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EXISTENCE OF GLOBAL ATTRACTOR FOR A MODEL OF SUSPENSION BRIDGE

FATIMA ZOHRA MAHDI, ALI HAKEM*, AND MOHAMED FERHAT

Abstract. The goal of this paper is to establish a well-posedness result and the existence of a finite-dimensional global attractor for the following model of a suspension bridge equations:

$$u_{tt} + \Delta^2 u - \Delta u + u_t - \int_0^\infty \mu(s)\Delta^2 u(x,y,t-s)\, ds + h(u) = f, \quad \text{in} \quad \Omega \times \mathbb{R}^+.$$

Furthermore, the regularity of global attractor is achieved. This results extend previous works

1. Introduction

From the physics point of view, the suspension bridge equation describes the transverse deflection of the roadbed in the vertical plane. The suspension bridge equations were presented by A.C. Lazer and P.J. McKenna[1] as new problems in the field of nonlinear analysis. Lately, similar models have been studied by many authors, but most of them have only concentrated on the existence of solutions, (see for instance [7,8] and the references therein), while the existence of the global attractors for the suspension bridge equations are most of our concern.

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* Corresponding author

Recently, S. Liu and Q. Ma [4] studied the long time dynamical behavior for the following extensible suspension bridge equations with past history

$$u_{tt} + u_t + \Delta^2 u + (\alpha - \beta\|\nabla u\|^2_{L^2(\Omega)})\Delta u - \int_\Omega \mu(s)\Delta^2 u(t-s)ds + ku^+ = g(x) \quad in \quad \Omega \times \mathbb{R}^+.$$

They proved the existence of the global attractors by using the contraction function method and the regularity. We point out here that C.K. Zhong and Q. Ma [9] proved the existence of strong solutions and global attractors for the suspension bridge equations. Related to this subject, we can mention the work of J.Y. Park and J.R. Kang [10,11]. In that papers, they obtained the existence of pullback attractor for the non autonomous suspension bridge equations and the existence of global attractors for the suspension bridge equations with nonlinear damping.

The recent work of A. Ferrero and F. Gazzola [2], suggested a rectangular plate model describing the displacement of a suspension bridge in the downward direction. The plate $\Omega = (0,\pi) \times (-l,l)$ is assumed to be partially hinged on the vertical edges

$$u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, \quad \forall y \in (-l,l),$$

and free on the horizontal edges

$$u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2-\sigma)u_{xxy}(x,\pm l) - u_y(x,\pm l) = 0, \quad \forall x \in (0,\pi).$$

They established the well-posedness and discussed several other stationary problems. We also recall the results by S.A. Messaoudi et al ([5,6]), where the authors investigated the following problem

$$u_{tt} + \Delta^2 u + h(u(x,y,t)) + \delta_1 u_t(x,y,t) + \delta_2 u_t(x,y,t-\tau) = f, \ in \quad \Omega \times (0,\infty),$$

which describes the downward displacement of a suspension bridge in the presence of a hanger restoring force $h(u)$ and external force f which includes gravity and a delay term which accounts for its history. They proved the existence of a finite-dimensional global attractor. For more details on suspension bridge models, we refer the reader to the new Book on mathematical models for suspension bridges by F. Gazzola [3]. Motivated by the previous works, in the present paper we investigate the problem (1.1) in which we contribute to the results obtained in the cited references.

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u + u_t - \int_0^\infty \mu(s)\Delta^2 u(x,y,t-s)ds \\ +h(u) = f, \quad in\ \Omega \times \mathbb{R}^+, \\ u(0,y,t) = u_{xx}(0,y,t) = 0, \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) - u_y(x,\pm l,t) = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u(x,y,t) = u_0(x,y), u_t(x,y,0) = u_1(x,y), \quad in\ \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,\pi) \times (-l,l) \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$. The memory kernel $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is an absolutely continuous function which may possibly blows up at 0.

2. Preliminaries

We present the following conditions about memory kernel

$$(H_1) : \mu \in \mathcal{C}^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu'(s) \le 0 \le \mu(s), \quad \forall s \in \mathbb{R}^+,$$

$$(H_2) : l = 1 - \int_0^\infty \mu(s)ds = 1 - \mu_0 > 0, \quad \forall s \in \mathbb{R}^+,$$

$$(H_3) : \mu'(s) + \delta\mu(s) \le 0, \quad \forall s \in \mathbb{R}^+, \quad \delta > 0.$$

Concerning the forcing term $h : \mathbb{R} \to \mathbb{R}$, we assume that

$$(G_1) : h(0) = 0, \quad \text{and} \quad |h(u) - h(v)| \le K_0 \left(1 + |u|^p + |v|^p\right)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{2.2}$$

where $K_0 > 0$ and $p > 0$. Condition $p > 0$ implies that $H_*^2(\Omega) \hookrightarrow L^{2(p+1)}(\Omega)$. In addition, we assume that

$$-K_1 \le H(u) \le h(u)u, \quad \forall u \in \mathbb{R}. \tag{2.3}$$

As in C.M. Dafermos [12], we introduce the relative displacement past history function as

$$\phi^t(x,y,s) = u(x,y,t) - u(x,y,t-s), \quad (x,y,s) \in \Omega \times \mathbb{R}^+, \quad t \ge 0. \tag{2.4}$$

Notice that ϕ satisfies the equation

$$\phi_t^t + \phi_s^t - u_t = 0, \tag{2.5}$$

with the boundary condition

$$\phi^t(x,y,0) = 0,$$

and the initial condition

$$\phi^0(x,y,s) = u_0(x,y) - u(x,y,-s) := w(s),$$

where w represents the history of u. Consequently, problem (1.1) becomes

$$\begin{cases} u_{tt} + \left(1 - \int_0^\infty \mu(s)ds\right)\Delta^2 u - \Delta u + u_t \\ + \int_0^\infty \mu(s)\Delta^2\phi^t(s)ds + h(u) = f \quad in\ \Omega\times(0,\infty), \\ \phi_t^t + \phi_s^t - u_t = 0, \quad in\ \Omega\times(0,\infty), \end{cases} \tag{2.6}$$

with the boundary conditions

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = 0 \quad for\ (y,t)\in(-l,l)\times\mathbb{R}^+, \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 \quad for\ (y,t)\in(-l,l)\times\mathbb{R}^+, \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) - u_y(x,\pm l,t) = 0, \quad (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ \phi^t(0,y,s) = \phi^t_{xx}(0,y,s) = 0, \quad for\ (y,s)\in(-l,l)\times\mathbb{R}^+, \\ \phi^t(\pi,y,s) = \phi^t_{xx}(\pi,y,s) = 0, \quad for\ (y,s)\in(-l,l)\times\mathbb{R}^+, \\ \phi^t_{yy}(x,\pm l,s) + \sigma\phi^t_{xx}(x,\pm l,s) = 0, \quad (x,s)\in(0,\pi)\times\mathbb{R}^+, \\ \phi^t_{yyy}(x,\pm l,s) + (2-\sigma)\phi^t_{xxy}(x,\pm l,s) - \phi^t_y(x,\pm l,t) = 0, \quad (x,s)\in(0,\pi)\times\mathbb{R}^+, \end{cases} \tag{2.7}$$

and the initial conditions

$$\begin{cases} u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \qquad in \quad \Omega \\ \\ \phi^0(x,y,s) = \phi_0(x,y,s) = u_0(x,y) - u(x,y,-s), \qquad in \quad \Omega\times[0,+\infty). \end{cases} \tag{2.8}$$

We will use the standard functional space and denote $(.,.)$ be a $L^2(\Omega)$- inner product and $\|.\|_p$ be $L^p(\Omega)$ norm. Especially, we take

$$H = V_0 = L^2(\Omega), \quad V = V_1 = H^2_*(\Omega),$$

with

$$H^2_*(\Omega) = \Big\{\xi \in H^2(\Omega), \xi = 0\ on\ \{0,\pi\}\times\{-l,l\}\Big\},$$

equipped with respective inner product and norm,

$$(u,v) = (\Delta u, \Delta v), \quad \|u\|_V = \|\Delta u\|_2.$$

Define

$$D(A) = \Big\{u \in H^4(\Omega) \quad \text{such that} \quad (2.7) \quad \text{holds}\Big\},$$

where $Au = \Delta^2 u$, and equip this space with the inner product (Au, Av), and the norm $\|Au\|_2^2 = (Au, Au)$. We have the following continuous dense injections

$$D(A) \subset V \subset H = H^* \subset V^*,$$

where H^*, V^* are the dual spaces of H, V respectively.

We consider the relative displacement ϕ as a new variable, we introduce the weighted L^2-space

$$L^2_\mu(\mathbb{R}^+, V_i) = \Big\{\phi : \mathbb{R}^+ \longrightarrow V_i \quad \text{such that} \quad \int_0^\infty \mu(s)\|\phi(s)\|^2_{V_i} ds < \infty\Big\},$$

which is a Hilbert space endowed with inner product and norm

$$(u, v)_{\mu,V_i} = \int_0^\infty \mu(r)(u(r), v(r))_{V_i} dr,$$

$$\|u\|^2_{\mu,V_i} = (u, u)_{\mu,V_i} = \int_0^\infty \mu(r)\|u(r)\|^2_{V_i} dr, \quad i = 0, 1, 2,$$

respectively, where $V_2 = D(A^{\frac{3}{4}})$ and $V_3 = D(A)$. Finally, we introduce the following Hilbert spaces

$$\mathcal{H}_0 = V \times H \times L^2_\mu(\mathbb{R}^+; V), \quad \mathcal{H}_1 = D(A) \times V \times L^2_\mu(\mathbb{R}^+; D(A)),$$

equipped with the norms

$$\|u, u_t, \phi\|_{\mathcal{H}_0} = \|\Delta u\|_2^2 + \|u_t\|_2^2 + \|\phi^t\|^2_{\mu,V},$$

and

$$\|u, u_t, \phi\|_{\mathcal{H}_1} = \|\nabla\Delta u\|_2^2 + \|\nabla u_t\| + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}.$$

Using the Poincaré inequality we obtain

$$\lambda_1\|v\|_2^2 \le \|\Delta v\|_2^2, \quad \forall v \in V,$$

where λ_1 denotes the first eigenvalue of $\Delta^2 v = \lambda v$ in Ω.

In order to obtain the global attractors of the problem (2.6)-(2.8), we need the following theorem. The well-posedness of problem (2.6)-(2.8) can be obtained by Faedo-Galerkin method (see[13]) and combining with a prior estimate of 3.1, we omit and only give the following theorem.

Theorem 2.1. *Assume that assumptions $(H_1) - (H_3)$, (G_1) hold and $f \in L^2(\Omega)$. Problem (2.6)-(2.8) has a weak solution $(u, u_t, \phi) \in C([0, T], \mathcal{H}_0)$ with initial data $(u_0, u_1, \phi^0) \in \mathcal{H}_0$, satisfying*

$$u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; H), \quad \phi \in L^\infty(0, T, L^2_\mu(\mathbb{R}^+, V)),$$

and the mapping $\{u_0, u_1, \phi^0\} \to \{u(t), u_t(t), \phi^t\}$ *is continuous in* $\mathcal{H}_0$. *In addition, if* $z^i(t) = (u^i(t), u^i_t(t), \phi^i)$ *is a weak solution of problem (2.6)-(2.8) corresponding to initial data* $z^i(0) = (u^i_0, u^i_1, \phi^i_0)$, $\quad i = 1, 2$, *then one has*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_0} \leq e^{ct}\|z_1(0) - z_2(0)\|_{\mathcal{H}_0}, \quad t \in [0, T],$$

for some constant $c \geq 0$.

The well-posedness of problem (2.6)-(2.8) implies that the family of operator $S(t) : \mathcal{H}_0 \to \mathcal{H}_0$ defined by

$$S(t)(u_0, u_1, \phi^0) = (u(t), u_t(t), \phi^t), \quad t \geq 0,$$

where $(u(t), u_t(t), \phi^t)$ is the unique weak solution of the problem (2.6)-(2.8), satisfies the semigroup properties and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on $\mathcal{H}_0$. Now, we recall some fundamentals of theory of infinite dimensional systems in mathematical physics. These abstract results will be used in our consideration.

Definition 2.1. *A dynamical system* $(\mathcal{H}, S(t))$ *is dissipative if it possesses a bounded absorbing set, that is, a bounded set* $\mathfrak{B} \subset \mathcal{H}$ *such that for any bounded set* $B \subset \mathcal{H}$ *there exists* $t_B \geq 0$ *satisfying*

$$S(t)B \subset \mathfrak{B}, \quad \forall t \geq t_B.$$

Definition 2.2. *Let* X *be Banach space and* B *a bounded subset of* X. *We call a function* $\Phi(.,.)$ *which is defined on* $X \times X$ *a contractive function on* $B \times B$ *if for any sequence* $\{x_n\}_{n=1}^{\infty} \subset B$, *there is a subsequence* $\{x_{nk}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, *such that*

$$\lim_{k\to\infty} \lim_{l\to\infty} \Phi(x_{nk}, x_{nl}) = 0. \tag{2.9}$$

Denote all such contractive functions on $B \times B$ *by* $\mathfrak{C}$.

Definition 2.3. *Let* $\{S(t)\}_{t\geq 0}$ *be a semi-group on a Banach space* $(X, \|.\|)$ *that has a bounded absorbing set* B_0. *Moreover, assume that for* $\epsilon > 0$ *there exist* $T = T(B_0, \epsilon)$ *and* $\Phi_T(.,.) \in \mathfrak{C}(B_0)$ *such that*

$$\|S(T)x - S(T)y\| \leq \epsilon + \Phi_T(x, y), \forall (x, y) \in B_0,$$

where Φ_T *depends on* T. *Then* $\{S(t)\}_{t\geq 0}$ *is asymptotically compact in* X, *i.e , for any bounded sequence* $\{y_n\}_{n=1}^{\infty} \subset X$ *and* $\{t_n\}$ *with* $t_n \to \infty$, $\{S(t_n)y_n\}_{n=1}^{\infty}$ *is precompact in* X.

Theorem 2.2. *[14] A dissipative dynamical system* $(\mathcal{H}, S(t))$ *has a compact global attractor if and only if it is asymptotically smooth.*

Our main result in the following

Theorem 2.3. *Assume that assumptions $(H_1)-(H_3)$ and (G_1) are fulfilled. Let $h \in C^1(\mathbb{R}, \mathbb{R})$ and $f \in L^2(\Omega)$ be given. Then the dynamical system $(\mathcal{H}_0, S(t))$ corresponding to the system $(2.6) - (2.8)$ has a compact global attractor $\mathcal{A} \subset \mathcal{H}_0$, which attracts any bounded set in $\mathcal{H}_0$ with $\|.\|_{\mathcal{H}_0}$.*

3. Global attractor in $\mathcal{H}_0$

In order to prove Theorem 2.3, we will apply the abstract results presented in Section 2. The first step is to show that the dynamical system $(\mathcal{H}_0, S(t))$ is dissipative. The second step is to verify the asymptotic compactness. Then the existence of compact global attractor is guaranteed by Theorem 2.2.

3.1. **A Priori estimates in $\mathcal{H}_0$.** First, taking the scalar product in H of the first equation of (2.6) with $v = u_t + \theta u$, after a computation, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + 2\int_\Omega H(u)dx - 2\int_\Omega fudx\right) + \theta l\|\Delta u\|_2^2 + \theta\|\nabla u\|_2^2 \\ &+(1-\theta)(u_t, v) + (\phi^t, u_t)_{\mu,v} + \theta(\phi^t, u)_{\mu,V} + \theta\int_\Omega h(u)udx - \theta\int_\Omega fudx = 0.\end{aligned} \tag{3.10}$$

Exploiting $(H_1) - (H_3)$ and Hölder inequality, we have

$$(1-\theta)(u_t, v) = (1-\theta)\|v\|_2^2 - \theta(1-\theta)(u, v),$$

$$\begin{aligned}(\phi^t, u_t)_{\mu,V} &= (\phi^t, \phi^t_t + \phi^t_s)_{\mu,V} = \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \int_0^\infty \mu(s)(\phi^t, \phi^t_s(s))_V ds \\ &= \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\|\phi^t\|^2_V ds \\ &= \frac{1}{2}\frac{d}{dt}\|\phi^t\|_{\mu,V} - \frac{1}{2}\int_0^\infty \mu'(s)\|\phi^t(s)\|^2_V ds \\ &\geq \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{\delta}{2}\int_0^\infty \mu(s)\|\phi^t\|^2_V ds = \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{\delta}{2}\|\phi^t\|^2_{\mu,V},\end{aligned} \tag{3.11}$$

$$\theta(\phi^t, u)_{\mu,V} \geq -\frac{\delta}{4}\|\phi^t\|^2_{\mu,V} - \frac{(1-l)\theta^2}{\delta}\|\Delta u\|_2^2.$$

We choose θ small enough, such that

$$1 - \frac{(1-l)\theta}{\delta} - \frac{\theta}{2\lambda_1} \geq 1 - \theta, \quad \frac{1}{2} - \theta \geq \frac{1}{4},$$

then combining with Hölder, Young and Poincar$\acute{e}$ inequalities, we obtain

$$\begin{aligned}
&\theta l\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\theta(1-\theta)(u,v)\\
&\geq \theta\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\frac{\theta}{\sqrt{\lambda_1}}\|\Delta u\|_2\|v\|_2\\
&\geq \theta l\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\left(\frac{\theta^2}{2\lambda_1}\|\Delta u\|_2^2+\frac{1}{2}\|v\|_2^2\right)\\
&=\theta l\left(1-\frac{(1-l)\theta}{\delta l}-\frac{\theta}{2\lambda_1 l}\right)\|\Delta u\|_2^2+\left(\frac{1}{2}-\theta\right)\|v\|_2^2\\
&\geq \theta l(1-\theta)\|\Delta u\|_2^2+\frac{1}{4}\|v\|_2^2.
\end{aligned} \tag{3.12}$$

Collecting with (3.11) and (3.12), there holds

$$\begin{aligned}
&\frac{d}{dt}\left(\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega H(u)dx-2\int_\Omega fudx\right)\\
&+\frac{1}{2}\|v\|_2^2+2\theta l(1-\theta)\|\Delta u\|_2^2+2\theta\|\nabla u\|_2^2+\frac{\delta}{2}\|\phi^t\|_{\mu,V}^2\\
&+2\theta\int_\Omega h(u)udx-2\theta\int_\Omega fudx\leq 0.
\end{aligned} \tag{3.13}$$

Provided that $\theta_0=\min\left\{2\theta l\left((1-\theta)-\frac{1}{4}\right),2\theta,\frac{1}{4},\frac{\delta}{2}\right\}$, let

$$E(t)=\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega H(u)dx-2\int_\Omega fudx, \tag{3.14}$$

and

$$I(t)=\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega h(u)udx-2\int_\Omega fudx. \tag{3.15}$$

We have

$$\frac{d}{dt}E(t)+\theta_0 I(t)\leq 0, \tag{3.16}$$

which implies

$$E(t)\leq -\theta_0\int_0^t I(\tau)d\tau+E(0), \tag{3.17}$$

where

$$E(0)=\|u_1+\theta u_0\|_2^2+l\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+\|\phi^0\|_{\mu,V}^2+2\int_\Omega H(u_0)dx-2\int_\Omega fu_0dx. \tag{3.18}$$

Noticing that (2.3)and (3.14)-(3.15), and using the compact Sobolev embedding theorem we get

$$E(t)\geq \|v\|_2^2+\left(l-\frac{\lambda_1+2\theta_0}{2\lambda_1}\right)\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}-M_1. \tag{3.19}$$

Similarly

$$I(t)\geq \|v\|_2^2+\left(l-\frac{\lambda_1+2\theta_0}{2\lambda_1}\right)\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}-M_1, \tag{3.20}$$

where $M_1 = \dfrac{2}{\lambda_1}\|f\|_2^2 + K_1|\Omega|$. Therefore, let $\dfrac{\lambda_1 + 2\theta_0}{2\lambda_1} < l$, and $0 < \theta_0 < \lambda_1(l - \frac{1}{2})$, we have

$$l - \frac{\lambda_1 + 2\theta_0}{2\lambda_1} > 0. \tag{3.21}$$

Associated with (3.19)-(3.20), there exists a positive constant C_1 such that

$$E(t) \geq C_1\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1, \tag{3.22}$$

$$I(t) \geq C_1\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1. \tag{3.23}$$

So we deduce from (3.22)-(3.23) and (3.17) that

$$\begin{aligned} &C_1\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1 \leq \\ &-\theta_0\int_0^t [C_1\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1]dt + E(0). \end{aligned} \tag{3.24}$$

Thus, for any $\rho_1^2 > \frac{M_1}{C_1}$, there exists $t_0 = t_0(B)$ such that

$$\|v(t_0)\|_2^2 + l\|\Delta u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2 + \|\phi^t(t_0)\|_{\mu,V} \leq \rho_1^2, \tag{3.25}$$

and we end up to.

Lemma 3.1. *Assume that assumptions $(H_1) - (H_3)$ and (G_1) hold and $h \in C(\mathbb{R},\mathbb{R}), f \in L^2(\Omega)$, then the ball of $\mathcal{H}_0$, $B_0 = B_{\mathcal{H}_0}(0,\rho_1)$, centered at 0 of radius ρ_1, is an absorbing set in $\mathcal{H}_0$ for the group $S(t)$. For any bounded subset B in $\mathcal{H}_0$, $S(t)B \subset B_0$ for $t \geq t_0$. There exists a positive constant $\mu_1 > \rho_1$ such that*

$$\|\Delta u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V} \leq \mu_1^2, \quad \forall t \geq t_0. \tag{3.26}$$

3.2. **Existence of global attractor.** First we prove an important Lemma.

Lemma 3.2. *Under the hypotheses of Theorem 2.3, there exists a constant $\mu_2 > \rho_1$, such that*

$$\|\nabla\Delta u\|_2^2 + \|\nabla u_t\|_2^2 + \|\phi^t\|_{\mu,D(A^{\frac{3}{4}})} \leq \mu_2^2, \quad \forall t \geq t_0. \tag{3.27}$$

Proof. Multiplying $(2.6)_1$ by $-\Delta\varsigma = -\Delta u_t - \theta\Delta u$ and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2\right) + \theta l\|\nabla\Delta u\|_2^2 + \theta\|\Delta u\|_2^2 \\ &+ (1-\theta)(u_t, -\Delta\varsigma) + (\phi^t, u_t)_{\mu,D(A^{\frac{3}{4}})} + \theta(\phi^t, u)_{\mu,D(A^{\frac{3}{4}})} \\ &+ (h(u), -\Delta\varsigma) + (f, \Delta\varsigma) = 0. \end{aligned} \tag{3.28}$$

Similar to previous estimates, we see that

$$(1-\theta)(u_t, -\Delta\varsigma) = (1-\theta)\|\nabla\varsigma\|_2^2 - \theta(1-\theta)(\nabla u, \nabla\varsigma),$$

$$(\phi^t, u_t)_{\mu,D(A^{\frac{3}{4}})} \geq \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} + \frac{\delta}{2}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})},$$

and

$$\theta(\phi^t, u)_{\mu,D(A^{\frac{3}{4}})} \geq -\frac{\delta}{4}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} - \frac{(1-l)\theta^2}{\delta}\|\nabla\Delta u\|_2^2.$$

Whereupon

$$\begin{aligned}&\theta\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|\nabla\Delta u\|_2^2 + (1-\theta)\|\nabla\varsigma\|_2^2 - \theta(1-\theta)(\nabla u, \nabla\varsigma)\\ &\geq \theta(1-\theta)\|\nabla\Delta u\|_2^2 + \frac{1}{4}\|\nabla\varsigma\|_2^2.\end{aligned}$$

Then we get from (3.28)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}\right) + \theta l(1-\theta)\|\nabla\Delta u\|_2^2\\ &+\frac{1}{4}\|\nabla\varsigma\|_2^2 + \theta\|\Delta u\|_2^2 + \frac{\delta}{4}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} \leq (h(u) - f, \Delta\varsigma).\end{aligned} \tag{3.29}$$

Similarly, exploiting the bound $\|u\|_2^2 \leq c$, which implies that $\|h(u)\|^2_{L^\infty} \leq c$ and

$$(h(u) - f, \Delta u_t + \theta\Delta u) \leq (\|h(u)\|^2_{L^\infty} + \|f\|_2^2)(\|\Delta u_t\|_2^2 + \|\Delta u\|_2^2) \leq c. \tag{3.30}$$

So, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}\right) + 2\theta(1-\theta)\|\nabla\Delta u\|_2^2\\ &+2\theta l\|\nabla\Delta u\|_2^2 + \frac{1}{2}\|\nabla\varsigma\|_2^2 + 2\theta\|\Delta u\|_2^2 + \frac{\delta}{2}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} \leq 2c.\end{aligned} \tag{3.31}$$

Thus, denote

$$F(t) = l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(\frac{3}{4})}.$$

We easily get

$$\frac{d}{dt}F(t) + \theta_0 F(t) \leq \tilde{C},$$

where $\theta_0 = \min\{2\theta(1-\theta), 2\theta, \frac{1}{2}, \frac{\delta}{2}\}$, $\quad \tilde{C} = 2c$. By the Gronwall Lemma, we get

$$F(t) \leq e^{-\theta_0 t}F(0) + \frac{\tilde{C}}{\theta_0}.$$

Using the fact that $F(t) \geq \|\nabla\Delta u\|_2^2 + \|\nabla u_t\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}$, then (3.27) holds. Next we show an essential inequality to prove Theorem 2.3 .

Lemma 3.3. *Under the hypotheses of Theorem 2.3, given a bounded set $B \subset \mathcal{H}_0$, let $z_1 = (u, u_t, \phi)$ and $z_2(t) = (v, v_t, \xi)$ be two weak solutions of problem (2.6)-(2.8) such that $z_1(0) = (u_0, u_1, \phi^0)$ and $z_2(0) = (v_0, v_1, \xi^0)$ are in B. Then, we have $\forall t \geq 0$*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}_0} \leq e^{-\nu_1 t}\|z_1(0) - z_2(0)\|^2_{\mathcal{H}_0} + C_3\int_0^t e^{-\nu_1(t-s)}\|u(s) - v(s)\|^2_{2(p+1)}ds, \tag{3.32}$$

where $\nu_1 > 0$ is a small constant and p, C_3 are positive constants.

Proof. Let us fix a bounded set $B \subset \mathcal{H}_0$. We set $w = u - v$ and $\zeta = \phi - \xi$. Then (w, ζ) satisfy

$$\begin{cases} w_{tt} + l\Delta^2 w - \Delta w + w_t + \displaystyle\int_0^\infty \mu(s)\Delta^2\zeta(s)ds + h(u) - h(v) = 0, \\ \zeta_t = -\zeta_s + w_t, \end{cases} \tag{3.33}$$

with initial conditions

$$w(0) = u_0 - v_0, w_t(0) = u_1 - v_1, \zeta^0 = \phi_0 - \xi_0.$$

Taking the scalar product in H of $(3.33)_1$ with $\varsigma = w_t + \theta w$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2\right) + \theta l\|\Delta w\|_2^2 + \theta\|\nabla w\|_2^2 + (1-\theta)(w_t, \varsigma) \\ &+(\zeta^t, w_t)_{\mu,V} + \theta(\zeta^t, w)_{\mu,V} + (h(u) - h(v), \varsigma) = 0. \end{aligned} \tag{3.34}$$

Combining with the previous discussion, we can obtain

$$(1-\theta)(w_t, \varsigma) = (1-\theta)\|\varsigma\|_2^2 - \theta(1-\theta)(w, \varsigma),$$

$$(\zeta^t, w_t)_{\mu,V} \geq \frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mu,V}^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,V}^2,$$

and

$$\theta(\zeta^t, w)_{\mu,V} \geq -\frac{\delta}{4}\|\zeta^t\|_{\mu,V} - \frac{(1-l)\theta^2}{\delta}\|\Delta w\|_2^2.$$

We have

$$\begin{aligned} &\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|\Delta w\|_2^2 + (1-\varsigma)\|\varsigma\|_2^2 - \theta(1-\theta)(w, \varsigma) \\ &\geq \theta l(1-\theta)\|\Delta w\|_2^2 + \frac{1}{4}\|\varsigma\|_2^2. \end{aligned}$$

Then we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}\right) + \theta l(1-\theta)\|\Delta w\|_2^2 \\ &+\frac{1}{4}\|\varsigma\|_2^2 + \theta\|\nabla w\|_2^2 + \frac{\delta}{4}\|\zeta^t\|_{\mu,V}^2 \leq -(h(u) - h(v), \varsigma). \end{aligned} \tag{3.35}$$

We use (2.2) and Young's inequality to obtain

$$\begin{aligned} &\left|-\int_\Omega h(u) - h(v)(w_t + \theta w)dx\right| \leq K_0\int_\Omega (1 + |u|^p + |v|^p)\,|w||w_t + \theta w|dx \\ &\leq K_0\int_\Omega \left(|\Omega|^{\frac{p}{2(p+2)}} + \|u\|_{2(p+1)}^p + \|v\|_{2(p+1)}^p\right)\|w\|_{2(p+1)}(\|w_t\|_2^2 + \theta\|w\|_2^2) \\ &\leq \left(\frac{K_0^2 c_B}{\theta} + \frac{2\theta K_0 c_B}{\lambda_1}\right)\|w\|_{2(p+1)}^2 + \frac{\theta}{4}\|\varsigma\|_2^2. \end{aligned} \tag{3.36}$$

In above inequality, we have used the fact that $\|w_t\|_2^2 = \|\varsigma - \theta w\|_2^2$ and $c_B > 0$ is an embedding constant for $L^{2(p+1)}(\Omega) \hookrightarrow L^2(\Omega)$. Integrating (3.33), we get from (3.35)

$$\begin{aligned}&\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}^2\right) + 2\theta l(1-\theta)\|\Delta w\|_2^2 \\ &+ \left(\frac{1}{2} - \frac{\theta}{2}\right)\|\varsigma\|_2^2 + 2\theta\|\nabla w\|_2^2 + \frac{\delta}{2}\|\zeta^2\|_{\mu,V}^2 \\ &\le \left(\frac{K_0^2 c_B}{\theta} + \frac{\theta 2K_0 c_B}{\lambda_1}\right)\|w\|_{2(p+1)}^2.\end{aligned} \tag{3.37}$$

Choosing θ small enough, such that

$$2\theta(1-\theta) > 0, \quad \frac{1}{2} - \frac{\theta}{2} > 0.$$

Thus, if we denote

$$W(t) = l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}^2,$$

then we easily find

$$\frac{d}{dt}W(t) + \nu_1 W(t) \le C_3\|w\|_{2(p+1)}^2,$$

where $\nu_1 = \min\Big\{2\theta(1-\theta), \frac{1-\theta}{2}, \frac{\delta}{2}\Big\}$, $C_3 = \dfrac{K_0^2 c_B}{\delta} + \dfrac{\theta 2K_0 c_B}{\lambda_1}$ which implies that

$$W(t) \le e^{-\nu_1 t}W(0) + C_3\int_0^t e^{-\nu_1(t-s)}\|w\|_{2(p+1)}^2 ds.$$

Invoking $W(t) \ge \|z_1(t) - z_2(t)\|_{\mathcal{H}_0}^2$, we deduce (3.32).

Lemma 3.4. *Under assumptions of Theorem 2.3, the dynamical system $(\mathcal{H}_0, S(t))$ corresponding to the problem (2.6)-(2.8) is asymptotically smooth.*

Proof. Let B be a bounded subset of $\mathcal{H}_0$ positively invariant with respect to $S(t)$. Denote by C_B serval positive constants that are dependent on B but not on t. For $z_0^1, z_0^2 \in B$, $S(t)z_0^1 = (u(t), u_t(t), \phi^t)$ and $S(t)z_0^1 = (v(t), v_t(t), \xi^t)$ are the solutions of (2.6)-(2.8). Then given $\epsilon > 0$, from inequality (3.27), we can choose $T > 0$ such that

$$\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \le \epsilon + C_B\left(\int_0^T \|u(s) - v(s)\|_{2(p+1)}^2 ds\right)^{\frac{1}{2}}, \tag{3.38}$$

where $C_B > 0$ is a constant which depends only on the size of B. The condition $p > 0$ implies that $2 < 2(p+1) < \infty$. Taking $\varrho = \frac{1}{2}(1 - \frac{1}{p+1})$ and applying Gagliardo-Nirenberg interpolation inequality, we have

$$\|u(t) - v(t)\|_{2(p+1)} \le C\|\Delta(u(t) - v(t))\|_2^{\varrho}\|u(t) - v(t)\|_2^{1-\varrho}.$$

Since $\|\Delta u(t)\|_2$ and $\|\Delta v(t)\|_2$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t) - v(t)\|_{2(p+1)}^2 \leq C_B \|u(t) - v(t)\|_2^{2(1-\varrho)}. \tag{3.39}$$

Then, from (3.38) and (3.39) we obtain

$$\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \leq \epsilon + \Phi_T(z_0^1, z_0^2),$$

with

$$\Phi_T(z_0^1, z_0^2) = C_B \left(\int_0^T \|u(s) - v(s)\|_2^{2(1-\varrho)} ds \right)^{\frac{1}{2}}.$$

The following proof $\Phi_T \in \mathfrak{C}$ namely Φ_T satisfies (2.9). Indeed , give a sequence $(z_0^n) = (u_0^n, u_1^n, \phi_0^n) \in B$, let us write $S(t)(z_0^n) = (u^n(t), u_t^n(t), \phi^{n,t})$ is uniformly bounded in $\mathcal{H}_0$. On the other hand,(u^n, u_t^n) is bounded in $C([0,T], V \times H), T > 0$.

By the compact embedding $V \subset H$, the Aubin lemma implies that there exists a subsequence (u^{nk}) that converges strongly in $C([0,T], H)$. Thus,

$$\lim_{k\to\infty} \lim_{l\to\infty} \int_0^T \|u^{nk}(s) - u^{nl}\|_2^{2(1-\varrho)} ds = 0.$$

Then (2.9) holds.

Proof of Theorem 2.3. Lemma 3.1 and Lemma 3.3 imply that $(\mathcal{H}_0, S(t))$ is a dissipative dynamical system which is asymptotically smooth. Then it has compact global attractor from theorem 2.2.

4. Asymptotic regular estimates

Theorem 4.1. *Under assumptions of Theorem 2.3, then the global attractor $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$.*

In order to prove Theorem 4.1, we fix a bounded set $B \subset \mathcal{H}_0$ and for $z = (u_0, u_1, \phi^0) \in B$, we split the solution $S(t)z = (u(t), u_t(t), \phi^t)$ of problem (2.6)-(2.8) into the sum

$$S(t)z = D(t)z + K(t)z,$$

where $D(t)z = z_1(t)$ and $K(t)z = z_2(t)$, namely $z = (u, u_t, \phi^t) = z_1 + z_2$. Furthermore,

$$u = v + w, \quad \phi^t = \zeta^t + \xi^t, \quad z_1 = (v, v_t, \zeta^t), \quad z_2 = (w, w_t, \xi^t),$$

where $z_1(t)$ satisfies for $(x,t) \in (0,\pi) \times \mathbb{R}^+$ and $(x,s) \in (0,\pi) \times \mathbb{R}^+$

$$\begin{cases} v_{tt} + l\Delta^2 v - \Delta v + v_t + \displaystyle\int_0^\infty \mu(s)\Delta^2\zeta^t(s)ds = 0, \\ \zeta_t^t = -\zeta_s^t + v_t, \\ v(0,y,t) = v_{xx}(0,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ v(\pi,y,t) = v_{xx}(\pi,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ v_{yy}(x,\pm l,t) + \sigma v_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ v_{yyy}(x,\pm l,t) + (2-\sigma)v_{xxy}(x,\pm l,t) - v_y(x,\pm l,t) = 0, \\ \zeta^t(0,y,s) = \zeta_{xx}^t(0,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \zeta^t(\pi,y,s) = \zeta_{xx}^t(\pi,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \zeta_{yy}^t(x,\pm l,s) + \sigma\zeta_{xx}^t(x,\pm l,s) = 0, \quad for\ (x,s) \in (0,\pi) \times \mathbb{R}^+, \\ \zeta_{yyy}^t(x,\pm l,s) + (2-\sigma)\zeta_{xxy}^t(x,\pm l,s) - \zeta_y^t(x,\pm l,t) = 0, \\ v(x,y,\tau) = u_\tau(x,y), \quad \zeta^\tau(x,y,s) = \phi_\tau(x,y,s). \end{cases} \tag{4.40}$$

And $z_2(t)$ satisfies for $(x,t) \in (0,\pi) \times \mathbb{R}^+$ and $(x,s) \in (0,\pi) \times \mathbb{R}^+$

$$\begin{cases} w_{tt} + l\Delta^2 w - \Delta w + w_t + \displaystyle\int_0^\infty \mu(s)\Delta^2\xi^t(s)ds + h(u) = f, \\ \xi_t^t = -\xi_s^t + w_t, \\ w(0,y,t) = w_{xx}(0,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ w(\pi,y,t) = w_{xx}(\pi,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ w_{yy}(x,\pm l,t) + \sigma w_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ w_{yyy}(x,\pm l,t) + (2-\sigma)w_{xxy}(x,\pm l,t) - w_y(x,\pm l,t) = 0, \\ \xi^t(0,y,s) = \xi_{xx}^t(0,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \xi^t(\pi,y,s) = \xi_{xx}^t(\pi,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \xi_{yy}^t(x,\pm l,s) + \sigma\xi_{xx}^t(x,\pm l,s) = 0, \quad for\ (x,s) \in (0,\pi) \times \mathbb{R}^+, \\ \xi_{yyy}^t(x,\pm l,s) + (2-\sigma)\xi_{xxy}^t(x,\pm l,s) - \xi_y^t(x,\pm l,t) = 0, \\ w(x,y,\tau) = 0, \quad \xi^\tau(x,y,s) = \xi_\tau(x,y,s) = 0. \end{cases} \tag{4.41}$$

The well-posedness of the problem (4.40) and (4.41) can be obtained by Faedo-Galerkin method. Furthermore, combining with a priori estimate of 3.1, about the solution $z_1(t)$ of equation (4.40), we have the following result:

Lemma 4.1. *Under assumptions of Theorem 2.3, there exists a constant $k_0 > 0$, such that the solution of* (4.40) *satisfies the following inequality*

$$\|D(t)z\|_{\mathcal{H}_0}^2 \leq Ce^{-k_0 t},$$

where C is a constant.

About the solution of equation (4.41), we have the following results:

Lemma 4.2. *Under the assumptions of Theorem 2.3, there exists a constant $N > 0$, such that the solution of (4.41) satisfies the inequality bellow*

$$\|K(t)z\|^2_{\mathcal{H}_1} \leq N.$$

Proof. Taking the scalar product in H of $(4.41)_1$ with $A\varsigma = Aw_t + \theta Aw$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2\right) + \theta l\|Aw\|_2^2 + \theta\|\nabla\Delta w\|_2^2\\ &+(1-\theta)(w_t, Aw) + (\xi^t, w_t)_{\mu,D(A)} + \theta(\xi^t, w)_{\mu,D(A)} + (h(u), A\varsigma) = (f, A\varsigma).\end{aligned} \tag{4.42}$$

Similar to the previous discussion, there yields

$$(1-\theta)(w_t, A\varsigma) = (1-\theta)\|\Delta\varsigma\|_2^2 - \theta(1-\theta)(Aw, \varsigma),$$

$$(\xi^t, w_t)_{\mu,D(A)} \geq \frac{1}{2}\frac{d}{dt}\|\xi^t\|^2_{\mu,D(A)} + \frac{\delta}{2}\|\xi^t\|^2_{\mu,D(A)},$$

$$\theta(\xi^t, w)_{\mu,D(A)} \geq -\frac{\delta}{4}\|\xi^t\|^2_{\mu,D(A)} - \frac{(1-l)\theta^2}{\delta}\|Aw\|_2^2.$$

Then, we get from (4.42)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)}\right)\\ &+\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|Aw\|_2^2 + (1-\theta)\|\Delta\varsigma\|_2^2 + \theta\|\nabla\Delta w\|_2^2 + \frac{\delta}{4}\|\xi^t\|^2_{\mu,D(A)}\\ &-\theta(1-\theta)(Aw, \varsigma) + (h(u), A\varsigma) = (f, A\varsigma).\end{aligned} \tag{4.43}$$

We have

$$\begin{aligned}&\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|Aw\|_2^2 + (1-\theta)\|\Delta\varsigma\|_2^2 - \theta(1-\theta)(Aw, \varsigma)\\ &\geq \theta l(1-\theta)\|Aw\|_2^2 + \frac{1}{4}\|\Delta\varsigma\|_2^2.\end{aligned} \tag{4.44}$$

By Lemma 3.1 and the Sobolev embedding theorem we know that $h(u)$, $h'(u)$ are uniformly bounded in L^∞ that there exists a constant $K_3 > 0$, such that

$$|h(u)| \leq K_3, and \quad |h'(u)| \leq K_3.$$

Combining with the Hölder, Young and Cauchy and (3.26), (3.27), it follows that

$$\begin{aligned}(h(u), A\varsigma) &= \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) \geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw)\\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - (h'(u)u_t, Aw)\\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - \int_\Omega h'(u)|u_t||Aw|dx\\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - K_3\mu_1\|Aw\|_2\\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - \frac{\theta l}{4}\|Aw\|_2^2 - \frac{K_3^2\mu_1^2}{\theta l},\end{aligned} \tag{4.45}$$

and

$$(f, A\varsigma) = (f, Aw_t + \theta Aw) = \frac{d}{dt}(f, Aw) + \theta(f, Aw). \tag{4.46}$$

Thus, collecting (4.44)-(4.46) from (4.43) yields

$$\begin{aligned} &\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right) \\ &+2l\left(\theta(1-\theta) - \frac{\theta}{4}\right)\|Aw\|_2^2 + \frac{1}{2}\|\Delta\varsigma\|_2^2 + 2\theta\|\nabla\Delta w\|_2^2 + \frac{\delta}{2}\|\xi^t\|^2_{\mu,D(A)} \\ &+2\theta(h(u), Aw) - 2\theta(f, Aw) \le \frac{K_3^2\mu_1^2}{\theta l}. \end{aligned} \tag{4.47}$$

Taking $\theta_0 = \min\left\{2\theta(1-\theta) - \frac{\theta}{2}, 2\theta, \frac{\delta}{2}, \frac{1}{2}\right\}$ we can obtain from (4.47)

$$\begin{aligned} &\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right) \\ &+\theta_0\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right) \\ &\le \frac{K_3^2\mu_1^2}{\theta l}. \end{aligned} \tag{4.48}$$

On the other hand, by the Hölder inequality, the Sobolev embedding theorem and (3.26), it follows that

$$\begin{aligned} &\frac{d}{dt}\|\frac{l}{2}\|Aw\|_2^2 + 2(h(u), Aw)\|_2^2 \ge \frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u)\|_2^2 \\ &-\frac{4}{l}\int_\Omega |h(u)|.|h'(u)||u_t|dx \ge \frac{d}{dt}\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u)\|_2^2 - \frac{4K_3^2\mu_1}{l}, \end{aligned} \tag{4.49}$$

and

$$\frac{d}{dt}\left(\frac{l}{2}\|Aw\|_2^2 + 2(f, Aw)\right) = \frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw - \sqrt{\frac{2}{l}}f\|_2^2. \tag{4.50}$$

Therefore, integrating with (4.49)-(4.50), we get from (4.48)

$$\begin{aligned} &\frac{d}{dt}\left(\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u) - \sqrt{\frac{2}{l}}f\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)}\right) \\ &+\theta_0\left(\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u) - \sqrt{\frac{2}{l}}f\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)}\right) \le \tilde{\tilde{C}}, \end{aligned} \tag{4.51}$$

where $\tilde{\tilde{C}} = K_3^2\mu_1^2\left(\frac{1}{\theta l} + \frac{4}{l}\right) + \frac{2\theta_0}{l}\left(K_3^2\mu_1^2 + \|f\|_2^2\right)$. Applying the Gronwell's Lemma, we can easily see that there exists a constant N such that

$$\|Aw\|_2^2 + \|\Delta w_t\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} \le N.$$

Proof of Theorem 4.1

By Lemma 4.1 and Lemma 4.2, we deduce that $(u, u_t, \phi^t) \in \mathcal{H}_1$ and we have

$$\|Au\|_2^2 + \|\Delta u_t\|_2^2 + \|\phi^t\|_{\mu,D(A)} \le N.$$

Now since $u(t,x)$ satisfies (2.6)-(2.8) with initial data (u_0, u_1, ϕ^0), we conclude that

$$\|(u_0, u_1, \phi^0\|_{\mathcal{H}_1} \leq \widehat{N}.$$

Thus $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$.

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Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria
E-mail address: mahdifatimazohra@yahoo.fr

Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria
E-mail address: hakemali@yahoo.com

Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria
E-mail address: ferhat22@hotmail.fr