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$\widetilde{L}_r-$BIHARMONIC NULL HYPERSURFACES IN GENERALIZED ROBERTSON-WALKER SPACETIMES

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(DEDICATED TO THE MEMORY OF PROF. DR. KRISHAN LAL DUGGAL (1929 - 2022))

ABSTRACT. In this paper, we derive $\widetilde{L}_r-$biharmonic equations for null hypersurfaces M in Generalized Robertson-Walker (GRW) spacetimes using linearized operators $\widetilde{L}_r$ $(0 \leq r \leq \dim(M))$ built uniquely from the rigged structure given by a timelike closed and conformal rigging vector field ζ. After providing a characterization for $\widetilde{L}_r-$harmonic null hypersurfaces we study $\widetilde{L}_r-$biharmonic null hypersurfaces for $r = 0$ and $r = 1$ in low dimensions: null surfaces and $3-$dimensional null hypersurfaces.

Keywords:Null hypersurface, $\widetilde{L}_r-$biharmonic, GRW spacetimes, Rigging vector field.

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1. Introduction

Consider an isometric immersion $\psi : M^n \rightarrow \mathbb{E}^m$ from a Riemannian manifold M^n into the Euclidean space $\mathbb{E}^m$. Denote by H and Δ the mean curvature vector field of M^n and the Laplace operator of M^n with respect to the induced Riemannian metric of $\mathbb{E}^m$. From the Beltrami's formula $\Delta\psi = nH$ we see that M is minimal in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic. Observe that $\Delta^2\psi = n\Delta H$. Manifolds with $\Delta H = 0$, or equivalently $\Delta^2\psi = 0$ are called biharmonic. Obviously, minimal submanifolds (i.e $H = 0$) are biharmonic. The question that arises is whether the class of biharmonic submanifolds is reduced to that of minimal submanifolds. Several authors have proved it in some cases (cf. [1, 16, 18, 20, 22] and notes in the report [14]).

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A well-known Bang-Yen Chen's conjecture says : Any biharmonic submanifold in pseudo-Euclidean space $\mathbb{E}_s^{n+p}$ is minimal. But in contrast to the Euclidean case ($s = 0$, where the conjecture is not entirely solved), the conjecture generally fails for submanifolds in a pseudo-Euclidean space. B.-Y. Chen and S. Ishikawa [13] gave examples of nonminimal biharmonic (also called proper biharmonic) space-like surfaces with constant mean curvature in pseudo-Euclidean spaces $\mathbb{E}_s^4$ ($s = 1, 2$) and proper biharmonic surfaces of signature $(1, 1)$ in $\mathbb{E}_s^4$ ($s = 1, 2, 3$) in [15]. Furthermore, in case of hypersurfaces, Chen has found a good relation between the finite type hypersurfaces and biharmonic ones [17, Chapter 11].

The Laplacian operator Δ involved in the biharmonicity can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)-$th mean curvature arising from normal variations of the hypersurface. They act on smooth functions by $L_r(f) = \mathrm{tr}(T_r \circ \nabla^2 f)$, where T_r is the $r-$th Newton transformation associated with the shape operator of the hypersurface, and $\nabla^2 f$ is the self-adjoint linear operator metrically equivalent to the Hessian of f. With this extension of the Laplace operator $\Delta = L_0$ and inspired by the Chen's conjecture, it appears natural to generalize the definition of biharmonic hypersurfaces replacing Δ by the L_r. Along these lines, the L_r-conjecture has been formulated (cf. [5]) as follows:

L_r-**Conjecture 1.1** : Every Euclidean hypersurface $\psi : M^n \to \mathbb{R}^{n+1}$ satisfying the condition $L_r^2 \psi = 0$ for some r, $0 \leq r \leq n-1$ has zero $(r+1)-$th mean curvature (equivalently, $(r+1)-$minimal).

This L_r-conjecture has been generalized (cf. [6]) for hypersurfaces of simply connected space forms as follows :

L_r-**Conjecture 1.2** : Let $\psi : M^n \to Q^{n+1}(c)$ be a hypersurface immersed into a simply connected space form $Q^{n+1}(c)$. If M is L_r-biharmonic then H_{r+1} is zero.

Recently, L_r-biharmonic hypersurfaces have been considered when the target space is pseudo-Riemannian and scrutinized by several authors [3, 19, 27, 28, 26] and references therein. In particular, it is shown in [27, Theorem 1.1] that on any L_k-biharmonic spacelike hypersurfaces in $\mathbb{E}_1^4$ with mutually distinct principal curvatures, if the $k-$th mean curvature H_k is constant then the same is for H_{k+1}. It is worth mentioning that all the hypersurfaces involved in the above quoted works are either spacelike or timelike, hence nondegenerate. To fill the gap, the present work focuses on L_r-biharmonic null (degenerate) hypersurfaces in generalized Robertson-Walker (GRW) spacetimes. As it is predictible due to the extra difficulties presented by the singularities of null hypersurfaces, our following results provide (partial) characterizations of such L_r-biharmonic null hypersurfaces, involving sometimes auxilliary screen foliations.

Theorem 1.1. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a timelike closed and conformal rigging vector field* ζ. *Then* M *is* $\widetilde{L}_r-$*harmonic for some* $0 \leq r < n$ *if and only if one of the following holds :*

(a) M *is* $r-$*maximal;*

(b) M *is* $(r+1)-$*maximal and* ζ *is parallele along* $M \subset \mathbb{R}^{n+2}_1$.

Theorem 1.2. *Let* $n \in \{1, 2\}$ *be integer,*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a non unit timelike closed and conformal rigging vector field* ζ.

(1) *For* $c = 0$, M *is biharmonic (i.e* $\widetilde{L}_0-$*biharmonic) if and only if it is totally geodesic, i.e null hyperplane. In particular the null mean curvature* H *vanishes.*

(2) *For* $c \neq 0$, *if* M *is biharmonic then the null mean curvature* H *is leafwise constant along the screen foliation induced by* ζ, *but not on the whole* M.

The following is a null version of the result in [27, Theorem 1.1] for $r = 1$ in generalized Robertson-Walker spaces.

Theorem 1.3. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a non unit timelike closed and conformal rigging vector field* ζ. *Then,*

(1) *For* $n = 1$, $\psi : M^2 \longrightarrow \overline{M}_1^3(c) \subseteq \mathbb{R}^{3+c^2}_{1+t}$ *is* $\widetilde{L}_1-$*biharmonic.*

(2) *For* $n = 2$, *if* M^3 *is* $\widetilde{L}_1-$*biharmonic and the null mean curvature function* $H := \overset{\star}{H}_1$ *is leafwise constant in the screen foliation* $\mathscr{F}$ *induced by* ζ *then the same is for the*

second order mean curvature $\overset{\star}{H}_2$. *Moreover, if* $\overset{\star}{H}_2$ *is constant on the whole null hypersurface* M^3 *then this constant is zero and* M *is* $2-$*maximal.*

Throughout the paper, all geometric objects (manifolds, metrics, connections, maps,...) are smooth. The Lie algebra of vector fields on a manifold N is denoted by $\mathfrak{X}(N)$.

2. Null hypersurfaces and rigged structures

A hypersurface M of a Lorentzian manifold $(\overline{M}, g)$ is null if the metric tensor is degenerate on it, i.e the induced structure from the Lorentzian ambient manifold is degenerate.

A rigging for a null hypersurface M is a vector field ζ defined in some open neighbourhood of M such that $\zeta_p \notin T_pM$ for all $p \in M$. If ζ is defined only over M, then we call it a restricted rigging. If a rigging exists, then we can take the unique null vector field $\xi \in \mathfrak{X}(M)$ such that $g(\zeta, \xi) = 1$ (called rigged vector field) and the (screen) distribution given by $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pM$ for all $p \in M$. We can also define the rigged metric as the Riemannian metric on M given by $\widetilde{g} = g + \omega \otimes \omega$, where $\omega = i^*\alpha$, α is the g-metrically equivalent one-form to ζ and $i : M \to \overline{M}$ is the canonical inclusion map. The rigged vector field ξ is unitary and orthogonal to $\mathcal{S}$ with respect to $\widetilde{g}$. Moreover, ω is $\widetilde{g}$-metrically equivalent to ξ, and is called the rigged one-form. The vector field $N = \zeta - \frac{1}{2}g(\zeta, \zeta)\xi$ is the unique null vector field defined on M, orthogonal to the screen distribution $\mathcal{S}$ and such that $g(N, \xi) = 1$.

Moreover, we have the following decompositions :

$$T_p\overline{M} = T_pM \oplus span(N_p), \quad T_pM = span\{\xi_p\} \oplus \mathcal{S}_p \tag{2.1}$$

for all $p \in M$.

The rigging technique presents two main advantages. The first one is that all the geometric objects defined above from the rigging are tuned together in a way that allows linking properties of the null hypersurface with properties of the ambient space. The second one is the presence of the Riemannian rigged metric $\widetilde{g}$, which geometry is reasonably well coupled with the ambient geometry in most cases and it allows us to use Riemannian tools for the study of the null hypersurface [23].

We get from decompositions (2.1)

$$\overline{\nabla}_U V = \nabla_U V + B(U, V)N, \qquad \overline{\nabla}_U N = -A(U) + \tau(U)N \tag{2.2}$$

where $\overline{\nabla}$, ∇ are the Levi-Civita connection of $\overline{M}$ and the induced (projected) connection on M, respectively. The induced connection ∇ is torsion free but, in general, is not metric, which makes it less useful in the theory. The second fundamental form B, the one-form τ (also called rotation one form) and the screen second fundamental form C are given by

$$B(U,V) = -g(\nabla_U \xi, V), \qquad \tau(U) = -g(\nabla_U \xi, \zeta),$$

$$C(U,V) = -g(\overline{\nabla}_U N, P(V)) = -g(\overline{\nabla}_U \zeta, P(V)),$$

for all $U, V \in \mathfrak{X}(M)$, where $P : TM \to \mathcal{S}$ is the canonical projection associated to the second decomposition in (2.1). The vector field $\overline{\nabla}_U \xi = \nabla_U \xi$ is tangent to the null hypersurface M and can be decomposed as

$$\overline{\nabla}_U \xi = -\tau(U)\xi - \overset{\star}{A}(U),$$

where $\overset{\star}{A}(U) \in \mathcal{S}$. The endomorphism $\overset{\star}{A}$ is the shape operator of $\mathcal{S}$ and satisfies

$$B(U,V) = g(\overset{\star}{A}(U), V) = g(U, \overset{\star}{A}(V)), \qquad B(\xi, U) = 0.$$

Some useful identities in the theory are the following:

$$-2C(U,X) = d\omega(U,X) + (L_\zeta g)(U,X) + g(\zeta,\zeta)B(U,X), \tag{2.3}$$

the Gauss-Codazzi equation

$$\begin{aligned} g(R_{UV}W, \xi) &= g(\left(\nabla_U \overset{\star}{A}\right)(V), W) - g(\left(\nabla_V \overset{\star}{A}\right)(U), W) \\ &\quad + \tau(U) g(\overset{\star}{A}(V), W) - \tau(V) g(\overset{\star}{A}(U), W), \end{aligned} \tag{2.4}$$

$$(L_\xi \widetilde{g})(X,Y) = -2B(X,Y) \tag{2.5}$$

for all $U, V, W \in \mathfrak{X}(M)$,$X, Y \in \mathcal{S}$, and the Raychaudhuri equation[9] :

$$\overline{Ric}(\xi,\xi) = \xi(H) + \tau(\xi)H - \| \overset{\star}{A} \|^2,$$

where H denotes the (non-normalized) null mean curvature of the null hypersurface given by

$$H_p = \sum_{i=1}^{n} B(e_i, e_i),$$

with $\{e_1, \dots, e_n\}$ an orthonormal basis in $\mathcal{S}_p$. In particular, $H = -\widetilde{\operatorname{div}}\xi$.

If $B = 0$, then it is said that M is totally geodesic and if $B = \rho g$ for certain $\rho \in C^\infty(M)$, then M is totally umbilical. Observe that these definitions do not depend on the chosen rigging, although the tensors B, τ and C do depend. Throughout, the Levi-Civita connection on the normalized rigged structure $(M, \widetilde{g})$ will be denoted $\widetilde{\nabla}$ and we have for all $X, Y, Z \in \mathcal{S}$

$$C(\xi, X) = -\tau(X) - \widetilde{g}(\widetilde{\nabla}_\xi \xi, X), \qquad \widetilde{\nabla}_X Y = \overset{\star}{\nabla}_X Y - \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\xi,$$

being $\overset{\star}{\nabla}$ the connection on the screen bundle $\mathcal{S}$. In particular

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) \quad \forall X, Y, Z \in \mathcal{S}.$$

From now on, we assume $\overline{M}$ to be a generalized Robertson-Walker (GRW) spacetime of constant sectional curvature $c \in \{-1, 0, 1\}$, which will be denoted $\overline{M}_1^{n+2}(c)$ throughout. It is known that such spacetime admits timelike closed and conformal vector field, say ζ. We have

$$\overline{M}_1^{n+2}(c) = (I \times_f F, \overline{g}), \quad \overline{g} = -dt^2 + f^2(t) g_F$$

where f (the warping function) is a smooth positive function on I, and the fiber (F, g_F) is an $(n+1)-$dimensional Riemannian manifold of constant sectional curvature c_F [29]. So, the target space $\overline{M}_1^{n+2}(c)$ of immersion is locally isometric to one of the modele spaces : a de Sitter spacetime $\mathbb{S}_1^{n+2}$ of curvature $c = 1$, the Lorentz-Minkowski spacetime $\mathbb{R}_1^{n+2}$ when $c = 0$ or the anti de Sitter spacetime $\mathbb{H}_1^{n+2}$ (actually the universal covering of this pseudohyperbolic space $\mathbb{H}_1^{n+2}$) of curvature $c = -1$. Hence, we consider the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^m$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$.

Due to the causal character (spacelike or null) of tangent vectors to a null hypersurface in Lorentzian space, the induced singular metric on the null hypersurface has signature $(0, n)$. So the timelike concircular vector field ζ can act as rigging vector field for M. The closed and conformal vector field ζ has the outstanding property that there exists a smooth function $\sigma \in C^\infty(\overline{M})$ (the conformal factor) such that $\overline{\nabla}_U \zeta = \sigma U$ for all $U \in \mathfrak{X}(\overline{M})$. In particular $L_\zeta \overline{g} = 2\sigma \overline{g}$. For a closed and conformal rigging, the rotation $1-$form vanishes identically $(\tau = 0)$ and ξ is $g-$geodesic. Moreover, due to the closedness of ζ, $\widetilde{\nabla}_U \xi = -\overset{\star}{A}(U)$ and

$$\widetilde{\nabla}_U V = \nabla_U V + [B(U, V) - C(U, PV)]\, \xi, \tag{2.6}$$

for all $U, V \in \mathfrak{X}(M)$. Also, using (2.3) we derive the following useful relation linking the shape operators A and $\overset{\star}{A}$.

$$A = -\frac{1}{2}\lambda \overset{\star}{A} - \sigma P, \tag{2.7}$$

where $\lambda = \overline{g}(\zeta, \zeta)$ denotes the length function of ζ.

For the closed rigging ζ, the screen distribution $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pM$ is integrable and gives rise to a foliation $\mathscr{F}$ on the null hypersurface. Moreover, we have shown in [11, Lemma 7] that the conformal factor σ and the length function λ are constant through the (screen) leaves $\mathscr{F}_p$, $p \in M$. In other words,

$$X \cdot \sigma = 0 \quad \text{and} \quad X \cdot \lambda = 0$$

for all $X \in \mathcal{S}$.

3. Rigged linearized operators $\widetilde{L}_r$ and technical lemmas

The shape operator $\overset{\star}{A}$ is self-adjoint and satisfies $\overset{\star}{A}\xi = 0$. Its $n+1$ real valued eigenfunctions $\overset{\star}{k}_0 = 0, \overset{\star}{k}_1, \dots, \overset{\star}{k}_n$ are the screen principal curvatures and we let $(X_0 = \xi, X_1, \dots, X_n)$ denote a $\widetilde{g}-$orthonormal basis of eigenvector fields of $\overset{\star}{A}$, with span$(X_1, \dots, Xn) = \mathcal{S}$. For $0 \leq r \leq n$, the $r-th$ null mean curvature $\overset{\star}{H}_r$ of the null hypersurface with respect to the shape operator $\overset{\star}{A}$ is given by

$$\binom{n+1}{r} \overset{\star}{H}_r = \sum_{0 \leq i_1 < \cdots < i_r \leq n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r} \qquad \text{and} \qquad \overset{\star}{H}_0 = 1,$$

and the null hypersurface is said to be $r-$maximal if $\overset{\star}{H}_r = 0$ identically on M. The following notations will be in use :

$$\overset{\star}{S}_r = \sum_{0 \leq i_1 < \cdots < i_r \leq n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}, \qquad \overset{\star}{S}{}_r^{\alpha} = \sum_{\substack{0 \leq i_1 < \cdots < i_r \leq n \\ i_1, \cdots, i_r \neq \alpha}} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r} .$$

In particular $\overset{\star}{S}_0 = 1$ and $\overset{\star}{S}_1 = H$ (the null mean curvature).

For $0 \leq r \leq n+1$, the $r-th$ Newton transformation $\overset{\star}{T}_r$ with respect to the shape operator $\overset{\star}{A}$ is the End$(\Gamma(TM))$ element given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r} (-1)^a \overset{\star}{S}_a \overset{\star}{A}{}^{r-a} .$$

Inductively,

$$\overset{\star}{T}_0 = I \text{ and } \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A} \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity of $\Gamma(TM)$ and $\overset{\star}{T}_{n+1}=0$ (follows Cayley-Hamilton's theorem). By algebraic computations, one shows the following.

Proposition 3.1 ([9]).

(1) $\overset{\star}{T}_r$ *is self-adjoint and commute with* $\overset{\star}{A}$ *for any* r*;*
(2) $\overset{\star}{T}_r X_\alpha = (-1)^r \overset{\star\alpha}{S_r} X_\alpha$ *(*for a fixed α);
(3) $\operatorname{tr}(\overset{\star}{T}_r) = (-1)^r (n+1-r) \overset{\star}{S}_r$;
(4) $\operatorname{tr}\left(\overset{\star}{A} \circ \overset{\star}{T}_{r-1}\right) = (-1)^{r-1} r \overset{\star}{S}_r$;
(5) $\operatorname{tr}\left(\overset{\star}{A}{}^2 \circ \overset{\star}{T}_{r-1}\right) = (-1)^{r-1}\left(\overset{\star}{S}_1 \overset{\star}{S}_r - (r+1)\overset{\star}{S}_{r+1}\right)$;
(6) $\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}) = (-1)^{r-1} X \cdot \overset{\star}{S}_r$.

Also, for the last item in Proposition 3.1, replacing ∇ by $\widetilde{\widetilde{\nabla}}$, it is easy to show by a straightforward computation that

$$\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \widetilde{\widetilde{\nabla}}_X \overset{\star}{A}) = (-1)^{r-1} X \cdot \overset{\star}{S}_r \ . \tag{3.8}$$

We recall the following from [9, Remark 3, Page 68].

Theorem 3.1. *Let* (M^{n+1}, ζ) *be a normalized null hypersurface of a Lorentzian space form* $(\overline{M}_1^{n+2}(c), \bar{g})$ *with rigged vector field* ξ *and* $\tau = 0$*. Then,*

$$\xi \cdot \overset{\star}{S}_r = (-1)^{r-1} tr\left(\overset{\star}{A}{}^2 \circ \overset{\star}{T}_{r-1}\right) \overset{Prop.\ 3.1\ (5)}{=} \left(\overset{\star}{S}_1 \overset{\star}{S}_r - (r+1)\overset{\star}{S}_{r+1}\right). \tag{3.9}$$

Consequently, if $\overset{\star}{S}_r = 0$ *for some* $r = 1, ..., n$*, then* $\overset{\star}{S}_k = 0$ *for all* $k \geq r$*.*

For each $0 \leq r \leq n$, the divergence of the operator $\overset{\star}{T}_r \colon \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ with respect to the rigged connection $\widetilde{\widetilde{\nabla}}$ is the vector field $\mathrm{div}^{\widetilde{\widetilde{\nabla}}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}) \in \mathfrak{X}(\mathrm{M})$ defined as the trace of $\widetilde{\widetilde{\nabla}} \overset{\star}{T}_r$, that is

$$\mathrm{div}^{\widetilde{\widetilde{\nabla}}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}) = \left(\widetilde{\widetilde{\nabla}}_\xi \overset{\star}{\mathrm{T}}_{\mathrm{r}}\right)(\xi) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\widetilde{\widetilde{\nabla}}_{\mathrm{X_i}} \overset{\star}{\mathrm{T}}_{\mathrm{r}}\right)(\mathrm{X_i}).$$

Using the iterative formula $\overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A} \circ \overset{\star}{T}_{r-1}$, we have

$$\mathrm{div}^{\widetilde{\widetilde{\nabla}}} \overset{\star}{\mathrm{T}}_{\mathrm{r}} = (-1)^{\mathrm{r}} \mathrm{div}^{\widetilde{\widetilde{\nabla}}}\left(\overset{\star}{\mathrm{S}}_{\mathrm{r}}\, \mathrm{I}\right) + \mathrm{div}^{\widetilde{\widetilde{\nabla}}}\left(\overset{\star}{\mathrm{A}} \circ \overset{\star}{\mathrm{T}}_{\mathrm{r-1}}\right).$$

But

$$\mathrm{div}^{\widetilde{\widetilde{\nabla}}}\left(\overset{\star}{\mathrm{S}}_{\mathrm{r}}\, \mathrm{I}\right) = \sum_{\alpha=0}^{n} \left(\widetilde{\widetilde{\nabla}}_{X_\alpha} \overset{\star}{S}_r I\right) X_\alpha = \sum_\alpha \left[\widetilde{\widetilde{\nabla}}_{X_\alpha}(\overset{\star}{S}_r X_\alpha) - \overset{\star}{S}_r \left(\widetilde{\widetilde{\nabla}}_{X_\alpha} X_\alpha\right)\right]$$

$$= \sum_\alpha (X_\alpha \cdot \overset{\star}{S}_r) X_\alpha = \widetilde{\nabla} \overset{\star}{S}_r \,.$$

On the other side,

$$\begin{aligned}
\operatorname{div}^{\widetilde{\nabla}} \left(\overset{\star}{\mathrm{A}} \circ \overset{\star}{\mathrm{T}}_{\mathrm{r}-1} \right) &= \sum_\alpha \left(\widetilde{\nabla}_{X_\alpha} \left(\overset{\star}{A} \circ \overset{\star}{T}_{r-1} \right) X_\alpha \right) \\
&= \sum_\alpha \left[\widetilde{\nabla}_{X_\alpha} \left(\overset{\star}{A} \circ \overset{\star}{T}_{r-1} (X_\alpha) \right) - \left(\overset{\star}{A} \circ \overset{\star}{T}_{r-1} \right) \left(\widetilde{\nabla}_{X_\alpha} X_\alpha \right) \right] \\
&= \sum_\alpha \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A} \right) \left(\overset{\star}{T}_{r-1} X_\alpha \right) + \overset{\star}{A} \left(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}-1} \right).
\end{aligned}$$

So, for all $U \in \mathfrak{X}(M)$,

$$\begin{aligned}
\widetilde{g} \left(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}}, \mathrm{U} \right) &= \widetilde{g} \left(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}-1}, \overset{\star}{\mathrm{A}}\, \mathrm{U} \right) + \sum_\alpha \widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A} \right) U \right) \\
&\quad + (-1)^r U \cdot \overset{\star}{S}_r \,.
\end{aligned} \tag{3.10}$$

We compute $\sum_\alpha \widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A} \right) U \right)$ using curvature relations. Before proceeding we note the following covariant derivative identity which is established by a direct computation. For all linear operator $T : \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $U, V \in \mathfrak{X}(M)$,

$$\begin{aligned}
\left(\widetilde{\nabla}_U T \right) (V) &= (\nabla_U T)(V) + [B(U, TV)\xi - B(U, V) T\xi] \\
&\quad - \frac{1}{2} \Big([\langle AU, TV \rangle + \langle U, A(TV) \rangle] \, \xi - [\langle AU, V \rangle + \langle U, AV \rangle] \, T\xi \Big).
\end{aligned} \tag{3.11}$$

Applying (3.11) with $T = \overset{\star}{A}$ and using the fact that $\overset{\star}{A} \xi = 0$ and $\overline{\nabla} \zeta = \sigma \otimes I$ we get :

$$\left(\widetilde{\nabla}_U \overset{\star}{A} \right) (V) = \left(\nabla_U \overset{\star}{A} \right) (V) + \left[\langle \overset{\star}{A} U, \overset{\star}{A}\, V \rangle - \langle AU, \overset{\star}{A}\, V \rangle \right] \xi. \tag{3.12}$$

So, for each $0 \le \alpha \le n$,

$$\begin{aligned}
\widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A} \right) U \right) &= \widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_{X_\alpha} \overset{\star}{A} \right) U \right) \\
&\quad + \left[g(\overset{\star}{A}\, X_\alpha, \overset{\star}{A} U) - g(A X_\alpha, \overset{\star}{A} U) \right] \times \widetilde{g}(X_{X_\alpha}, \overset{\star}{T}_{r-1}\, \xi).
\end{aligned}$$

Using item (ii) in Proposition 3.1 and (2.7) we see that the last term in above equality vanishes. Hence, in closed and conformal setting,

$$\begin{aligned}
\widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A} \right) U \right) &= \widetilde{g} \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_{X_\alpha} \overset{\star}{A} \right) U \right) \\
&= g \left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_{X_\alpha} \overset{\star}{A} \right) U \right) + \omega \left(\overset{\star}{T}_{r-1} X_\alpha \right) \omega \left(\left(\nabla_{X_\alpha} \overset{\star}{A} \right) U \right).
\end{aligned}$$

We show that the last term vanishes. Indeed,

$$\begin{aligned} \omega\left(\overset{\star}{T}_{r-1} X_\alpha\right) &= \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \xi\right) = (-1)^{r-1} \overset{\star}{S}{}^{\alpha}_{r-1} \widetilde{g}(X_\alpha, \xi) \\ &= \begin{cases} 0 & \text{if} \quad \alpha \neq 0 \\ (-1)^{r-1} \overset{\star}{S}{}^{0}_{r-1} = (-1)^{r-1} \overset{\star}{S}_{r-1} & \text{if} \quad \alpha = 0, \end{cases} \end{aligned} \tag{3.13}$$

where we use the fact that $\overset{\star}{S}{}^{0}_{r-1} = \overset{\star}{S}_{r-1}$ due to $\overset{\star}{k}_0 = 0$. From (3.13) we need to compute the second factor just for $\alpha = 0$.

$$\begin{aligned} \omega\left(\left(\nabla_{X_\alpha} \overset{\star}{A}\right) U\right) &= \widetilde{g}\left((\nabla_\xi \overset{\star}{A})(U), \xi\right) \\ &= \widetilde{g}\left(\nabla_\xi (\overset{\star}{A} U) - \overset{\star}{A} (\nabla_\xi U), \xi\right) = \widetilde{g}\left(\nabla_\xi (\overset{\star}{A} U), \xi\right) \\ &= \widetilde{g}\left(\overset{\star}{\nabla}_\xi (\overset{\star}{A} U) + C(\xi, \overset{\star}{A} U)\xi, \xi\right) = C(\xi, \overset{\star}{A} U) \overset{(2.7)}{=} 0. \end{aligned}$$

Hence, for $0 \leq \alpha \leq n$,

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) = g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_{X_\alpha} \overset{\star}{A}\right) U\right). \tag{3.14}$$

Now, Gauss-Codazzi equation (2.4) with $\tau = 0$ provides

$$\overline{g}\left(\overline{R}(U, V)W, \xi\right) = g\left((\nabla_U \overset{\star}{A})V, W\right) - g\left((\nabla_V \overset{\star}{A})U, W\right),$$

for all $U, V, V \in \mathfrak{X}(M)$, where we make use of the identity

$$\left(\nabla_U B\right)(V, W) = g\left((\nabla_U \overset{\star}{A})V, W\right) + \omega(W) g(\overset{\star}{A} U, \overset{\star}{A} V).$$

Hence, (3.14) becomes

$$\begin{aligned} \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) &= \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) \\ &\quad + g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_U \overset{\star}{A}\right)(X_\alpha)\right). \end{aligned}$$

From (3.12), the following equation holds

$$\left(\nabla_U \overset{\star}{A}\right)(X_\alpha) = \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha) - \left[g(\overset{\star}{A} U, \overset{\star}{A} X_\alpha) - g(AU, \overset{\star}{A} X_\alpha)\right]\xi$$

and we get

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) = \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) + g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right).$$

But

$$g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) = \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) - \omega(\overset{\star}{T}_{r-1} X_\alpha)\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right). \tag{3.15}$$

Due to the relation $\overset{\star}{T}_{r-1} X_\alpha = (-1)^{r-1} \overset{\star}{S}{}^{\alpha}_{r-1} X_\alpha$, we see that for $\alpha \neq 0$, $\omega(\overset{\star}{T}_{r-1} X_\alpha) = 0$. Also, for $\alpha = 0$,

$$\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(\xi)\right) = \widetilde{g}\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(\xi), \xi\right) = \widetilde{g}\left(\widetilde{\nabla}_U(\overset{\star}{A}\xi) - \overset{\star}{A}(\widetilde{\nabla}_U \xi), \xi\right) = -\widetilde{g}\left(\overset{\star}{A}(\widetilde{\nabla}_U \xi), \xi\right) = 0,$$

hence the product $\omega(\overset{\star}{T}_{r-1} X_\alpha)\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right)$ in (3.15)vanishes identically and we get

$$g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) = \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right).$$

Therefore, for $0 \leq \alpha \leq n$,

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right)(U)\right) = \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) + \widetilde{g}\left(X_\alpha, \left(\overset{\star}{T}_{r-1} \circ \left(\widetilde{\nabla}_U \overset{\star}{A}\right)\right)(X_\alpha)\right).$$

Returning back to (3.10) we have

$$\begin{aligned}
\widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}}, \mathrm{U}\right) &= \widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r-1}}, \overset{\star}{\mathrm{A}}\mathrm{U}\right) + \sum_\alpha \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) \\
&\quad + \sum_\alpha \widetilde{g}\left(\left(\overset{\star}{T}_{r-1} \circ (\widetilde{\nabla}_U \overset{\star}{A})\right) X_\alpha, X_\alpha\right) + (-1)^r U \cdot \overset{\star}{S}_r \\
&= \widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r-1}}, \overset{\star}{\mathrm{A}}\mathrm{U}\right) + \sum_\alpha \overline{g}\left(\overline{R}(X_\alpha, \xi) \overset{\star}{T}_{r-2} X_\alpha, \overset{\star}{A}U\right) \\
&\quad + \overline{g}\left(\overline{R}(X_\alpha, \xi) \overset{\star}{T}_{r-1} X_\alpha, U\right).
\end{aligned}$$

By iterating this process, we get the following.

Lemma 3.1.

$$\widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}}, \mathrm{U}\right) = \sum_{i=0}^{r-1} \sum_{\alpha=0}^{n} \overline{g}\left(\overline{R}(X_\alpha, \xi) \overset{\star}{T}_i X_\alpha, \overset{\star}{A}{}^{r-1-i} U\right) \tag{3.16}$$

Corollary 3.1. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^{n+2+c^2}$$

be a isometric immersion of a null hypersurface in $\overline{M}_1^{n+2}(c)$ where $t = c(c-1)/2$ with $c = 1, 0, -1$, furnished with a closed and conformal rigging vector field ζ. Then, for all $f \in C^\infty(M)$.

$$\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}} = 0 \quad \textit{and} \quad \mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}} \widetilde{\nabla}\mathrm{f}\right) = tr\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}} \circ \widetilde{\nabla}^2 \mathrm{f}\right). \tag{3.17}$$

Proof. When the ambient Lorentzian manifold $\overline{M}^{n+2}$ has constant sectional curvature c, we have for a fixed r, each $i = 0 \dots r-1$ and $\alpha = 0, \dots, n$ the term $\overline{g}\left(\overline{R}(X_\alpha, \xi) \overset{\star}{T}_i X_\alpha, \overset{\star}{A}^{r-1-i} U\right)$ in (3.16) vanishes identically. So $\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}} = 0$. By definition,

$$\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) = \mathrm{tr}\left(\widetilde{\nabla}\, \overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) = \sum_{\alpha=0}^{\mathrm{n}} \widetilde{\mathrm{g}}\left(\widetilde{\nabla}_{\mathrm{X}_\alpha}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\widetilde{\nabla}\mathrm{f}), \mathrm{X}_\alpha\right),$$

and

$$\widetilde{\nabla}_{X_\alpha}(\overset{\star}{T}_r \widetilde{\nabla} f) = \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{T}_r\right)\widetilde{\nabla} f + \overset{\star}{T}_r \left(\widetilde{\nabla}_{X_\alpha} \widetilde{\nabla} f\right).$$

So,

$$\begin{aligned}
\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) &= \sum_{\alpha=0}^{n} \widetilde{g}\left(\widetilde{\nabla} f, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{T}_r\right)(X_\alpha)\right) + \sum_{\alpha=0}^{n} \widetilde{g}\left(\overset{\star}{T}_r \left(\widetilde{\nabla}_{X_\alpha}\widetilde{\nabla} f\right), X_\alpha\right) \\
&= \widetilde{g}\left(\widetilde{\nabla} f, \mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}})\right) + \mathrm{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 f\right)
\end{aligned}$$

and the second claim in (3.17) follows from $\mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}) = 0$. □

For the sake of comparison, note that in [9] using the projected (induced) connection ∇ we established the following.

Proposition 3.2. [9, Proposition 3] $\forall X \in \mathfrak{X}(M)$,

$$\begin{aligned}
g(div^{\nabla} \overset{\star}{T}_r, U) &= \sum_{a=0}^{r-1}\sum_{i=1}^{n} \bar{g}\left(\bar{R}(X_i, \xi)\, \overset{\star}{T}_a\, X_i, \overset{\star}{A}_\xi^{\,r-1-a}\, U\right) \\
&\quad + \sum_{a=0}^{r-1}\left(\tau(\overset{\star}{A}_\xi^{\,r-1-a}\, U) tr(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a) - \tau(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a\, U))\right) \\
&\quad + (-1)^r \omega(U)\left(\sum_{i=1}^{n} \overset{\star}{S}{}^{i}_{r-1} \overset{\star}{k}{}^{2}_{i} - \xi(\overset{\star}{S}_r)\right).
\end{aligned} \tag{3.18}$$

Taking $r = 2$ and $U = \xi$ in (3.18) leads to

$$\begin{aligned}
0 = g(div^{\nabla} \overset{\star}{T}_2, \xi) &= \sum_{i=1}^{n} \overline{g}\left(\overline{R}(X_i, \xi)\, \overset{\star}{T}_i X_i, \xi\right) + \tau(\xi)\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) \\
&\quad - \tau(\overset{\star}{A} \circ \overset{\star}{T}_1\, \xi) + \sum_{i=1}^{n} \overset{\star}{S}{}^{i}_{1} \overset{\star}{k}{}^{2}_{i} - \xi \cdot \overset{\star}{S}_2 \\
&= \sum_{i=1}^{n} \overset{\star}{S}{}^{i}_{1}\, \overline{K}_\xi(\Pi_i) + \tau(\xi)\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) \\
&\quad - \tau(\overset{\star}{A} \circ \overset{\star}{T}_1\, \xi) + \sum_{i=1}^{n} \overset{\star}{S}{}^{i}_{1} \overset{\star}{k}{}^{2}_{i} - \xi \cdot \overset{\star}{S}_2
\end{aligned}$$

where $\overline{K}_\xi(\Pi_i) = \dfrac{\overline{g}(\overline{R}(\xi, X_i)X_i, \xi)}{\overline{g}(X_i, X_i)} = \overline{g}(\overline{R}(\xi, X_i)X_i, \xi)$ stands for the null sectional curvature of the null plane $\Pi_i = \mathrm{span}(X_i, \xi)$. But $\overset{\star}{A} \circ \overset{\star}{T}_1 \xi = 0$ and $\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) = -2 \overset{\star}{S}_2$, so

$$\sum_{i=1}^{n} \overset{\star}{S}{}_1^{i}\, \overline{K}_\xi(\Pi_i) = \xi\cdot \overset{\star}{S}_2 + 2\tau(\xi) \overset{\star}{S}_2 - \sum_{i=1}^{n} \overset{\star}{S}{}_1^{i} \overset{\star}{k}{}_i^{2} \ . \tag{3.19}$$

Therefore, we can state the following.

Lemma 3.2. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c)$$

be a isometric immersion of a null hypersurface in a space $\overline{M}_1^{n+2}(c)$ of constant curvature c, furnished with a conformal rigging vector field ζ. Then

$$\xi\cdot \overset{\star}{S}_2 = \sum_{i=1}^{n} \overset{\star}{S}{}_1^{i} \overset{\star}{k}{}_i^{2} \ . \tag{3.20}$$

In particular, for $n = 2$

$$\xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2 \ . \tag{3.21}$$

Proof. For constant sectional curvature, $\overline{K}_\xi(\Pi_i) = 0$, $i = 1, \ldots, n$ and since $\tau(\xi) = 0$, we obtain (3.20) from (3.19). □

Now, for $n = 2$,

$$\sum_{i=1}^{2} \overset{\star}{S}{}_1^{i} \overset{\star}{k}{}_i^{2} = \overset{\star}{k}_2 \overset{\star}{k}{}_1^{2} + \overset{\star}{k}_1 \overset{\star}{k}{}_2^{2} = \overset{\star}{k}_1 \overset{\star}{k}_2 \,(\overset{\star}{k}_1 + \overset{\star}{k}_2) = \overset{\star}{S}_1 \overset{\star}{S}_2 \ .$$

For each Newton transformation $\overset{\star}{T}_r$, we can consider the second-order linear differential operator $\widetilde{L}_r : C^\infty(M) \to C^\infty(M)$ given by

$$\widetilde{L}_r(f) = \mathrm{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 f\right) \tag{3.22}$$

where $\widetilde{\nabla}^2 f := \widetilde{\nabla}\widetilde{\nabla} f$ stands for the $\widetilde{g}-$dual of the Hessian $\widetilde{Hess}\ f$ of f with respect to $\widetilde{g}$ on M. Observe that when $r = 0$, $\widetilde{L}_0 = \widetilde{\Delta}$ is nothing but the Laplacian operator on the Riemannian rigged structure $(M, \widetilde{g})$. Also, the second-order linear differential operator $\widetilde{L}_r$ defined here in (3.22) is different from $L_r(f) = \mathrm{tr}\left(\overset{\star}{T}_r \circ \nabla(\widetilde{\nabla} f)\right)$ as defined in [25] where a hybrid use of the (projected) induced connection ∇ and the rigged Levi-Civita connection $\widetilde{\nabla}$ on $(M, \widetilde{g})$ is made. But these two connections do not coincide in general. Indeed, the equality $\widetilde{\nabla} = \nabla$ holds if and only if $B = C$ and $\tau = 0$ (cf. [10, Theorem 4.1]).

From (3.22) and (3.17) and using divergence properties, we get

Lemma 3.3. *For all* $f, h \in C^\infty(M)$,

$$\widetilde{L}_r(fh) = f\widetilde{L}_r(h) + h\widetilde{L}_r(f) + 2\widetilde{g}\left(\widetilde{\nabla} f, \overset{\star}{T}_r\ \widetilde{\nabla} h\right). \tag{3.23}$$

For the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$, we will calculate $\widetilde{L}_r$ acting on the coordinate components of the immersion ψ, i.e a function given by $\left\langle \psi, a \right\rangle$ where $a \in \mathbb{R}^m_{1+t}$ is an arbitrary fixed vector. We let $\overset{0}{\nabla}$ and $\overline{\nabla}$ denote the Levi-Civita connections on $\mathbb{R}^{n+2+c^2}_{1+t}$ and $\overline{M}_1^{n+2}(c)$, respectively. For all $U, V \in \mathfrak{X}(M)$,

$$\overset{0}{\nabla}_U\ V = \overline{\nabla}_U V - c\overline{g}(U,V)\psi$$

which, by use of (2.6) gives

$$\overset{0}{\nabla}_U\ V = \widetilde{\nabla}_U V + B(U,V)(N-\xi) + g(AU,V)\xi - cg(U,V)\psi. \tag{3.24}$$

In particular, for all $U \in \mathfrak{X}(M)$,

$$\overset{0}{\nabla}_U \xi = \overline{\nabla}_U \xi = \widetilde{\nabla}_U \xi = -\overset{\star}{A} U,$$

Lemma 3.4. *Set* $h = \left\langle \psi, a \right\rangle$, $a \in \mathbb{R}^{n+2+c^2}_{1+t}$ *with* $c = -1, 0, 1$ *and* $\lambda = \langle \zeta, \zeta \rangle$. *Then,*

$$\widetilde{\nabla} h = a^T - \langle a, N-\xi \rangle \xi = a - \langle a, N-\xi\rangle\xi - \langle a, \xi\rangle N - c\langle a, \psi\rangle\psi; \tag{3.25}$$

$$\begin{aligned} \widetilde{L}_r h &= (-1)^{r+1}\left[(n-r)\sigma\ \overset{\star}{S}_r + (r+1)(\lambda+1)\ \overset{\star}{S}_{r+1}\right]\langle \xi, a\rangle + (-1)^r (r+1)\ \overset{\star}{S}_{r+1}\ \langle \zeta, a\rangle \\ &\quad + (-1)^{r+1}(n-r)c\ \overset{\star}{S}_r\ \langle \psi, a\rangle, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \widetilde{L}_r \psi &= (-1)^{r+1}\left[(n-r)\sigma\ \overset{\star}{S}_r + (r+1)(\lambda+1)\ \overset{\star}{S}_{r+1}\right]\xi + (-1)^r (r+1)\ \overset{\star}{S}_{r+1}\ \zeta \\ &\quad + (-1)^{r+1}(n-r)c\ \overset{\star}{S}_r\ \psi. \end{aligned} \tag{3.27}$$

Proof. The function h is smooth on M and for all $X \in \mathfrak{X}(M)$,

$$\widetilde{g}(X, \widetilde{\nabla} h) = X \cdot h = X \cdot \langle \psi, a\rangle = \left\langle \overset{0}{\nabla}_X\ \psi, a \right\rangle = \langle X, a\rangle.$$

But

$$a = a^T + \langle \xi, a\rangle N + c\langle \psi, a\rangle \psi, \tag{3.28}$$

where $a^T \in \mathfrak{X}(M)$ is the tangential component of the vector a projected on M in the direction span(N, ψ). So, noting that $\omega(a^T) = \langle a, N\rangle$,

$$\begin{aligned} \widetilde{g}(X, \widetilde{\nabla} h) &= \left\langle X, a^T + \langle \xi, a\rangle N + c\langle \psi, a\rangle \psi \right\rangle = g(X, a^T) + \langle \xi, a\rangle \widetilde{g}(\xi, X) \\ &= \widetilde{g}(X, a^T) - \omega(X)\omega(a^T) + \langle \xi, a\rangle \widetilde{g}(\xi, X) = \widetilde{g}\left(X, a^T - \langle a, N-\xi\rangle \xi\right), \end{aligned}$$

and we get $\widetilde{\nabla} h = a^T - \langle a, N-\xi\rangle\xi$ and the last equality in (3.25) follows from (3.28).

Further, note that

$$\overset{0}{\nabla}_U N = -AU - c\omega(U)\psi \quad \overset{0}{\nabla}_U \xi = -\overset{\star}{A}U \quad \text{and} \quad \overset{0}{\nabla}_U \psi = U,$$

hence, a straightforward computation using (3.25) leads to

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla} h &= -c\langle \psi, a\rangle PU + \langle AU - \overset{\star}{A}U, a\rangle \xi + \langle N - \xi, a\rangle \overset{\star}{A}U \\ &\quad + \langle \overset{\star}{A}U, a\rangle N + \langle \xi, a\rangle AU - c\langle PU, a\rangle \psi. \end{aligned} \tag{3.29}$$

On the other hand, applying (3.24) with $V = \widetilde{\nabla} h$ leads to

$$\overset{0}{\nabla}_U \widetilde{\nabla} h = \widetilde{\nabla}_U \widetilde{\nabla} h + \langle a, \overset{\star}{A}U\rangle (N - \xi) + \langle a, AU\rangle \xi - c\langle PU, a\rangle \psi. \tag{3.30}$$

Therefore, using (3.29), (3.30) and (2.7) we get

$$\widetilde{\nabla}_U \widetilde{\nabla} h = \left\langle N - \frac{1}{2}(2+\lambda)\xi, a\right\rangle \overset{\star}{A}U - \langle \sigma\xi + c\psi, a\rangle PU, \tag{3.31}$$

which in terms of ζ reads

$$\widetilde{\nabla}_U \widetilde{\nabla} h = \left\langle \zeta - (1+\lambda)\xi, a\right\rangle \overset{\star}{A}U - \langle \sigma\xi + c\psi, a\rangle PU. \tag{3.32}$$

It follows from (3.32) that

$$\begin{aligned} \widetilde{L}_r h &= \operatorname{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 h\right) = \sum_\alpha \widetilde{g}\left(\overset{\star}{T}_r\left(\widetilde{\nabla}_{X_\alpha}\widetilde{\nabla} h\right), X_\alpha\right) \\ &= \sum_\alpha \left[\langle \zeta - (1+\lambda)\xi, a\rangle\, \widetilde{g}(\overset{\star}{T}_r \overset{\star}{A} X_\alpha, X_\alpha) - \langle \sigma\xi + c\psi, a\rangle \widetilde{g}(\overset{\star}{T}_r PX_\alpha, X_\alpha)\right] \\ &= \langle \zeta - (1+\lambda)\xi, a\rangle \operatorname{tr}(\overset{\star}{A} \circ \overset{\star}{T}_r) - \langle \sigma\xi + c\psi, a\rangle \left(\operatorname{tr}(\overset{\star}{T}_r) - (-1)^r \overset{\star}{S}_r\right) \\ &= (-1)^r (r+1)\, \langle \zeta - (1+\lambda)\xi, a\rangle\, \overset{\star}{S}_{r+1} + (-1)^r (n-r) \langle \sigma\xi + c\psi, a\rangle\, \overset{\star}{S}_r\, . \end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{L}_r h &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\langle \xi, a\rangle + (-1)^r (r+1)\overset{\star}{S}_{r+1}\langle \zeta, a\rangle \\ &\quad + (-1)^{r+1}(n-r)c\overset{\star}{S}_r\langle \psi, a\rangle,\end{aligned}$$

which is (3.26). Extend $\widetilde{L}_r$ to the $\mathbb{R}^m_t-$valued function ψ by setting

$$\widetilde{L}_r\psi = \left(\widetilde{L}_r\psi_1, \ldots, \widetilde{L}_r\psi_m\right)$$

where $\psi_i = \varepsilon_i\langle \psi, e_i\rangle$ and $(e_1, \ldots, e_m)$ stands for an orthonormal basis of $\mathbb{R}^m_{1+t}$ with $m = n+2+c^2$, $t = c(c-1)/2$ and $\varepsilon_i = \langle e_i, e_i\rangle = \pm 1$. We have

$$\begin{aligned}\widetilde{L}_r\psi &= \sum_{i=1}^m \varepsilon_i \widetilde{L}_r\langle \psi, e_i\rangle e_i \\ &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\sum_{i=1}^m \varepsilon_i\langle \xi, e_i\rangle e_i \\ &\quad + (-1)^r(r+1)\overset{\star}{S}_{r+1}\sum_{i=1}^m \varepsilon_i\langle \zeta, e_i\rangle e_i + (-1)^{r+1}(n-r)c\overset{\star}{S}_r\sum_{i=1}^m \varepsilon_i\langle \psi, e_i\rangle e_i, \\ &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\xi \\ &\quad + (-1)^r(r+1)\overset{\star}{S}_{r+1}\zeta + (-1)^{r+1}(n-r)c\overset{\star}{S}_r\psi,\end{aligned}$$

which completes the proof. □

Remark 3.1. *Due to $\overset{\star}{S}_{n+1} = 0$, we see from above expression (3.27) that $\widetilde{L}_n\psi = 0$ and that (M, ζ) is (trivially) $\widetilde{L}_n-$harmonic.*

Lemma 3.5. *Let $a \in \mathbb{R}^m_t$ be a fixed constant vector and $U \in \mathfrak{X}(M)$. Then,*

$$\widetilde{\nabla}\langle \xi, a\rangle = -\overset{\star}{A}a^T, \tag{3.33}$$

where $a^T = a - \langle a, \xi\rangle N - c\langle \psi, a\rangle\psi$.

$$\begin{aligned}\widetilde{\nabla}_U\widetilde{\nabla}\langle \xi, a\rangle &= -\left(\widetilde{\nabla}_{a^T}\overset{\star}{A}\right)U - \left[\langle \overset{\star}{A}{}^2U, a^T\rangle + \langle \frac{1}{2}\overset{\star}{A}{}^2U + \sigma\overset{\star}{A}U, a^T\rangle\right]\xi \\ &\quad + \langle \xi, a\rangle\left(\frac{1}{2}\overset{\star}{A}{}^2U + \sigma\overset{\star}{A}U\right) + c\langle \psi, a\rangle\overset{\star}{A}U;\end{aligned} \tag{3.34}$$

$$\begin{aligned}\widetilde{L}_r\langle \xi, a\rangle &= (-1)^{r+1}\langle \widetilde{\nabla}\overset{\star}{S}_{r+1}, a\rangle + (-1)^r(r+1)c\overset{\star}{S}_{r+1}\langle \psi, a\rangle \\ &\quad + (-1)^{r+1}\left(\left[\frac{1}{2}\lambda\overset{\star}{S}_1 - (r+1)\sigma\right]\overset{\star}{S}_{r+1} + \frac{1}{2}(r+2)\lambda\overset{\star}{S}_{r+2} - \xi\cdot\overset{\star}{S}_{r+1}\right)\langle \xi, a\rangle\end{aligned} \tag{3.35}$$

and

$$\widetilde{L}_r\xi = (-1)^{r+1}\widetilde{\nabla}\overset{\star}{S}_{r+1} + (-1)^r(r+1)c\overset{\star}{S}_{r+1}\psi + (-1)^{r+1}\left(\left[\frac{1}{2}\lambda\overset{\star}{S}_1 - (r+1)\sigma\right]\overset{\star}{S}_{r+1} + \frac{1}{2}(r+2)\lambda\overset{\star}{S}_{r+2} - \xi\cdot\overset{\star}{S}_{r+1}\right)\xi. \quad (3.36)$$

Proof. Set $\nu = \langle\xi, a\rangle$. For $U \in \mathfrak{X}(M)$,

$$\begin{aligned}\widetilde{g}\left(\widetilde{\nabla}\nu, U\right) &= U\cdot\nu = U\cdot\langle\xi, a\rangle = \langle\overset{0}{\nabla}_U\xi, a\rangle = \langle -\overset{\star}{A}U, a\rangle \\ &= \langle -\overset{\star}{A}U, a^T\rangle = \langle U, -\overset{\star}{A}a^T\rangle = \widetilde{g}(U, -\overset{\star}{A}a^T).\end{aligned}$$

Therefore, $\widetilde{\nabla}\langle\xi, a\rangle = -\overset{\star}{A}a^T$. Using this expression, we get by direct computation that for all U, $W \in \mathfrak{X}(M)$,

$$\begin{aligned}\langle\overset{0}{\nabla}_U\widetilde{\nabla}\nu, W\rangle &= -\left\langle a^T + \langle a,\xi\rangle N, \left(\nabla_U\overset{\star}{A}\right)W\right\rangle \\ &\quad -\langle\overset{\star}{A}{}^2U, W\rangle\omega(a^T) + c\langle\overset{\star}{A}U, W\rangle\langle\psi, a\rangle\end{aligned}$$

It is easy to check that if $T \in End(TM)$ is a self-adjoint operator with respect to g then

$$\begin{aligned}\left\langle(\nabla_UT)V, W\right\rangle &= \left\langle V, (\nabla_UT)W\right\rangle + \omega(V)B(U, TW) \\ &\quad -\omega(TV)B(U, W) - \omega(W)B(U, TV) + \omega(TW)B(U, V).\end{aligned}$$

Applying this for $T = \overset{\star}{A}$ leads to

$$\begin{aligned}\langle\overset{0}{\nabla}_U\widetilde{\nabla}\nu, W\rangle &= -\left(\left\langle\left(\nabla_U\overset{\star}{A}\right)a^T, W\right\rangle + \omega(W)B(U\,\overset{\star}{A}\,a^T) - \omega(a^T)B(U, \overset{\star}{A}W)\right. \\ &\quad \left.+\langle a,\xi\rangle\left\langle\left(\nabla_U\overset{\star}{A}\right)W, N\right\rangle\right) - \langle\overset{\star}{A}{}^2U, W\rangle\omega(a^T) + c\langle\psi, a\rangle\langle\overset{\star}{A}U, W\rangle.\end{aligned}$$

But $\left\langle\left(\nabla_U\overset{\star}{A}\right)W, N\right\rangle = \langle\overset{\star}{A}AU, W\rangle$ and due to (2.7), we get

$$\left\langle\left(\nabla_U\overset{\star}{A}\right)W, N\right\rangle = \left\langle -\frac{1}{2}\lambda\overset{\star}{A}{}^2U - \sigma\overset{\star}{A}U, W\right\rangle$$

where $\lambda = \langle\zeta, \zeta\rangle$. Also, by Gauss-Codazzi equation with $\tau = 0$, the following equation holds,

$$\langle\overline{R}(U, V)\xi, W\rangle = -\langle\overline{R}(U, V)W, \xi\rangle = \langle(\nabla_V\overset{\star}{A})U - (\nabla_U\overset{\star}{A})V, W\rangle,$$

and since the ambient space has constant sectional curvature c, the left hand side vanishes, which leads to $\langle(\nabla_V\overset{\star}{A})U, W\rangle = \langle(\nabla_U\overset{\star}{A})V, W\rangle$. Therefore, $\left\langle\left(\nabla_U\overset{\star}{A}\right)a^T, W\right\rangle =$

$\left\langle \left(\nabla_{a^T} \overset{\star}{A}\right) U, W \right\rangle$ and

$$\langle \overset{0}{\nabla}_U \widetilde{\nabla}\nu, W\rangle = \Big\langle -\left(\nabla_{a^T}\overset{\star}{A}\right)U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right)\langle \xi, a\rangle + c\langle\psi, a\rangle\overset{\star}{A}U, W\Big\rangle.$$

and this leads to

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= -\left(\nabla_{a^T}\overset{\star}{A}\right)U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) \\ &\quad + c\langle\psi, a\rangle\overset{\star}{A}U + \beta(U)\xi + \gamma(U)\psi. \end{aligned} \tag{3.37}$$

Taking respectively ξ and ψ components both side leads to $\beta(U) = 0$ and $\gamma(U) = c\left\langle \overset{\star}{A}U, a^T\right\rangle$. Hence,

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= -\left(\nabla_{a^T}\overset{\star}{A}\right)U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) \\ &\quad + c\langle\psi, a\rangle\overset{\star}{A}U + c\left\langle \overset{\star}{A}U, a^T\right\rangle\psi. \end{aligned} \tag{3.38}$$

Computing the same term $\overset{0}{\nabla}_U \widetilde{\nabla}\nu$ using the right hand side of (3.24), we get

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= \widetilde{\nabla}_U\widetilde{\nabla}\nu - \left\langle \overset{\star}{A}^2 U, a^T\right\rangle (N - \xi) \\ &\quad - \left\langle -\frac{1}{2}\lambda \overset{\star}{A}^2 U - \sigma \overset{\star}{A} U, a^T\right\rangle \xi + c\left\langle \overset{\star}{A}U, a^T\right\rangle \psi. \end{aligned} \tag{3.39}$$

By comparing (3.38)‘and (3.39) and using (2.7) we get,

$$\begin{aligned} \widetilde{\nabla}_U\widetilde{\nabla}\nu &= -(\nabla_{a^T}\overset{\star}{A})U - \left[\left\langle \overset{\star}{A}^2 U, a^T\right\rangle + \left\langle \frac{1}{2}\lambda\overset{\star}{A}^2 U + \sigma \overset{\star}{A} U, a^T\right\rangle\right]\xi \\ &\quad + \langle\xi, a\rangle\left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) + c\langle\psi, a\rangle \overset{\star}{A}U. \end{aligned} \tag{3.40}$$

Finally, taking into account that

$$(\nabla_{a^T} \overset{\star}{A})U = (\widetilde{\nabla}_{a^T} \overset{\star}{A})U - [\langle \overset{\star}{A}^2 U - \overset{\star}{A} AU, a^T\rangle\rangle]\xi,$$

we get the desired relation (3.34). Now,

$$\begin{aligned} \widetilde{L}_r\langle\xi, a\rangle &= \operatorname{tr}(\overset{\star}{T}_r \widetilde{\nabla}^2\nu) \\ &\overset{3.34}{=} -\operatorname{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}_{a^T} \overset{\star}{A}\right) - 0 + \langle\xi, a\rangle\left(\frac{1}{2}\lambda\operatorname{tr}(\overset{\star}{T}_r \circ \overset{\star}{A}^2) + \sigma\operatorname{tr}(\overset{\star}{T}_r \circ \overset{\star}{A})\right) + \\ &\quad c\langle\psi, a\rangle\operatorname{tr}\left(\overset{\star}{T}_r \circ \overset{\star}{A}\right), \end{aligned}$$

and (3.35) is straightforward from Proposition 3.1. The last claim (3.36) follows from

$$\widetilde{L}_r\xi = \sum_{i=1}^{m} \varepsilon_i\Big(\widetilde{L}_r\langle\xi, e_i\rangle\Big)e_i$$

where we use (3.35) componentwise. □

Before the next statement, we recall the following from [11, Lemma 4 (4)], where ζ is a closed and conformal vector field.

$$\overline{Ric}(U,\zeta) = -(n+1)U\cdot\sigma, \tag{3.41}$$

forall $U\in\mathfrak{X}(M)$. Since our ambient space $\overline{M}^{n+2}(c)$ has constant sectional curvature c, it follows from (3.42) that

$$U\cdot\sigma = -c\omega(U) \quad \text{for all} \quad U\in\mathfrak{X}(M). \tag{3.42}$$

Taking $U=\xi$ provides

$$\xi\cdot\sigma = -c. \tag{3.43}$$

It turns out that

$$\widetilde{\nabla}\sigma = (\xi\cdot\sigma)\xi = -c\xi. \tag{3.44}$$

Furthermore, for $U\in\mathfrak{X}(M)$,

$$\widetilde{\nabla}_U\widetilde{\nabla}\sigma \;\;=\;\; \widetilde{\nabla}_U(-c\xi) = c\overset{\star}{A}U, \tag{3.45}$$

and we get

$$\widetilde{L}_r\sigma = tr\left(\overset{\star}{T}_r\ (\widetilde{\nabla}_U\widetilde{\nabla}\sigma)\right) = (-1)^r(r+1)c\overset{\star}{S}_{r+1}\ . \tag{3.46}$$

As for σ, the function $\lambda=\langle\zeta,\zeta\rangle$ is (screen) leafwise constant and $\overline{\nabla}\lambda = 2\sigma\zeta$. Therefore,

$$\widetilde{\nabla}\lambda = 2\sigma\xi. \tag{3.47}$$

Hence, for all $U\in\mathfrak{X}(M)$,

$$\widetilde{\nabla}_U\widetilde{\nabla}\lambda = -2c\omega(U)\xi - 2\sigma\overset{\star}{A}U, \tag{3.48}$$

and

$$\widetilde{L}_r\lambda = \mathrm{tr}\left(\overset{\star}{T}_r\ (\widetilde{\nabla}_U\widetilde{\nabla}\lambda)\right) = (-1)^{r+1}\left(2c\overset{\star}{S}_r + 2(r+1)\sigma\overset{\star}{S}_{r+1}\right). \tag{3.49}$$

Following the same steps as above for the function $\nu=\langle\xi,a\rangle$, we establish the following.

Lemma 3.6. *Let $a \in \mathbb{R}^m_t$ be a fixed constant vector and $U \in \mathfrak{X}(M)$. Then,*

$$\widetilde{\nabla}\langle \zeta, a\rangle = \sigma a^T + \langle \sigma(\xi - N) - c\psi, a\rangle\, \xi, \tag{3.50}$$

or equivalently

$$\widetilde{\nabla}\langle \zeta, a\rangle = \sigma a + \langle \sigma(\xi - N) - c\psi, a\rangle\, \xi - \sigma\langle \xi, a\rangle N - c\sigma\langle a, \psi\rangle\psi; \tag{3.51}$$

$$\begin{aligned}\widetilde{\nabla}_U\widetilde{\nabla}\langle \zeta, a\rangle &= -c\omega(U)a^T - \sigma\left(\sigma\langle a, \xi\rangle + c\langle a, \psi\rangle\right) PU \\ &\quad - [(\lambda+1)\sigma\langle a, \xi\rangle - \sigma\langle a, \zeta\rangle - c\langle a, \psi\rangle]\, \overset{\star}{A}\, U \\ &\quad -c\left\langle \frac{1}{2}(2+\lambda)\omega(U)\xi - \omega(U)\zeta + U, a\right\rangle \xi;\end{aligned}$$

$$\begin{aligned}\widetilde{L}_r\langle \zeta, a\rangle &= (-1)^{r+1}\left[\left((n-r)\sigma^2 + 2c\right) \overset{\star}{S}_r + (r+1)(\lambda+1)\sigma \overset{\star}{S}_{r+1}\right]\langle \xi, a\rangle \\ &\quad + (-1)^{r+1}\left[(n-r)c\sigma \overset{\star}{S}_r - (r+1)c \overset{\star}{S}_{r+1}\right]\langle \psi, a\rangle \\ &\quad + (-1)^r(r+1)\sigma \overset{\star}{S}_{r+1}\langle \zeta, a\rangle;\end{aligned}$$

and

$$\begin{aligned}\widetilde{L}_r\zeta &= (-1)^{r+1}\left[\left((n-r)\sigma^2 + 2c\right) \overset{\star}{S}_r + (r+1)(\lambda+1)\sigma \overset{\star}{S}_{r+1}\right]\xi \\ &\quad + (-1)^{r+1}\left[(n-r)c\sigma \overset{\star}{S}_r - (r+1)c \overset{\star}{S}_{r+1}\right]\psi \\ &\quad + (-1)^r(r+1)\sigma \overset{\star}{S}_{r+1}\, \zeta.\end{aligned} \tag{3.52}$$

Now, we compute $\widetilde{L}^2_r\psi$. Starting from (3.26),

$$\begin{aligned}\widetilde{L}^2_r\langle \psi, a\rangle &= (-1)^r(r+1)\widetilde{L}_r\left(\overset{\star}{S}_{r+1}\langle \zeta, a\rangle\right) + (-1)^{r+1}c(n-r)\widetilde{L}_r\left(\overset{\star}{S}_r\langle \psi, a\rangle\right) \\ &\quad + (-1)^{r+1}(r+1)\widetilde{L}_r\left((\lambda+1)\overset{\star}{S}_{r+1}\langle \xi, a\rangle\right) + (-1)^{r+1}(n-r)\widetilde{L}_r\left(\sigma \overset{\star}{S}_r\langle \xi, a\rangle\right).\end{aligned}$$

We compute each term using Lemma 3.3, (3.33) (3.36), (3.50), (3.52), (3.44), (3.46), (3.47) and (3.49):

$$\begin{aligned}(-1)^r(r+1)\widetilde{L}_r\left(\overset{\star}{S}_{r+1}\langle \zeta, a\rangle\right) &= 2(-1)^r(r+1)\sigma\left\langle \overset{\star}{T}_r\, \widetilde{\nabla}\, \overset{\star}{S}_{r+1}, a\right\rangle \\ &\quad + (r+1)\left[(-1)^r\widetilde{L}_r\, \overset{\star}{S}_{r+1} + (r+1)\sigma\, \overset{\star}{S}{}^2_{r+1}\right]\langle \zeta, a\rangle \\ &\quad - (r+1)\Big[\left((n-r)\sigma^2 + 2c\right)\overset{\star}{S}_r\overset{\star}{S}_{r+1} \\ &\quad + (r+1)(\lambda+1)\sigma\, \overset{\star}{S}{}^2_{r+1}\Big]\langle \xi, a\rangle \\ &\quad - (r+1)\Big[(n-r)c\sigma\, \overset{\star}{S}_r\overset{\star}{S}_{r+1} - (r+1)c\, \overset{\star}{S}{}^2_{r+1}\end{aligned}$$

$$+2c \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_{r+1})\Big]\langle\psi, a\rangle;$$

$$\begin{aligned}(-1)^{r+1}c(n-r)\widetilde{L}_r\left(\overset{\star}{S}_r \langle\psi, a\rangle\right) &= (-1)^{r+1}(n-r)c\Big[\widetilde{L}_r \overset{\star}{S}_r +(-1)^{r+1}(n-r)c \overset{\star}{S}{}_r^2\Big]\langle\psi, a\rangle\\ &+(n-r)c\Big[(n-r)\sigma \overset{\star}{S}{}_r^2 +(r+1)(\lambda+1) \overset{\star}{S}_r\overset{\star}{S}_{r+1}\Big]\langle\xi, a\rangle\\ &-(n-r)(r+1)c \overset{\star}{S}_r\overset{\star}{S}_{r+1} \langle\zeta, a\rangle\\ &+2(-1)^{r+1}(n-r)c\Big\langle \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_r, a\Big\rangle;\end{aligned}$$

$$\begin{aligned}(-1)^{r+1}(r+1)\widetilde{L}_r\left((\lambda+1) \overset{\star}{S}_{r+1} \langle\xi, a\rangle\right) &= (r+1)(\lambda+1) \overset{\star}{S}_{r+1} \Big\langle\widetilde{\nabla} \overset{\star}{S}_{r+1}, a\Big\rangle\\ &+2(-1)^r(r+1)(\lambda+1)\Big\langle\overset{\star}{T}_r \circ\overset{\star}{A}\widetilde{\nabla} \overset{\star}{S}_{r+1}, a\Big\rangle\\ &-(r+1)^2(\lambda+1)c \overset{\star}{S}{}_{r+1}^2 \langle\psi, a\rangle\\ &+\Big[\Big(\frac{1}{2}\lambda(\lambda+1)(r+1) \overset{\star}{S}_1\\ &-(r+1)^2(\lambda+1)\sigma\Big) \overset{\star}{S}{}_{r+1}^2\\ &+\frac{1}{2}\lambda(\lambda+1)(r+1)(r+2) \overset{\star}{S}_{r+1}\overset{\star}{S}_{r+1}\\ &-(r+1)(\lambda+1) \overset{\star}{S}_{r+1} (\xi\cdot \overset{\star}{S}_{r+1})\\ &+(-1)^{r+1}(r+1)(\lambda+1)\widetilde{L}_r \overset{\star}{S}_{r+1}\\ &+2(r+1)c \overset{\star}{S}_r\overset{\star}{S}_{r+1} +2(r+1)^2\sigma \overset{\star}{S}{}_{r+1}^2\\ &-4(r+1)\sigma \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_{r+1})\Big]\langle\xi, a\rangle;\end{aligned}$$

$$\begin{aligned}(-1)^{r+1}(n-r)\widetilde{L}_r\left(\sigma \overset{\star}{S}_r \langle\xi, a\rangle\right) &= (n-r)\sigma \overset{\star}{S}_r \Big\langle\widetilde{\nabla} \overset{\star}{S}_{r+1}, a\Big\rangle\\ &+2(-1)^r(n-r)\sigma\Big\langle\overset{\star}{T}_r\circ \overset{\star}{A} \widetilde{\nabla} \overset{\star}{S}_r, a\Big\rangle\\ &-(n-r)(r+1)\sigma c \overset{\star}{S}_r\overset{\star}{S}_{r+1} \langle\psi, a\rangle\\ &+(n-r)\Big[(-1)^{r+1}\sigma\widetilde{L}_r \overset{\star}{S}_r\\ &+(n-r)\Big(\frac{1}{2}\lambda\sigma \overset{\star}{S}_1 -(r+1)(c+\sigma^2)\Big) \overset{\star}{S}_r\overset{\star}{S}_{r+1}\\ &+2(n-r)c \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_r) + \frac{1}{2}\lambda(n-r)(r+2)\sigma \overset{\star}{S}_r\overset{\star}{S}_{r+2}\\ &-(n-r)\sigma \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_{r+1})\Big]\langle\xi, a\rangle.\end{aligned}$$

Putting all the above together, we get the following.

Proposition 3.3. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}{}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a isometric immersion of a null hypersurface in the Robertson-Walker space $\overline{M}{}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a timelike closed and* *conformal rigging vector field* ζ. *If* $\lambda = \langle \zeta, \zeta \rangle$ *denotes the squared length function of* ζ *and* σ *its conformal factor, Then,*

$$\begin{aligned} \widetilde{L}_r^2 \langle \psi, a \rangle &= \Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + (n-r)\sigma \Big] \Big\langle \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\ &+ 2(-1)^r (r+1)(\lambda+1) \Big\langle (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\ &+ 2(-1)^r (n-r)\sigma \Big\langle (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_r, a \Big\rangle \\ &+ 2(-1)^r (r+1)\sigma \Big\langle \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\ &+ 2(-1)^{r+1} (n-r) c \Big\langle \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_r, a \Big\rangle \\ &+ \Lambda_r^{\xi} \langle \xi, a \rangle + \Lambda_r^{\zeta} \langle \zeta, a \rangle + \Lambda_r^{\psi} \langle \psi, a \rangle \end{aligned} \tag{3.53}$$

for a fixed $a \in \mathbb{R}^m_{1+t}$; *and*

$$\begin{aligned} \widetilde{L}_r^2 \psi &= \Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + (n-r)\sigma \Big] \widetilde{\nabla} \overset{\star}{S}_{r+1} \\ &+ 2(-1)^r (r+1)(\lambda+1) (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_{r+1} \\ &+ 2(-1)^r (n-r)\sigma (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_r \\ &+ 2(-1)^r (r+1)\sigma \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_{r+1} \\ &+ 2(-1)^{r+1} (n-r) c \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_r \\ &+ \Lambda_r^{\xi} \xi + \Lambda_r^{\zeta} \zeta + \Lambda_r^{\psi} \psi; \end{aligned} \tag{3.54}$$

with Λ_r^{ξ}, Λ_r^{ζ} *and* Λ_r^{ψ} *as follows :*

$$\begin{aligned} \Lambda_r^{\xi} &= (-1)^{r+1}(r+1)(\lambda+1) \widetilde{L}_r \overset{\star}{S}_{r+1} + (-1)^{r+1} \sigma (n-r) \widetilde{L}_r \overset{\star}{S}_r \\ &+ (r+1)\lambda \left(\frac{1}{2}(\lambda+1) \overset{\star}{S}_1 - 2(r+1)\sigma \right) \overset{\star}{S}{}^2_{r+1} + c(n-r)^2 \sigma \overset{\star}{S}{}^2_r \\ &+ (n-r) \left(\frac{1}{2} \lambda \sigma \overset{\star}{S}_1 + (r+1)(c\lambda - 2\sigma^2) \right) \overset{\star}{S}_r \overset{\star}{S}_{r+1} \\ &+ \frac{1}{2}(r+1)(r+2)\lambda(\lambda+1) \overset{\star}{S}_{r+1} \overset{\star}{S}_{r+2} \\ &+ \frac{1}{2}(r+2)(n-r)\lambda\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+2} + 2(n-r)c \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_r) \\ &- \Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + \sigma(n+3r+4) \overset{\star}{S}_r \Big] (\xi \cdot \overset{\star}{S}_{r+1}); \end{aligned} \tag{3.55}$$

$$\Lambda_r^{\zeta} = (r+1) \Big[(-1)^r \widetilde{L}_r \overset{\star}{S}_{r+1} + (r+1)\sigma \overset{\star}{S}{}^2_{r+1} - (n-r)c \overset{\star}{S}_r \overset{\star}{S}_{r+1} \Big] \tag{3.56}$$

and

$$\begin{aligned} \Lambda_r^{\psi} &= c \Big[(-1)^{r+1}(n-r) \widetilde{L}_r \overset{\star}{S}_r + (n-r)^2 c \overset{\star}{S}{}^2_r - (r+1)^2 \lambda \overset{\star}{S}{}^2_{r+1} \\ &- 2(r+1)(n-r)\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+1} - 2(r+1) \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_{r+1}) \Big]. \end{aligned} \tag{3.57}$$

Remark 3.2. *Observe that*

$$\widetilde{\nabla}\overset{\star}{S}_r = P\widetilde{\nabla}\overset{\star}{S}_r + (\xi\cdot\overset{\star}{S}_r)\xi, \quad \overset{\star}{T}_r\,\widetilde{\nabla}\overset{\star}{S}_r = P\left[\overset{\star}{T}_r\,\widetilde{\nabla}\overset{\star}{S}_r\right] + (-1)^r\,\overset{\star}{S}_r\,(\xi\cdot\overset{\star}{S}_r)\xi$$

and similar formulas for $\widetilde{\nabla}\overset{\star}{S}_{r+1}$ *and* $\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_{r+1}$. *So we get the following useful equivalent formula for (3.54)*

$$\begin{aligned}\widetilde{L}_r^2\psi &= \left[(r+1)(\lambda+1)\,\overset{\star}{S}_{r+1} + (n-r)\sigma\right]P\widetilde{\nabla}\,\overset{\star}{S}_{r+1}\\ &\quad +2(-1)^r(r+1)(\lambda+1)(\overset{\star}{T}_r\circ\overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_{r+1}\\ &\quad +2(-1)^r(n-r)\sigma(\overset{\star}{T}_r\circ\overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_r\\ &\quad +2(-1)^r(r+1)\sigma P\,\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_{r+1}\\ &\quad +2(-1)^{r+1}(n-r)cP\,\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_r\\ &\quad +\overset{\star}{\Lambda}{}_r^{\xi}\,\xi + \Lambda_r^{\zeta}\zeta + \Lambda_r^{\psi}\psi; \end{aligned} \tag{3.58}$$

with

$$\begin{aligned}\overset{\star}{\Lambda}{}_r^{\xi} &= (-1)^{r+1}(r+1)(\lambda+1)\widetilde{L}_r\overset{\star}{S}_{r+1} + (-1)^{r+1}\sigma(n-r)\widetilde{L}_r\overset{\star}{S}_r\\ &\quad +(r+1)\lambda\left(\frac{1}{2}(\lambda+1)\,\overset{\star}{S}_1 - 2(r+1)\sigma\right)\overset{\star}{S}{}^2_{r+1} + c(n-r)^2\sigma\,\overset{\star}{S}{}^2_r\\ &\quad +(n-r)\left(\frac{1}{2}\lambda\sigma\,\overset{\star}{S}_1 + (r+1)(c\lambda-2\sigma^2)\right)\overset{\star}{S}_r\overset{\star}{S}_{r+1}\\ &\quad +\frac{1}{2}(r+1)(r+2)\lambda(\lambda+1)\,\overset{\star}{S}_{r+1}\overset{\star}{S}_{r+2}\\ &\quad +\frac{1}{2}(r+2)(n-r)\lambda\sigma\,\overset{\star}{S}_r\overset{\star}{S}_{r+2} - 2(r+1)\sigma\,\overset{\star}{S}_r\,(\xi\cdot\overset{\star}{S}_{r+1}). \end{aligned} \tag{3.59}$$

Definition 3.1. *A connected isometric immersion*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_q$$

of a null hypersurface in $\overline{M}_1^{n+2}(c)$ *furnished with a rigging vector field* ζ *is said to be* $\widetilde{L}_r-$*biharmonic if the position vector field* ψ *satisfies the condition* $\widetilde{L}_r^2\psi = 0$.

Remark 3.3. *Based on (3.58),(3.59), (3.56), (3.57) and Theorem 3.1, a* $r-$*maximal null hypersurface*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_q$$

is biharmonic. For this, we fix that proper $\widetilde{L}_r-$*biharmonic null hypersurfaces are* $\widetilde{L}_r-$ *biharmonic, but not* $r-$*maximal.*

4. Examples

Example 4.1 (Null cone torus). *Let $n \geq m \geq 2$ be integers. Consider*

$$M = \{x \in \mathbb{L}^{n+3} \mid -x_0^2 + x_1^2 + \cdots + x_{m+1}^2 = 0, \quad x_{m+2}^2 + \cdots + x_{n+2}^2 = 1\} \cap \{x_0 > 0\}.$$

It is easy to see that $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is a null hypersurface of the De Sitter spacetime $\mathbb{S}_1^{n+2}$ given by the product of the lightcone Λ_0^{m+1} of dimension $m+1$ with the $n-m$ standard sphere $\mathbb{S}^{n-m}$ (a null cone torus). A timelike closed and conformal rigging for M is given by

$$\zeta = \partial_0 + x_0 x,$$

with (null) rigged vector field

$$\xi = -\frac{1}{x_0} \cdot \Big(x_0, x_1, \ldots, x_{m+1}, 0, \ldots, 0\Big).$$

Then the shape operator is

$$\overset{\star}{A} \simeq \left[\begin{array}{c|c|c} 0 & \cdots \quad \cdots & 0 \\ \vdots & \frac{1}{x_0} I_m & 0 \\ \hline \vdots & 0 & 0_{n-m} \\ 0 & & \end{array}\right],$$

and we get that

$$\overset{\star}{H}_r = \begin{cases} \binom{n+1}{r}^{-1} \binom{m}{r} \cdot \dfrac{1}{(x_0)^r} & \text{if} \quad 0 \leq r \leq m \\ \\ 0 & \text{if} \quad m+1 \leq r \leq n+1 \end{cases} \tag{4.60}$$

Based on Remark 3.3, we see that $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is $\widetilde{L}_k-$biharmonic for $m+1 \leq k \leq n+1$.

Example 4.2 (Null cone cylinder). *Let $1 \leq m \leq n-1$ be integers, and*

$$M = \{x \in \mathbb{L}^{n+2} \mid -x_0^2 + x_1^2 + \cdots + x_{m+1}^2 = 0, \quad x_0 > 0\}.$$

This null cone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ is a null hypersurface in $\mathbb{L}^{n+2}$, for which a natural timelike closed and conformal rigging is given by the constant vector field

$$\zeta = \partial_1$$

with corresponding rigged vector field

$$\xi = -\frac{1}{x_0} \cdot \Big(x_0, x_1, \ldots, x_{m+1}, 0, \ldots, 0\Big).$$

Similar computations as in above Example 4.1, show that the hight order mean curvatures are given as in (4.60) and $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ *is* $\widetilde{L}_k-$*biharmonic for* $m+1 \leq k \leq n+1$.

5. Proofs of main results

5.1. **Proof of Theorem 1.1.** The L_r-harmonicity condition reads

$$\begin{aligned} 0 = \widetilde{L}_r\psi &= (-1)^{r+1}\Big[(n-r)\sigma \overset{\star}{S}_r + (r+1)\lambda \overset{\star}{S}_{r+1}\Big]\xi + \Big[(-1)^r(r+1) \overset{\star}{S}_{r+1}\Big]\zeta \\ &\quad + (-1)^{r+1}\Big[c(n-r) \overset{\star}{S}_r\Big]\psi. \end{aligned}$$

This is equivalent to

$$\overset{\star}{S}_{r+1} = 0, \quad \sigma \overset{\star}{S}_r = 0 \quad \text{and} \quad c \overset{\star}{S}_r = 0.$$

Obviously, due to Theorem 3.1, if $\overset{\star}{S}_r = 0$ the above system is satisfied. Assume $\overset{\star}{S}_r \neq 0$. Then, $\overset{\star}{S}_{r+1} = 0$ and $\sigma = 0$ and the latter implies $c = 0$ due to (3.43). □

5.2. **Proof of Theorem 1.2.** We prove cases $n = 1$ and $n = 2$ separately.

- Case $n = 1$.

  From (3.54) with $n = 1$ and $r = 0$,

$$\widetilde{L}_0^2\psi = \Big[(\lambda+1) \overset{\star}{S}_1 + 3\sigma\Big] P\widetilde{\nabla}\overset{\star}{S}_1 + 2(\lambda+1) \overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_1 + \overset{\star}{\Lambda}{}_0^{\xi}\, \xi + \Lambda_0^{\zeta}\zeta + \Lambda_0^{\psi}\psi,$$

with

$$\begin{aligned} \overset{\star}{\Lambda}{}_0^{\xi} &= -(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \frac{\lambda}{2}\Big[(\lambda+1) \overset{\star}{S}_1 - 3\sigma\Big] \overset{\star}{S}{}_1^2 \\ &\quad + (c\lambda - 2\sigma^2) \overset{\star}{S}_1 - 2\sigma(\xi \cdot \overset{\star}{S}_1) + c\sigma, \end{aligned} \tag{5.61}$$

$$\Lambda_0^{\zeta} = \widetilde{\Delta} \overset{\star}{S}_1 + \sigma \overset{\star}{S}{}_1^2 - c \overset{\star}{S}_1 \tag{5.62}$$

and

$$\Lambda_0^{\psi} = c\Big[c - \lambda \overset{\star}{S}{}_1^2 - 2\sigma \overset{\star}{S}_1 - 2(\xi \cdot \overset{\star}{S}_1)\Big] \tag{5.63}$$

where we used $\overset{\star}{S}_2 = 0$. Therefore, the condition $\widetilde{L}_0^2\psi = 0$ is equivalent to

$$\overset{\star}{A} P\widetilde{\nabla}\overset{\star}{S}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 3\sigma}{2(\lambda+1)} P\widetilde{\nabla}\overset{\star}{S}_1 \tag{5.64}$$

$$\widetilde{\Delta}\overset{\star}{S}_1 + \sigma\overset{\star}{S}{}_1^2 - c\overset{\star}{S}_1 = 0 \tag{5.65}$$

$$c\Big[c - \lambda\overset{\star}{S}{}_1^2 - 2\sigma\overset{\star}{S}_1 - 2(\xi\cdot\overset{\star}{S}_1)\Big] = 0 \tag{5.66}$$

$$-(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \frac{\lambda}{2}\Big[(\lambda+1)\overset{\star}{S}_1 - 3\sigma\Big]\overset{\star}{S}{}_1^2 + (c\lambda - 2\sigma^2)\overset{\star}{S}_1 - 2\sigma(\xi\cdot\overset{\star}{S}_1) + c\sigma = 0. \tag{5.67}$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_1 \neq 0$. Then, we see that $P\widetilde{\nabla}\overset{\star}{S}_1$ is an eigenvector field of $\overset{\star}{A}$ with eigenfunction (a screen principal curvature)

$$\overset{\star}{k} = -\frac{(\lambda+1)\overset{\star}{S}_1 + 3\sigma}{2(\lambda+1)}.$$

Since the null surface M is $2-$dimensional, it follows that $\overset{\star}{k} = 0$ or $\overset{\star}{k} = \overset{\star}{S}_1$. But each of the two cases implies $\overset{\star}{S}_1 = \overset{\star}{S}_1(\sigma,\lambda)$ which leads to a condradiction since σ and λ are leafwise constant. We conclude that $P\widetilde{\nabla}\overset{\star}{S}_1 = 0$ and $\overset{\star}{S}_1$ is leafwise constant. Observe that by the Raychaudhuri equation (2), if $\overset{\star}{S}_1$ is constant on the whole M, this constant is zero. But the case $c \neq 0$ implies $\overset{\star}{S}_1 \neq 0$. Indeed, $\overset{\star}{S}_1 = 0$ in (5.67) leads to $\sigma = 0$ on M and $c = -\xi\cdot\sigma = 0$ which is a contradiction. Hence, for $c \neq 0$, $\overset{\star}{S}_1$ is not constant on the whole M. To go further, let (ξ, X) be a local $\widetilde{g}-$orthonormal basis of M. Since $\widetilde{\nabla}\overset{\star}{S}_1 = (\xi\cdot\overset{\star}{S}_1)\xi = \overset{\star}{S}{}_1^2\,\xi$ we get

$$\widetilde{\Delta}\overset{\star}{S}_1 = \widetilde{g}(\widetilde{\nabla}_\xi\widetilde{\nabla}\overset{\star}{S}_1, \xi) + \widetilde{g}(\widetilde{\nabla}_X\widetilde{\nabla}\overset{\star}{S}_1, X = \widetilde{g}(\widetilde{\nabla}_\xi(\overset{\star}{S}{}_1^2\,\xi), \xi) + \widetilde{g}(\widetilde{\nabla}_X(\overset{\star}{S}{}_1^2\,\xi), X) = \overset{\star}{S}{}_1^3. \tag{5.68}$$

Consider the case where $c = 0$ and assume $\overset{\star}{S}_1 \neq 0$. From (5.65) and (5.68) we get $(\overset{\star}{S}_1 + \sigma)\overset{\star}{S}{}_1^2 = 0$. Therefore $\overset{\star}{S}_1 = -\sigma$. Then we get

$$\sigma^2 = \overset{\star}{S}{}_1^2 = \xi\cdot\overset{\star}{S}_1 = -\xi\cdot\sigma = c = 0.$$

Therefore, $\sigma = 0$ on M and $\overset{\star}{S}_1 = -\sigma = 0$ which is a contradiction.

- Case $n = 2$.

With $r = 0$, equation (3.54) reads

$$\widetilde{L}_0^2\psi = \Big[(\lambda+1)\overset{\star}{S}_1 + 4\sigma\Big]P\widetilde{\nabla}\overset{\star}{S}_1 + 2(\lambda+1)\overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_1 + \overset{\star}{\Lambda}{}_0^{\xi}\,\xi + \Lambda_0^\zeta\zeta + \Lambda_0^\psi\psi, \tag{5.69}$$

with

$$\overset{\star}{\Lambda}{}_0^{\xi} = -(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \lambda\Big[\frac{1}{2}(\lambda+1)\overset{\star}{S}_1 - 2\sigma\Big]\overset{\star}{S}{}_1^2 + 2\Big[\frac{1}{2}\lambda\sigma\overset{\star}{S}_1 + c\lambda - 2\sigma^2\Big]\overset{\star}{S}_1 + \lambda(\lambda+1)\overset{\star}{S}_1\overset{\star}{S}_2 + 2\lambda\sigma\overset{\star}{S}_2 + 4c\sigma - 2\sigma(\xi\cdot\overset{\star}{S}_1). \tag{5.70}$$

$$\Lambda_0^{\zeta} = \widetilde{\Delta}\overset{\star}{S}_1 + \sigma\overset{\star}{S}{}_1^2 - 2c\overset{\star}{S}_1, \tag{5.71}$$

and

$$\Lambda_0^{\psi} = c\Big[4c - \lambda\overset{\star}{S}{}_1^2 - 4\sigma\overset{\star}{S}_1 - 2(\xi\cdot\overset{\star}{S}_1)\Big]. \tag{5.72}$$

Therefore, the biharmonicity condition amounts to

$$\overset{\star}{A}P\widetilde{\nabla}\overset{\star}{S}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)}P\widetilde{\nabla}\overset{\star}{S}_1, \quad \overset{\star}{\Lambda}{}_0^{\xi} = 0, \quad \Lambda_0^{\zeta} = 0 \quad \text{and} \quad \Lambda_0^{\psi} = 0. \tag{5.73}$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_1 \neq 0$. Then we see from the first equation in (5.73) that

$$\overset{\star}{k}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)}$$

is a screen principal curvature. Also, it is easy to see that the screen shape operator is (with $\overset{\star}{k}_0 = 0$),

$$\overset{\star}{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overset{\star}{k}_1 & 0 \\ 0 & 0 & \overset{\star}{k}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)} & 0 \\ 0 & 0 & \frac{3(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)} \end{bmatrix}.$$

From Raychaudury equation (2) and due to $\tau(\xi) = 0$ and $\overline{Ric}(\xi,\xi) = 0$, we have

$$\xi\cdot\overset{\star}{S}_1 = \frac{1}{2(\lambda+1)^2}\Big[5(\lambda+1)^2\overset{\star}{S}{}_1^2 + 16(\lambda+1)\sigma\overset{\star}{S}_1 + 16\sigma^2\Big]. \tag{5.74}$$

Now, we treat the cases $c = 0$ and $c \neq 0$ separately.

Assume $c \neq 0$. Eq. (5.74) in the last equation in (5.73) yields

$$(\lambda+5)\overset{\star}{S}{}_1^2+\frac{4\sigma(\lambda+5)}{\lambda+1}\overset{\star}{S}_1+\frac{16\sigma^2}{(\lambda+1)^2}-4c=0.$$

But, $\lambda+5\not\equiv 0$, otherwise we get $c=0$ from (3.43) and (3.47) which is a contradiction. So, $\overset{\star}{S}_1=\overset{\star}{S}_1(\lambda,\sigma)$. Therefore, since λ and σ are (screen) leafwise constant, the same is for $\overset{\star}{S}_1$ and we get $P\widetilde{\nabla}\overset{\star}{S}_1=0$ which is a contradiction.

Assume now that $c=0$. It follows from the third equation in (5.73) that

$$\widetilde{\Delta}\overset{\star}{S}_1=-\sigma\overset{\star}{S}{}_1^2. \tag{5.75}$$

Also,

$$\overset{\star}{S}_2=\frac{1}{2}\left(\overset{\star}{S}{}_1^2-\xi\cdot\overset{\star}{S}_1\right)\overset{(5.74)}{=}-\frac{3}{4}\overset{\star}{S}{}_1^2-\frac{4\sigma}{\lambda+1}\overset{\star}{S}_1-\frac{4\sigma^2}{(\lambda+1)^2}. \tag{5.76}$$

Therefore, by replacing the expressions (5.75), (5.76) and (5.74) in the second equation in (5.73) we get

$$\frac{1}{4}\lambda(\lambda+1)\overset{\star}{S}{}_1^3+\frac{1}{2}(11\lambda+8)\sigma\overset{\star}{S}{}_1^2+\frac{4\sigma^2}{\lambda+1}(4\lambda+5)\overset{\star}{S}_1+\frac{8\sigma^3}{(\lambda+1)^2}(\lambda^2+2\lambda+2)=0$$

which is polynomial in $\overset{\star}{S}_1$ with degree 3 since $\lambda(\lambda+1)\neq 0$. Therefore, $\overset{\star}{S}_1=\overset{\star}{S}_1(\lambda,\sigma)$ which implies again a contradiction $P\widetilde{\nabla}\overset{\star}{S}_1=0$ since λ and σ are (screen) leafwise constant. Finally, we conclude that $P\widetilde{\nabla}\overset{\star}{S}_1=0$ and $\overset{\star}{S}_1$ is (screen) leafwise constant. Now we are interested in knowing whether $\overset{\star}{S}_1$ can be globally constant over the whole hypersurface M, in which case this constant would necessarily be zero. For this, observe that due to (5.72) and the last equation in (5.73), $c\neq 0$ implies $\|\overset{\star}{A}\|^2=\xi\cdot\overset{\star}{S}_1\neq 0$ and the answer is negative. It remains to analyze the case where $c=0$. Use (5.71) and the third equation in (5.73) to get

$$\widetilde{\Delta}\overset{\star}{S}_1=-\sigma\overset{\star}{S}{}_1^2. \tag{5.77}$$

Also, $0=c=-\xi\cdot\sigma$ and being leafwise constant, we see that σ restricts to a constant over the whole M. Assume this constant to be zero. From the second equation in (5.73) and (5.70) we get

$$\overset{\star}{S}_1(\overset{\star}{S}{}_1^2+2\overset{\star}{S}_2)=0. \tag{5.78}$$

In this relation, assume $\overset{\star}{S}_2\neq 0$, then by Theorem 3.1, $\overset{\star}{S}_1\neq 0$ and we get

$$\frac{1}{2}\left(\overset{\star}{S}{}_1^2-\xi\cdot\overset{\star}{S}_1\right)=\overset{\star}{S}_2=-\frac{1}{2}\overset{\star}{S}{}_1^2,$$

i.e $\xi\cdot \overset{\star}{S}_1 = 2\overset{\star}{S}{}_1^2$. Before we go further, we note the following. Choose a local $\widetilde{g}$−orthonormal frame (X_0, X_1, X_2) consisting of eigenvectors of $\overset{\star}{A}$ such that $X_0 = \xi$ and $X_1, X_2 \in \Gamma(\mathcal{S})$. Then, by a straightforward computation, $\widetilde{\Delta}\overset{\star}{S}_1 = \xi\cdot(\xi\cdot \overset{\star}{S}_1) - (\xi\cdot \overset{\star}{S}_1)\overset{\star}{S}_1$. Therefore, $0 = \widetilde{\Delta}\overset{\star}{S}_1 = 4\overset{\star}{S}{}_1^3 - 2\overset{\star}{S}{}_1^3$, thus, $\overset{\star}{S}_1 = 0$: a contradiction. So, in (5.78), we have $\overset{\star}{S}_2 = 0$ and consequently $\overset{\star}{S}_1 = 0$. Now, assume that σ restricts on M to a non zero constant. Substituting (5.77) and $\overset{\star}{S}_2$ in the second equation in (5.73) yields

$$\lambda(\lambda+1)\overset{\star}{S}{}_1^3 + (\lambda+1)\sigma\overset{\star}{S}{}_1^2 - 4\sigma^2\overset{\star}{S}_1 - \left[\lambda(\lambda+1)\overset{\star}{S}_1 + (\lambda+2)\sigma\right](\xi\cdot\overset{\star}{S}_1) = 0.$$

Taking again derivative with respect to ξ both side leads to

$$\begin{aligned}\lambda(\lambda+1)(\xi\cdot\overset{\star}{S}_1)^2 + \left[-2\lambda(\lambda+1)\overset{\star}{S}{}_1^2 + (3\lambda+2)\sigma\overset{\star}{S}_1 + 6\sigma^2\right](\xi\cdot\overset{\star}{S}_1)&\\ - \left[(5\lambda+2)\overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma\right]\sigma\overset{\star}{S}{}_1^2 = 0&\end{aligned} \tag{5.79}$$

Observe that since $\xi\cdot\overset{\star}{S}_1 = 0$ implies $\overset{\star}{S}_1 = 0$, we infer that $\xi\cdot\overset{\star}{S}_1$ is solution of Eq. (5.79). Consequently, we have

$$\xi\cdot\overset{\star}{S}_1 = 0 \quad \text{or} \quad \left\{\begin{array}{l}\xi\cdot\overset{\star}{S}_1 = \dfrac{2\lambda(\lambda+1)\overset{\star}{S}{}_1^2 - (3\lambda+2)\sigma\overset{\star}{S}_1 - 6\sigma^2}{\lambda(\lambda+1)}\\ \left[(5\lambda+2)\overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma\right]\sigma\overset{\star}{S}{}_1^2 = 0\end{array}\right. . \tag{5.80}$$

Observe that $\overset{\star}{S}_1 = 0$ is incompatible with the second system in (5.80) as it implies $\sigma = 0$ which is a contradiction. So, for this system, $\overset{\star}{S}_1 \neq 0$ and we get

$$(5\lambda+2)\overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma = 0.$$

But $5\lambda + 2 \not\equiv 0$, otherwise $2\sigma = \xi\cdot\lambda = 0$ and $\sigma = 0$, a contradiction. Therefore,

$$\overset{\star}{S}_1 = -\frac{\lambda^2 + \sigma\lambda + 4\sigma}{5\lambda+2}, \tag{5.81}$$

from which we get

$$\xi\cdot\overset{\star}{S}_1 = \frac{-2\sigma}{(5\lambda+2)^2}\left[5\lambda^2 + 4\lambda - 18\sigma\right]. \tag{5.82}$$

Replacing (5.81) in the first equation of the system in (5.80) yields

$$\xi\cdot\overset{\star}{S}_1 = 2\left(\frac{\lambda^2+\sigma\lambda+4\sigma}{5\lambda+2}\right)^2 + \frac{3\lambda+2}{\lambda(\lambda+1)}\frac{\lambda^2+\sigma\lambda+4\sigma}{5\lambda+2}\sigma - \frac{6\sigma^2}{\lambda(\lambda+1)}. \tag{5.83}$$

From (5.82) and (5.83) we see that $\lambda=\lambda(\sigma)=$ const i.e $0=\xi\cdot\lambda=2\sigma$ i.e $\sigma=0$ and this is a contradiction. So the second expression of $\xi\cdot\overset{\star}{S}_1$ is not admissible and we conclude that $\xi\cdot\overset{\star}{S}_1=0$ is the only one solution, and this implies $\overset{\star}{S}_1=0$ and the proof is complete. □

5.3. **Proof of Theorem 1.3.** From (3.54) with $n=1$ and $r=1$,

$$\widetilde{L}_1^2\psi=2(\lambda+1)\overset{\star}{S}_2 P\widetilde{\nabla}\overset{\star}{S}_2-4(\lambda+1)(\overset{\star}{T}_1\circ\overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_2-4\sigma P\overset{\star}{T}_1\widetilde{\nabla}\overset{\star}{S}_2+\overset{\star\xi}{\Lambda_1}\xi+\Lambda_1^{\zeta}\zeta+\Lambda_1^{\psi}\psi, \quad (5.84)$$

with

$$\overset{\star\xi}{\Lambda_1}=2(\lambda+1)\widetilde{L}_1\overset{\star}{S}_2+2\lambda\Big(\frac{1}{2}(\lambda+1)\overset{\star}{S}_1-4\sigma\Big)\overset{\star 2}{S_2}+3\lambda(\lambda+1)\overset{\star}{S}_2\overset{\star}{S}_3-4\sigma\overset{\star}{S}_1(\xi\cdot\overset{\star}{S}_2)$$

$$\Lambda_1^{\zeta}=2\Big[-\widetilde{L}_1\overset{\star}{S}_2+2\sigma\overset{\star}{S}_2\Big],\quad \text{and } \Lambda_1^{\psi}=-4c\Big[\lambda\overset{\star 2}{S_2}+\overset{\star}{S}_1(\xi\cdot\overset{\star}{S}_2)\Big].$$

But for the null surface M^2, we have $\overset{\star}{S}_2=\overset{\star}{k}_0\overset{\star}{k}_1=0$. So, $\widetilde{L}_1^2\psi=0$ and M^2 is $\widetilde{L}_1-$biharmonic and item (1) is proved.

Let $n=2$ and $r=1$ in (3.54). We treat separately the cases $\sigma=0$ and $\sigma\neq0$.

- For $\sigma=0$ we see that $c=0$ and $\lambda=$ cste. So,

$$\widetilde{L}_1^2\psi=2(\lambda+1)\overset{\star}{S}_2 P\widetilde{\nabla}\overset{\star}{S}_2-4(\lambda+1)(-\overset{\star}{S}_1\overset{\star}{A}+\overset{\star 2}{A})P\widetilde{\nabla}\overset{\star}{S}_2+\overset{\star\xi}{\Lambda_1}\xi+\Lambda_1^{\zeta}\zeta+\Lambda_1^{\psi}\psi, \quad (5.85)$$

with

$$\overset{\star\xi}{\Lambda_1}=\lambda(\lambda+1)\overset{\star}{S}_1\overset{\star}{S}_2,\quad \Lambda_1^{\zeta}=\widetilde{L}_1\overset{\star}{S}_2\quad\text{and}\quad\Lambda_1^{\psi}=0, \quad (5.86)$$

where we used $\overset{\star}{S}_3=0$. From the $\widetilde{L}_1-$biharmonicity condition, the first equality in (5.86) yields $\overset{\star}{S}_1\overset{\star}{S}_2=0$ which implies $\overset{\star}{S}_2=0$. Indeed, if $\overset{\star}{S}_1\neq0$ then $\overset{\star}{S}_2=0$. Now, by Theorem 3.1 $\overset{\star}{S}_1=0$ implies $\overset{\star}{S}_2=0$.

- For $\sigma\neq0$,

$$\begin{aligned}\widetilde{L}_1^2\psi &= \Big[2(\lambda+1)\overset{\star}{S}_2+4\sigma\overset{\star}{S}_1+\sigma\Big]P\widetilde{\nabla}\overset{\star}{S}_2+4(\lambda+1)\left[\left(\overset{\star}{S}_1-\frac{\sigma}{\lambda+1}\right)\overset{\star}{A}-\overset{\star 2}{A}\right]P\widetilde{\nabla}\overset{\star}{S}_2\\ &\quad -2c\overset{\star}{S}_1 P\widetilde{\nabla}\overset{\star}{S}_1+2\sigma\left[\left(\overset{\star}{S}_1+\frac{c}{\sigma}\right)\overset{\star}{A}-\overset{\star 2}{A}\right]P\widetilde{\nabla}\overset{\star}{S}_1+\overset{\star\xi}{\Lambda_1}\xi+\Lambda_1^{\zeta}\zeta+\Lambda_1^{\psi}\psi.\end{aligned}$$

Assume $\overset{\star}{S}_1$ is $\mathscr{F}-$leafwise constant. Set

$$\overset{\star}{D}=\left(\overset{\star}{S}_1-\frac{\sigma}{\lambda+1}\right)\overset{\star}{A}-\overset{\star 2}{A}.$$

The $\widetilde{L}_1-$bihamonicity condition implies

$$\overset{\star}{D}\, P\widetilde{\nabla}\overset{\star}{S}_2 = -\left[\frac{1}{2}\overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\overset{\star}{S}_1 + 1)\right] P\widetilde{\nabla}\overset{\star}{S}_2\,. \tag{5.87}$$

Observe that ξ is also an eigenvector field of $\overset{\star}{D}$ associated to the eigenvalue $\overset{\star}{\lambda}_0 = 0$. Also, $\overset{\star}{D}$ is diagonalizable and

$$\mathrm{trace}(\overset{\star}{D}) = 2\overset{\star}{S}_2 - \frac{\sigma}{\lambda+1}\overset{\star}{S}_1\,.$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_2 \neq 0$. It follows from (5.87) that

$$\overset{\star}{\lambda}_1 = -\left[\frac{1}{2}\overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\overset{\star}{S}_1 + 1)\right]$$

is an eigenfunction for $\overset{\star}{D}$. Observe that $\overset{\star}{\lambda}_1 \neq 0$. Otherwise, $\overset{\star}{S}_2 = \dfrac{-\sigma}{2(\lambda+1)}(4\overset{\star}{S}_1 + 1)$ which implies $P\widetilde{\nabla}\overset{\star}{S}_2 = 0$ and this is a contradiction. We find that the third eigenfunction of $\overset{\star}{D}$ is

$$\overset{\star}{\lambda}_2 = \mathrm{trace}(\overset{\star}{D}) - \overset{\star}{\lambda}_1 = \frac{5}{2}\overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}.$$

Without losing generality we can choose a local $\widetilde{g}-$orthonormal frame field consisting of eigenvector fields of $\overset{\star}{D}$ such that

$$X_0 = \xi, \quad X_1 = \frac{P\widetilde{\nabla}\overset{\star}{S}_2}{\left\|P\widetilde{\nabla}\overset{\star}{S}_2\right\|} \in \Gamma(\mathcal{S}) \quad \text{and} \quad X_2 \in \Gamma(\mathcal{S}).$$

In this local frame, $\overset{\star}{D}$ takes the form

$$\overset{\star}{D} = \begin{bmatrix} \overset{\star}{\lambda}_0 & 0 & 0 \\ 0 & \overset{\star}{\lambda}_1 & 0 \\ 0 & 0 & \overset{\star}{\lambda}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\left[\frac{1}{2}\overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\overset{\star}{S}_1 + 1)\right] & 0 \\ 0 & 0 & \frac{5}{2}\overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)} \end{bmatrix}$$

Taking into account the ξ, ζ and ψ components we also derive the following equations :

$$\begin{aligned} \overset{\star}{\Lambda}{}_1^{\xi} &= 2(\lambda+1)\widetilde{L}_1\overset{\star}{S}_2 + \sigma\widetilde{L}_1\overset{\star}{S}_1 + 2\lambda\left[\frac{1}{2}(\lambda+1)\overset{\star}{S}_1 - 4\sigma\right]\overset{\star}{S}_2 + c\sigma\overset{\star}{S}{}_1^2 \\ &+ \left[\frac{1}{2}\lambda\sigma\overset{\star}{S}_1 + 2c(c\lambda - 2\sigma^2)\right]\overset{\star}{S}_1\overset{\star}{S}_2 - 4\sigma\overset{\star}{S}_1(\xi\cdot\overset{\star}{S}_2) = 0; \end{aligned} \tag{5.88}$$

$$2\Lambda_1^{\zeta} = -\widetilde{L}_1 \overset{\star}{S}_2 + 2\sigma \overset{\star}{S}{}_2^2 - c \overset{\star}{S}_1 \overset{\star}{S}_2 = 0; \tag{5.89}$$

$$\Lambda_1^{\psi} = c\left[\widetilde{L}_1 \overset{\star}{S}_1 + c \overset{\star}{S}_1 + c \overset{\star}{S}_1 - 4\lambda \overset{\star}{S}{}_2^2 - 4\sigma \overset{\star}{S}_1 \overset{\star}{S}_2 - 4 \overset{\star}{S}_1 (\xi\cdot \overset{\star}{S}_2)\right] = 0. \tag{5.90}$$

Observe that since $\overset{\star}{S}_3 = 0$, we have $\xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2$. Let us compute $\widetilde{L}_1 \overset{\star}{S}_1$.

$$\widetilde{L}_1 \overset{\star}{S}_1 = \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{\xi} \widetilde{\nabla} \overset{\star}{S}_1, \xi\right) + \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X_1} \widetilde{\nabla} \overset{\star}{S}_1, X_1\right) + \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X2} \widetilde{\nabla} \overset{\star}{S}_1, X_2\right)$$

where $\widetilde{\nabla} \overset{\star}{S}_1 = (\xi\cdot \overset{\star}{S}_1)\xi = (\overset{\star}{S}{}_1^2 - 2 \overset{\star}{S}_2)\xi$. Computing each term leads to

$$\widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{\xi} \widetilde{\nabla} \overset{\star}{S}_1, \xi\right) = -2 \overset{\star}{S}{}_1^4 + 6 \overset{\star}{S}{}_1^2 \overset{\star}{S}_2;$$

$$\widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X_1} \widetilde{\nabla} \overset{\star}{S}_1, X_1\right) = \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X2} \widetilde{\nabla} \overset{\star}{S}_1, X_2\right) = \overset{\star}{S}_2 (\overset{\star}{S}{}_1^2 - 2 \overset{\star}{S}_2).$$

So,

$$\widetilde{L}_1 \overset{\star}{S}_1 = 8 \overset{\star}{S}{}_1^2 \overset{\star}{S}_2 - 2 \overset{\star}{S}{}_1^4 - 4 \overset{\star}{S}{}_2^2 . \tag{5.91}$$

Assume $c \neq 0$. From (5.90) and (5.91),

$$-4(\lambda+1) \overset{\star}{S}{}_2^2 + [4 \overset{\star}{S}{}_1^2 - 4\sigma \overset{\star}{S}_1] \overset{\star}{S}_2 - 2 \overset{\star}{S}{}_1^4 + c \overset{\star}{S}_1 = 0.$$

Hence, since $\lambda + 1 \neq 0$ we see that $\overset{\star}{S}_2 = \overset{\star}{S}_2 (\overset{\star}{S}_1, \lambda, \sigma)$ and this implies $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ which is a contradiction.

Assume $c = 0$. We get from (5.89), $\widetilde{L}_1 \overset{\star}{S}_2 = 2\sigma \overset{\star}{S}{}_2^2$ with σ constant on M. Using (5.88), we derive

$$4\lambda \overset{\star}{S}{}_2^2 + \left[(4 + \frac{1}{2}\lambda)\sigma \overset{\star}{S}{}_1^2 + (\lambda^2 + \lambda - 4\sigma^2) \overset{\star}{S}_1 - 4\sigma\lambda\right] \overset{\star}{S}_2 - 2\sigma \overset{\star}{S}{}_1^4 = 0.$$

But $\lambda < 0$ since ζ is timelike. So, $\overset{\star}{S}_2 = \overset{\star}{S}_2 (\overset{\star}{S}_1, \lambda, \sigma)$ and this implies $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ which is again a contradiction.

Finally, we conclude that $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ i.e $\overset{\star}{S}_2$ is leafwise constant in the screen foliation $\mathscr{F}$.

Assume that $\overset{\star}{S}_2$ and hence $\overset{\star}{H}_2$ is constant on the whole null hypersurface M^3. Then $0 = \xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2$ and this implies again $\overset{\star}{S}_2 = 0$ as shown in previous argument above. □

Discussion. Consider the case where the rigging is a unit timelike vector field, i.e $\lambda = \langle\zeta, \zeta\rangle = -1$. Due to $\widetilde{\nabla}\lambda = 2\sigma\xi$ and $\xi \cdot \sigma = -c$, we get $\sigma = 0$ on the null hypersurface M and $c = 0$. Hence, when the rigging ζ is a timelike unit closed and conformal vector field, the target space of immersion is necessarily Minkowskian, and ζ is a Killing vector field in a neighbourhood of the null hypersurface. Moreover,

$$\widetilde{L}_r^2\psi = \left[(-1)^r(r+1)\widetilde{L}_r \overset{\star}{S}_{r+1}\right]\zeta.$$

Consequently, the null hypersurface connected isometric immersion $\psi : M^{n+1} \longrightarrow \mathbb{R}_1^{n+2}$ furnished with a timelike unit closed and conformal vecor field (a Killing rigging) ζ is $r-$biharmonic if and only if $\widetilde{L}_r\overset{\star}{S}_{r+1}= 0$.

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