WITH ANTI-TORQUED VECTOR FIELDS

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Abstract. The kinematic and dynamic properties of relativistic spacetime in the context of relativity can be modelled by three distinct classes: shrinking, steady, and expanding. This physical framework bears a resemblance to conformal Ricci flow, where solitons serve as fixed points. Notably, within the solar system, the gravitational effects predicted by Ricci flow align with those of Einstein's gravity, ensuring consistency with all classical tests. In this article, we investigate conformal solitons, which extend the concept of Ricci solitons, within the framework of a magnetized spacetime manifold equipped with an anti-torqued vector field ζ. An anti-torqued vector field is defined as one that resists rotational deformation within the fluid-spacetime structure, effectively encoding a type of constrained rotational symmetry relevant in magneto-fluid dynamics. We demonstrate that whether these conformal solitons are steady, expanding, or shrinking depends on intricate relationships among key physical parameters, including magnetic permeability, magneto-fluid density, isotropic pressure, magnetic flux, and the strength of the magnetic field.

Keywords: Soliton, Spacetime, Energy momentum tensor.

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1. Introduction

In modern physics, space and time are inseparable, at least in the process of representing physical things through ourselves, where these two dimensions play an important role in imagining and conceptualizing the connections of all physical things. In 1915, Einstein

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developed the theory of gravity known as general relativity, which exposes the fundamental role of the physics and geometry of spacetime. It plays an important role in Engineering when applied to day to day life. If we consider general relativity, then the space-time in the four-dimensional pseudo-Riemannian manifold with Lorentzian metric (M^4, g), where g is considered to be perfectly liquid space-time. Perfect fluids are used in cosmology to model the idealized distributions of matter. It is defined by various thermodynamical variables (variables are: particle number density, energy density, pressure, temperature, and entropy per particle). These variables are spacetime scalar fields whose values represent measurements made in the rest frame of the isotropic or star.

On the other hand, the Ricci flow was first introduced by Hamilton [9]. Over the last decades, many differential geometers progressively studied Ricci flow [3, 14]. Fischer [8] proposed a modified version known as conformal Ricci flow, which differs from the classical Ricci flow in its constraints. While the original Ricci flow preserves unit volume, the conformal Ricci flow instead imposes a scalar curvature constraint. Interestingly, the conformal Ricci flow equations exhibit structural similarities to the Navier-Stokes equations in fluid dynamics. In this analogy, the time-dependent scalar field p acts as a conformal pressure—unlike physical pressure, which ensures fluid incompressibility, conformal pressure influences the deformation of the metric under the flow. The fixed points of this system correspond to Einstein metrics with a specific constant $\frac{-1}{n}$. Building on these concepts, Catino and Mazzieri [6] introduced Einstein solitons, which provide self-similar solutions to the Einstein flow. Extending this framework, Roy et al. [18] developed the notion of conformal Einstein solitons. Both conformal Ricci and conformal Einstein solitons generate self-similar solutions, offering a deeper understanding of geometric flows in mathematical physics. The conformal Ricci and conformal Einstein flow respectively are given by:

$$\frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - \phi g \quad and \quad r = -1 \quad and \quad \frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g). \tag{1.1}$$

A matter is assumed to be fluid, having pressure, density, and kinematic and dynamical quantities like verticity, shear, velocity, acceleration, and expansion [25, 1]. The energy-momentum tensor acts a big role in the matter content of spacetime (universe). The energy-momentum tensor applications are cosmology and stellar structure, and examples are electromagnetism and scalar field theory. The study of the kinematic and dynamic nature of relativistic space-time application in relativity has a physical model of three classes, namely: shrinking, steady, and expanding. Such a physical model are similar to conformal flow. Also, for the solar system, conformal flow gravity effects are not different from Einstein's gravity, and hence it obeys all the classical tests.

Over the last decades, many differential geometers [19, 17] progressively studied the various geometric flows in Relativistic perfect fluid spacetime (briefly RPFS). The study of Ricci solitons and their geometric properties in RPFS was first explored by Ali and Ahsan [2]. Subsequently, Blaga [5] investigated the geometric characteristics of RPFS in the context of Ricci solitons, Einstein solitons, and their extensions—namely, π-Ricci solitons and π-Einstein solitons. Further contributions were made in [26], where the authors examined Ricci soliton structures in RPFS with a torse-forming timelike velocity vector field ζ. D. Siddiqi and A. Siddiqui [23] later analyzed the geometric structure of RPFS using conformal Ricci solitons. Siddiqi and De [24] extended these investigations to relativistic magneto-fluid spacetimes (RMFS). More recently, Praveena et al. [16, 15] studied Ricci, Einstein, and conformal Ricci solitons in almost pseudo-symmetric K$\ddot{a}$hlerian and Kähler-Norden spacetimes, incorporating various curvature tensors. Additionally, Bhattacharyya et al. [18] examined conformal Einstein solitons in para-K$\ddot{a}$hler manifolds.

Inspired by these developments, the present work explores the geometric behavior of conformal Ricci and Einstein flows in RMFS with an anti-torqued vector field.

2. Relativistic magneto fluid spacetime

A relativistic magneto-fluid (RMF) is a continuum medium whose physical state can be fully described by several key parameters: the fluid's rest frame, mass density, isotropic pressure, magnetic flux, and magnetic field strength. In general relativity, such magneto-fluids serve as fundamental models for idealized matter distributions, including stellar interiors and homogeneous cosmological models.

The RMF framework makes several simplifying assumptions - the medium exhibits zero shear stress, negligible viscosity, and no thermal conduction. Mathematically, its behavior is governed by a magnetic energy-momentum tensor T with specific symmetric properties that capture these physical characteristics. This formulation provides a valuable theoretical tool for analyzing relativistic plasma systems where electromagnetic and gravitational interactions play equally important roles. T is in the form [13, 12]:

$$\mathfrak{T} = \rho g + (\sigma + \rho)\mathcal{A} \otimes \mathcal{A} + \nu\{\mathcal{H}\left(\mathcal{A} \otimes \mathcal{A} + \frac{1}{2}g\right) - \mathcal{B} \otimes \mathcal{B}\}, \tag{2.2}$$

where $\nu, \sigma, \rho, \mathcal{B}, \mathcal{H}$ are the magnetic permeability, magneto-fluid density, isotopic pressure, magnetic flux, strength of the magnetic field, respectively and $\mathcal{A}(\cdot) = g(\cdot, \zeta)$, $g(\cdot, \xi) = \mathcal{B}(\cdot)$ are two non-zero 1-forms. Also, ζ and ξ, are unit timelike vector field ζ such that $g(\zeta, \zeta) = -1$ and spacelike magnetic flux vector field ξ such that $g(\xi, \xi) = 1$. Therefore, ζ and ξ are orthogonal vector fields generate the magneto-fluid spacetime.

Einstein's gravitational equation with cosmological constant is given as [12]

$$k\mathfrak{T} = S + \left(\lambda - \frac{r}{2}\right) g, \tag{2.3}$$

for any $E, F \in \chi(M)$, where λ, k are the cosmological constant and gravitational constant, respectively.

In view of (2.2), equation (2.3) takes the form

$$\begin{aligned} S &= \left[-\lambda + \frac{r}{2} + k\left(\frac{\nu\mathcal{H}}{2} + \rho\right)\right] g \\ &\quad + k(\nu\mathcal{H} + \sigma + \rho)\mathcal{A} \otimes \mathcal{A} - k\nu\mathcal{B} \otimes \mathcal{B}. \end{aligned} \tag{2.4}$$

3. Characteristics of relativistic magneto fluid spacetime with anti-torqued vector field

Let (M^4, g) be a relativistic magneto fluid spacetime (briefly $RMFS$) satisfying (2.4). Contracting the equation (2.4) provides

$$r = 4\lambda - k[\nu(\mathcal{H} - 1) + 3\rho - \sigma]. \tag{3.5}$$

Using the above equation in (2.4), we have

$$\begin{aligned} S(E, F) &= \left(\lambda + \frac{k}{2}(\nu + \sigma - \rho)\right) g(E, F) + k(\nu\mathcal{H} + \sigma + \rho)\mathcal{A}(E)\mathcal{A}(F) \\ &\quad - k\nu\mathcal{B}(E)\mathcal{B}(F), \end{aligned} \tag{3.6}$$

which also implies

$$QE = aE + b\mathcal{A}(E)\zeta + c\mathcal{B}(E)\xi, \tag{3.7}$$

where $a = \lambda + \frac{k}{2}(\nu + \sigma - \rho), b = k(\nu\mathcal{H} + \sigma + \rho), c = -k\nu$.

We consider the special case when ζ is an anti-torqued vector field [7] of the form:

$$\nabla_E \zeta = f(E - \mathcal{A}(E)\zeta), \tag{3.8}$$

for a vector field E on M^4, where $\mathcal{A}$ is one form dual to unit anti-torqued vector field and f is a non-zero smooth function.

Theorem 3.1. *On a RMFS with an anti-torqued vector field ζ, the following relations hold:*

$$(\nabla_X\mathcal{A})(E) = f[g(E,F)-\mathcal{A}(E)\mathcal{A}(F)], \tag{3.9}$$

$$\mathcal{A}(\nabla_\zeta\zeta) = 2, \qquad \nabla_\zeta\zeta = 2\zeta, \tag{3.10}$$

$$R(E,F)\zeta = f^2[\mathcal{A}(E)F-\mathcal{A}(F)E]+E(f)[F-\mathcal{A}(F)\zeta]-F(f)[E-\mathcal{A}(E)\zeta], \tag{3.11}$$

$$R(E,\zeta)\zeta = f^2[E+\mathcal{A}(E)\zeta]+2E(f)\zeta-\zeta(f)[E-\mathcal{A}(E)\zeta], \tag{3.12}$$

$$\mathcal{A}(R(E,F)D) = f^2[\mathcal{A}(F)g(X,D)-\mathcal{A}(E)g(F,D)]-E(f)[g(F,D)-\mathcal{A}(F)\mathcal{A}(D)] +F(f)[g(E,D)-\mathcal{A}(E)\mathcal{A}(D)], \tag{3.13}$$

$$(\pounds_\zeta g)(E,F) = 2f[g(E,F)-\mathcal{A}(E)\mathcal{A}(F)]. \tag{3.14}$$

Proof. Compute $(\nabla_E\mathcal{A})(F) = E(\mathcal{A}(F))-\mathcal{A}(\nabla_E F) = E(g(F,\zeta))-g(\nabla_E F,\zeta) = g(F,\nabla_E\zeta) = f[g(E,F)-\mathcal{A}(E)\mathcal{A}(F)]$. Specifically, $(\nabla_\zeta\mathcal{A})E = 0$. The relation (3.9) can be obtained by (3.8).

Now, utilizing (3.8) in $R(E,F)\zeta = \nabla_E\nabla_F\zeta-\nabla_F\nabla_E\zeta-\nabla_{[E,F]}\zeta$ and from direct computation we obtain the relation (3.11). Additionally (3.12) and (3.13) follows from (3.11). Now differentiating g along ζ, then by simple calculation we get (3.14). □

4. Conformal Ricci soliton in a $RMFS$

This section is devoted to studying the conformal Ricci soliton in the context of $RMFS$. Conformal Ricci solitons, which are defined as [4]:

$$\pounds_V g+2S+\left[2\Lambda-\left(\pi+\frac{2}{n}\right)\right]g = 0, \tag{4.15}$$

where S,π,Λ are the Ricci tensor, the conformal pressure, a constant respectively and $\pounds_V$ is the Lie-derivative operator along the vector field V on spacetime. The conformal Ricci soliton becomes shrinking (resp. steady, expanding) for $\Lambda<0$ (resp. $\Lambda=0$, $\Lambda>0$).

Taking ζ instead of V in (4.15) and then using (3.14) yields

$$S(E,F) = -\left[\Lambda-\frac{1}{2}\left(\pi+\frac{1}{2}\right)+f\right]g(E,F)+f\mathcal{A}(E)\mathcal{A}(F).$$

Making use of (2.4) in the above equation, we obtain

$$\left[-\lambda+\frac{r}{2}+k\left(\frac{\nu\mathcal{H}}{2}+\rho\right)\right]g(E,F)+k(\nu\mathcal{H}+\sigma+\rho)\mathcal{A}(E)\mathcal{A}(F) -k\nu\mathcal{B}(E)\mathcal{B}(F) = -\left[\Lambda-\frac{1}{2}\left(\pi+\frac{1}{2}\right)+f\right]g(E,F)+f\mathcal{A}(E)\mathcal{A}(F).$$

Setting $E = F = \zeta$ in the foregoing equation and then making use of (3.5) yields

$$\Lambda = -\lambda + k\nu\left(\mathcal{H} - \frac{1}{2}\right) + \frac{3}{2}k\rho + \frac{k\sigma}{2} + \frac{\pi}{2} - 2f + \frac{1}{4}. \tag{4.16}$$

Theorem 4.1. *A $RMFS$ with anti-torqued vector field ζ admitting a conformal Ricci soliton is shrinking, steady, or expanding accordingly cosmological constant* $\lambda \lesseqgtr k\nu\left(\mathcal{H} - \frac{1}{2}\right) + \frac{3}{2}k\rho + \frac{k\sigma}{2} + \frac{\pi}{2} - 2f + \frac{1}{4}$ *respectively.*

Let us consider a spacetime in the absence of a cosmological constant i.e. $\lambda = 0$. Then it yields $S(\zeta,\zeta) = \frac{k}{2}[\nu(2\mathcal{H} - 1) + \sigma + 3\rho]$. If the characteristic vector field is timelike then in a spacetime $S(\zeta,\zeta) > 0$, which implies $\nu(2\mathcal{H} - 1) + \sigma + 3\rho > 0$, the spacetime obeys the cosmic strong force condition.

In view of the above converse and Eq. (4.16), we can state the following theorem.

Theorem 4.2. *A $RMFS$ with anti-torqued vector field ζ admitting a conformal Ricci soliton which satisfies timelike convergence condition in the absence of a cosmological constant is expanding.*

5. Conformal $\mathcal{A}$-Ricci Soliton in a $RMFS$

Consider the equation

$$\pounds_V g + 2S + \left[2\Lambda - \left(\pi + \frac{2}{n}\right)\right] g + 2\Omega\mathcal{A}\otimes\mathcal{A} = 0, \tag{5.17}$$

where Λ, Ω are real constants and π, S are same as defined in (4.15). The quadruple $(g, \zeta, \Lambda, \Omega)$ which satisfy the equation (5.17) is said to be a conformal $\mathcal{A}$-Ricci soliton in M [21]. In particular if $\Omega = 0$, then it reduces to a conformal Ricci soliton [4] and it becomes shrinking (resp. steady, expanding) for $\Lambda < 0$ (resp. $\Lambda = 0$, $\Lambda > 0$) [9].

Writing the Lie derivative $\pounds_\zeta g$ explicitly, we have $\pounds_\zeta g = g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)$. Then (5.17) takes the form

$$S(E,F) = -\left[\Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right)\right] g(E,F) - \Omega\mathcal{A}(E)\mathcal{A}(F) - \frac{1}{2}[g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)], \tag{5.18}$$

for any $E, F \in \chi(M^4)$.

From (2.4) and (5.18), we have

$$\left[\lambda + \frac{k}{2}(\nu + \sigma - \rho) + \Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right)\right] g(E,F) + [k(\nu\mathcal{H} + \sigma + \rho) + \Omega]\mathcal{A}(E)\mathcal{A}(F)$$
$$-k\nu\mathcal{B}(E)\mathcal{B}(F) + \frac{1}{2}[g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)] = 0. \tag{5.19}$$

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and $\zeta = \sum_{i=1}^{4} \zeta^i e_i$. We have $\sum_{i=1}^{4} \epsilon_{ii}(\zeta^i)^2 = -1$ and multiplying (5.19) by ϵ_{ii} and summing over i for $E = F = e_i$, we obtain

$$4\Lambda - \Omega = -4\lambda + k[\nu(\mathcal{H} - 1) - \sigma + 3\rho) + 2\pi + 1 - div\zeta. \tag{5.20}$$

Plugging $E = F = \zeta$ in (5.19), we obtain

$$\Lambda - \Omega = -\lambda + \frac{k}{2}[\nu(2\mathcal{H} - 1) + \sigma + 3\rho] + \frac{1}{2}\left(\pi + \frac{1}{2}\right). \tag{5.21}$$

On solving (5.20) and (5.21), we have

$$\Lambda = -\lambda - \frac{k}{2}\left(\frac{\nu}{3} + \sigma - \rho\right) + \frac{\pi}{2} + \frac{1}{4} - \frac{div\zeta}{3},$$

$$\Omega = -k\left[\nu\left(\mathcal{H} - \frac{1}{3}\right) + \sigma + \rho\right] - \frac{div\zeta}{3}.$$

Thus, we have the following theorem:

Theorem 5.1. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and let $\mathcal{A}$ be the g-dual 1-form of the gradient vector field $\zeta = grad(\phi)$ with $g(\zeta,\zeta) = -1$. If (5.17) define a conformal $\mathcal{A}$-Ricci soliton in M^4, then the Laplacian equation satisfied by ϕ becomes*

$$\Delta(\phi) = -3\Omega - k\left[\nu\left(\mathcal{H} - \frac{1}{3}\right) + \sigma + \rho\right].$$

Remark 5.1. *If $\Omega = 0$ in (5.17), then we obtain the conformal Ricci soliton with $\Lambda = -\lambda + k\left[\nu\left(\mathcal{H} + \frac{1}{6}\right) + \frac{\sigma+\rho}{2}\right] + \frac{1}{2}\left(\pi + \frac{1}{2}\right)$, which is expanding, steady, or shrinking accordingly*

$$\lambda \mathop{\lesseqgtr} k\left[\nu\left(\mathcal{H} + \frac{1}{6}\right) + \frac{\sigma + \rho}{2}\right] + \frac{1}{2}\left(\pi + \frac{1}{2}\right)$$

respectively.

6. Conformal Einstein Soliton in a $RMFS$

Consider the equation

$$\pounds_V g + 2S + \left[2\Lambda - r + \left(\pi + \frac{2}{n}\right)\right] g = 0, \tag{6.22}$$

where $g, \xi, \Lambda, S, r, \mathcal{A}$ are same as defined in (4.15) and π is a scalar non-dynamical field. The triplet (g, ζ, Λ) which satisfy the equation (6.22) is said to be a conformal Einstein soliton in M [18]. It is called shrinking (resp. steady or expanding) for $\Lambda < 0$ (resp. $\Lambda = 0$ or $\Lambda > 0$). Taking ζ instead of V in (6.22) and then making use of (3.14) yields

$$S(E,F) = -\left[\Lambda - \frac{r}{2} + \frac{1}{2}\left(\pi + \frac{1}{2}\right) + f\right] g(E,F) + f\mathcal{A}(E)\mathcal{A}(F).$$

Utilizing (2.4) in the foregoing equation, one can easily obtain

$$\left[-\lambda+\frac{r}{2}+k\left(\frac{\nu\mathcal{H}}{2}+\rho\right)\right]g(E,F)+k(\nu\mathcal{H}+\sigma+\rho)\mathcal{A}(E)\mathcal{A}(F)-k\nu\mathcal{B}(E)\mathcal{B}(F)$$
$$=-\left[\Lambda-\frac{r}{2}+\frac{1}{2}\left(\pi+\frac{1}{2}\right)+f\right]g(E,F)+f\mathcal{A}(E)\mathcal{A}(F).$$

Setting $E=F=\zeta$ in the above equation provides

$$\Lambda=\lambda+k\left(\frac{\nu\mathcal{H}}{2}+\sigma\right)-\frac{\pi}{2}-2f-\frac{1}{4}.$$

Theorem 6.1. *A RMFS with anti-torqued vector field ζ admitting a conformal Einstein soliton is shrinking, steady, or expanding accordingly cosmological constant* $\lambda \gtreqless +\frac{\pi}{2}+2f+\frac{1}{4}-k\left(\frac{\nu\mathcal{H}}{2}+\sigma\right)$ *respectively.*

7. Conformal $\mathcal{A}$-Einstein Soliton in a $RMFS$

Consider the equation

$$\pounds_V g+2S+\left[2\Lambda-r+\left(\pi+\frac{2}{n}\right)\right]g+2\Omega\mathcal{A}\otimes\mathcal{A}=0, \tag{7.23}$$

where Λ, Ω are real constants and r,π,S are same as defined in (6.22). The quadruple (g,ζ,Λ,Ω) which satisfy the equation (7.23) is said to be a conformal $\mathcal{A}$-Einstein soliton in M. In particular if $\Omega=0$, (g,ζ,Λ) is a conformal Einstein soliton [18] and it becomes shrinking (resp. steady, expanding) for $\Lambda<0$ (resp. $\Lambda=0$, $\Lambda>0$) [9].

Writing the Lie derivative $\pounds_\zeta g$ explicitly, we have $\pounds_\zeta g=g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)$ and from (7.23) we obtain:

$$S(E,F)=-\left[\Lambda-\frac{r}{2}+\frac{1}{2}\left(\pi+\frac{1}{2}\right)\right]g(E,F)-\Omega\mathcal{A}(E)\mathcal{A}(F)-\frac{1}{2}[g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)], \tag{7.24}$$

for any $E,F\in\chi(M^4)$.

From (2.4) and (7.24), we have

$$\left[-\lambda+k\left(\frac{\nu\mathcal{H}}{2}+\rho\right)+\Lambda-\frac{1}{2}\left(\pi+\frac{1}{2}\right)\right]g(E,F)+[k(\nu\mathcal{H}+\sigma+\rho)+\Omega]\mathcal{A}(E)\mathcal{A}(F)$$
$$-k\nu\mathcal{B}(E)\mathcal{B}(F)+\frac{1}{2}[g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)]=0. \tag{7.25}$$

Consider $\{e_i\}_{1\le i\le 4}$ an orthonormal frame field and $\zeta=\sum_{i=1}^4\zeta^i e_i$. We have $\sum_{i=1}^4\epsilon_{ii}(\zeta^i)^2=-1$ and multiplying (7.25) by ϵ_{ii} and summing over i for $X=Y=e_i$, we obtain

$$4\Lambda-\Omega=4\lambda+k(\nu\mathcal{H}+3\rho+\nu+\sigma)+2\pi+1-div\zeta. \tag{7.26}$$

Plugging $E = F = \zeta$ in (7.25), we obtain

$$\Lambda - \Omega = \lambda + k(\frac{\nu\mathcal{H}}{2} + \sigma) + \left(\frac{\pi}{2} + \frac{1}{4}\right). \tag{7.27}$$

On solving (7.26) and (7.27), we have

$$\Lambda = \lambda + k\left[\frac{\nu\mathcal{H}}{6} + \rho + \frac{\nu}{3}\right] + \frac{\pi}{2} + \frac{1}{4} - \frac{div\zeta}{3},$$

$$\Omega = -k\left(\frac{\nu\mathcal{H}}{3} - \sigma + \rho - \frac{\nu}{3}\right) - \frac{div\zeta}{3}.$$

Thus, we have the following theorem:

Theorem 7.1. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and let $\mathcal{A}$ be the g-dual 1-form of the gradient vector field $\zeta = grad(\phi)$ with $g(\zeta, \zeta) = -1$. If (7.23) define a conformal $\mathcal{A}$-Einstein soliton in M^4, then the Laplacian equation satisfied by ϕ becomes*

$$\Delta(\phi) = -3\left[\Omega + k\left(\frac{\nu\mathcal{H}}{3} - \sigma + \rho - \frac{\nu}{3}\right)\right].$$

Remark 7.1. *If $\Omega = 0$ in (7.23), then we obtain the conformal Ricci soliton with $\Lambda = \lambda + k\left[\frac{\nu\mathcal{H}}{2} - \sigma + 2\rho\right] + \frac{\pi}{2} + \frac{1}{4}$, which is expanding, steady or shrinking accordingly*

$$\lambda \underset{<}{\overset{\geq}{=}} -k\left[\frac{\nu\mathcal{H}}{2} - \sigma + 2\rho\right] - \frac{\pi}{2} - \frac{1}{4}$$

respectively.

8. Conclusion

In the framework of general relativity, the energy-momentum tensor T fundamentally characterizes the matter distribution within spacetime. Conventional cosmological models typically represent the universe's matter content as a perfect fluid within a 4-dimensional Lorentzian manifold. Within this paradigm, Einstein's field equations serve as the foundational tool for constructing viable cosmological models.

Relativistic magneto-fluid spacetime (RMFS) models hold particular significance across multiple disciplines, including astrophysics, nuclear physics, and plasma physics. Recent investigations have revealed that geometric flows provide powerful tools for characterizing the intrinsic structures of RMFS. Of special interest are soliton solutions - those metric configurations evolving through dilations and diffeomorphisms, which emerge naturally in the singularity analysis of these flows. These self-similar solutions find applications not only in physics but also in chemistry, biology, and economics (see [20], [27], [28]).

This work systematically examines various classes of solitons in RMFS endowed with an anti-torqued vector field. We establish precise conditions under which these solitons exhibit expanding, steady, or shrinking behavior. Furthermore, we derive the Laplace equation for such RMFS configurations admitting conformal $\mathcal{A}$-Ricci and $\mathcal{A}$-Einstein solitons.

The investigation of conformal solitons gains additional importance from the remarkable similarity between conformal Ricci flow equations and the Navier-Stokes equations of fluid dynamics. In this correspondence, the time-dependent scalar field p functions as a conformal pressure - distinct from conventional fluid pressure that preserves incompressibility, as it directly influences metric deformation under the flow.

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