



ON THE QUADRATIC VARIATION OF SOME SEQUENCES OF FRACTIONAL PARTS

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Abstract. Let a, b be real numbers such that $b > a \geq 0$, and let $c = b - a$. Let x be a real number and n a positive integer ($n \in \mathbb{N}^*$). The present study focuses on proving the following asymptotic formula

$$\sum_{n \geq 1} \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2 = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O\left(c^{139/378} x^{61/126}\right),$$

with $c = b - a$, uniformly for $x \geq 100c^{-25/3}(1+b)^{29}$ generalization of the following result

$$\sum_{n \geq 1} \left(\left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right)^2 = \frac{\zeta(3/2)}{\pi} \sqrt{x} + O\left(x^{3/7}\right) \quad (x > 0),$$

established in previous literature for the specific case $a = 0$ and $b = 1$.

Keywords: Arithmetic function, Integer part function, Fractional part function, Quadratic variation, Asymptotic behavior.

MSC (2020): 11N37, 11K38.

1. INTRODUCTION

The sums involving the integer part function were introduced by Dirichlet in the middle of the 19th century when he proved that

$$\sum_{n \leq x} \sum_{d|n} 1 = \sum_{n \leq x} [x/n] = x \log x + x(2\gamma - 1) + O(x^{1/2}), \quad x > 0,$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}$, and γ is Euler's constant (see [1], [7]). For the reader's convenience, we note that the notation $f = O(g)$ (Landau notation) should be understood in the sense that there exists a constant $C > 0$ and a real number x_0 such that for all $x \geq x_0$, $|f(x)| \leq Cg(x)$ (see [9]). Later, this research topic became very popular among mathematicians, and there is a vast amount of literature focusing on estimating sums of the form

$$\sum_{n \leq x} f([x/n]),$$

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where f is an arithmetic function; see [4], [5], [6], [8], [11], [12], [13] and the references therein. These references are considered essential as they provide historical and language-specific insights into sums involving integer part functions. This diverse array of references forms a solid foundation for readers seeking a nuanced understanding of these kinds of sums. The significance of the results obtained in both [6] and especially in [5] sparked our curiosity about the possibility of other similar cases that might offer a broader extension of the established results

$$S_f(x; a) := \sum_{n \leq \frac{x}{1+a}} f\left(\left\lfloor \frac{x}{n} - a \right\rfloor\right), \quad (1.1)$$

where a is a positive real number and $x > 0$. This, in particular, shaped our goal of investigating and resolving all challenges that contribute to formulating an ideal approximate expression for sums of the form (1.1). In our study, f is an arithmetic function satisfying some assumptions to be made precise later on. We recall that some techniques used in the previously cited articles are primarily based on the well-known Ramanujan hypothesis:

$$(H) : |f(n)| \ll n^\varepsilon, \quad n \geq 1, \varepsilon > 0.$$

Here, the notation $f \ll g$ (Vinogradov notation) means that there exists a constant $C > 0$ such that $|f(n)| \leq Cg(n)$ for all sufficiently large n (see [9]). In this work, we opt for the use of another improved version of this hypothesis given by

$$(H') : |f(n)| \ll n(\log en)^{-\alpha}, \quad n \geq 1, \alpha \in \mathbb{R}.$$

Observe here that hypothesis (H') is more general than hypothesis (H) . Additionally, we denote by $\{t\} = t - \lfloor t \rfloor$ the fractional part of $t \in \mathbb{R}$. Before proceeding further in our study, we need to introduce the parameter $m \in [1, x/4]$ chosen according to certain requirements, and the integer $k = \lfloor x/n - a \rfloor$. Immediately, one has

$$x/n - (a + 1) < k \leq x/n - a \quad \text{and} \quad x/(k + a + 1) < n \leq x/(k + a).$$

This allows us to write the sum (1.1) as follows:

$$S_f(x; a) = \sum_{k \leq x-a} f(k) (\lfloor x/(k + a) \rfloor - \lfloor x/(k + a + 1) \rfloor) = S_f^\sharp(x; a, m) + S_f^*(x; a, m), \quad (1.2)$$

where

$$S_f^\sharp(x; a, m) := \sum_{k \leq m} f(k) \left(\left\lfloor \frac{x}{k + a} \right\rfloor - \left\lfloor \frac{x}{k + a + 1} \right\rfloor \right)$$

and

$$S_f^*(x; a, m) := \sum_{m < k \leq x-a} f(k) (\lfloor x/(k + a) \rfloor - \lfloor x/(k + a + 1) \rfloor).$$

Keeping in mind the hypothesis (H') , we easily deduce that

$$S_f^*(x; a, m) \ll \sum_{n \leq x/(m+a)} (x/n)(\log m)^{-\alpha} \ll x(\log m)^{-\alpha} \log(x/(m + a)). \quad (1.3)$$

We will now discuss the estimation of the sum $S_f^\sharp(x; a, m)$. Let $\{t\} = t - \lfloor t \rfloor$ denote the fractional part of t . We can then rewrite

$$S_f^\sharp(x; a, m) = M_f(x; a, m) + R_f(x; a, m), \quad (1.4)$$

where

$$M_f(x; a, m) := \sum_{k \leq m} \frac{xf(k)}{(k+a)(k+a+1)}$$

$$\text{and } R_f(x; a, m) := \sum_{k \leq m} f(k) \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right).$$

As a direct consequence of the hypothesis (H') , we deduce that

$$M_f(x; a, m) = x \sum_{k \geq 1} \frac{f(k)}{(k+a)(k+a+1)} + O(x(\log m)^{-\alpha+1}). \quad (1.5)$$

At this level, the use of the Cauchy–Schwarz inequality leads to

$$R_f(x; a, m) \leq \left(\sum_{k \leq m} |f(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k \geq 1} \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right)^2 \right)^{\frac{1}{2}}.$$

Observe that if the function f satisfies hypothesis (H') , then

$$R_f(x; a, m) \leq m^{3/2}(\log m)^{-\alpha} \left(\sum_{k \geq 1} \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right)^2 \right)^{1/2}. \quad (1.6)$$

Our strategy now is to provide an explicit estimate for the quantity $R_f(x; a, m)$, referenced in (1.6). Achieving this is equivalent to deriving an asymptotic formula for the sum $Q(x; a, b)$, which is defined as follows

$$Q(x; a, b) := \sum_{n \geq 1} \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2, \quad b > a \geq 0. \quad (1.7)$$

Certainly, understanding the behavior of the function $Q(x; a, b)$ has been pivotal in deriving an asymptotic formula for the sum referenced in (1.4). This progress would not have been achievable without recognizing the critical role played by the sum in (1.5), which serves as a cornerstone for this development. A notable aspect of this work lies in the investigation of the intermediate sum (1.7). Indeed, in [3, p. 3] and [2], significant results were established for the specific case $Q(x; 0, 1)$, where it was demonstrated that

$$Q(x; 0, 1) = \frac{\zeta(3/2)}{\pi} \sqrt{x} + O(x^{3/7}), \quad x > 0.$$

Building on these findings, our focus shifts to estimating the more general sum $Q(x; a, b)$ as highlighted in (1.7). This sum plays a central role in our analysis, providing the necessary tools to describe the asymptotic behavior of the quantity in (1.1), particularly when considering the estimate given in (1.3). This work can be considered a generalization of M. Balazard’s approach [3]. Our main result is stated as follows

Theorem 1.1. *Let a and b be two real numbers such that $b > a \geq 0$ and $c = b - a$. For $x \geq 100c^{-25/3}(1+b)^{29}$, one has*

$$Q(x; a, b) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O\left(c^{139/378} x^{61/126}\right).$$

2. PROOF OF THEOREM 1.1

2.1. About the sum $Q(x; a, b)$. In this transitional step, we provide an alternative and useful formulation of the sum $Q(x; a, b)$ defined previously by the formula (1.7). This plays an important role in our study and helps us justify our main results, namely Theorem 1. To simplify the notation, we will use $Q(x)$ to denote the sum $Q(x; a, b)$. Let us introduce the real parameter D (with $D \leq x$), a sufficiently large integer, such that the sum $Q(x)$ can be expressed as follows:

$$Q(x) = Q_0(x) + \sum_{0 < d \leq D} Q_d(x) + \sum_{d > D} Q_d(x),$$

where

$$Q_d(x) = \sum_{n \geq 1} \varrho_d(h, k) \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2.$$

Here, ϱ_d is the classical indicator function defined by

$$\varrho_d(h, k) = \begin{cases} 1 & \text{if } (h, k) \in E_d, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$k = \left\lfloor \frac{x}{n+a} \right\rfloor, \quad h = \left\lfloor \frac{x}{n+b} \right\rfloor.$$

and

$$E_d = \{(h, k) \in \mathbb{N}^2 : 0 \leq h \leq k \leq \frac{x}{1+a}, k - h = d\}, \quad (2.8)$$

At this level, observe that

$$\max \left(\frac{x}{k+1} - a, \frac{x}{h+1} - b \right) < n \leq \min \left(\frac{x}{k} - a, \frac{x}{h} - b \right),$$

where $0 \leq h \leq k \leq x$ (with the convention $x/0 = \infty$). Furthermore, the introduction of the interval of values of n , denoted by $I_{(h,k)}$, defined as

$$I_{(h,k)} = \mathbb{N}^* \cap \left[\frac{x}{k+1} - a, \frac{x}{k} - a \right] \cap \left[\frac{x}{h+1} - b, \frac{x}{h} - b \right],$$

allows us to write $Q_d(x)$ as follows

$$Q_d(x) = \sum_{(h,k) \in E_d} \sum_{n \in I_{(h,k)}} \left(\frac{cx}{(n+a)(n+b)} - k + h \right)^2; \quad 0 \leq k - h < \frac{cx}{ab} + 1.$$

2.2. Study of the sum $Q_0(x)$. Our main result concerning the quantity $Q_0(x)$ reads as follows.

Proposition 2.1. *For $x \geq \frac{9b^2}{c}$, we have*

$$Q_0(x) = \frac{2}{15} \sqrt{cx} + \delta_0 + O \left(\frac{(1+b)^2}{c} \right),$$

where

$$\delta_0 = \frac{c^2}{x^2} \sum_{1 \leq k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right), \quad x > 0. \quad (2.9)$$

In order to give the proof of this results, we need to use the following auxiliary results

Lemma 2.1. For $x \geq a^2/c$, one has

$$Q_0(x) = c^2 x^2 F\left(\frac{x}{K+1} - a\right) + c^2 x^2 \sum_{1 \leq k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right)\right), \quad (2.10)$$

where

$$F(t) = \sum_{n>t} \frac{1}{(n+a)^2(n+b)^2}, \quad (2.11)$$

and

$$K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

Proof. Taking into account (2.8), we see that $E_0(x) = \{(k, k), k \in \mathbb{N}, 0 \leq k \leq \frac{x}{1+a}\}$. Therefore,

$$Q_0(x) = \sum_{0 \leq k \leq \frac{x}{1+a}} Q(k, k) = \sum_{0 \leq k \leq \frac{x}{1+a}} \sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k} - b} \frac{(cx)^2}{(n+a)^2(n+b)^2}.$$

Note that if $k(k+1) > \frac{x}{c}$, the sum

$$\sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k} - b} \frac{(cx)^2}{(n+a)^2(n+b)^2}$$

is not well defined. Therefore, let K be the largest integer k such that $k(k+1) \leq \frac{x}{c}$, i.e., $K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor$. From which, we observe that $K \leq \sqrt{\frac{x}{c}}$ when $x \geq \frac{a^2}{c}$. Now, for $t \geq 0$, consider the function

$$F(t) = \sum_{n>t} \frac{1}{(n+a)^2(n+b)^2},$$

with $F(\infty) = 0$ and $F(t) = F(0)$ if $t < 0$. Consequently, for $x \geq a$, we deduce that

$$Q_0(x) = c^2 x^2 F\left(\frac{x}{K+1} - a\right) + c^2 x^2 \sum_{1 \leq k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right)\right).$$

□

Lemma 2.2. Let $0 \leq a < b$ and $t > 0$. Then,

$$F(t) = \frac{1}{3t^3} - \frac{\{t\} - 2(a+b) - \frac{1}{2}}{t^4} + O\left(\frac{b^2}{t^5}\right), \quad (2.12)$$

Proof. It suffices to take into account the implication (2.11) and the fact that

$$\frac{1}{(n+a)^2(n+b)^2} = \frac{1}{n^4} - \frac{2(a+b)}{n^5} + O\left(\frac{b^2}{n^6}\right), \quad n \in \mathbb{N}^*.$$

□

Before giving the next technical result, we mention that the quantity $x/(K+1)$ is well defined. In fact, since $K+1 = \sqrt{\frac{x}{c}} + \theta_0$, where $|\theta_0| \leq 1$ and $\sqrt{\frac{x}{c}} \geq 3$, then $\frac{x}{K+1} - a = \frac{x}{\sqrt{\frac{x}{c}} + \theta_0} - a = \sqrt{cx} + O(a)$. More precisely, $\frac{x}{K+1} = \sqrt{cx}(1 + \theta_1)$, with $\theta_1 = O(\sqrt{\frac{c}{x}})$ and $\theta_1 \geq -\frac{5}{6}$. This is justified by the fact that

$$\frac{x}{K+1} - c \geq \frac{\sqrt{cx}}{6}, \quad (2.13)$$

when $x \geq \frac{9b^2}{c}$. Utilizing the result (2.12) we arrive at

Lemma 2.3. *For $x \geq \frac{9b^2}{c}$, we have*

$$F\left(\frac{x}{K+1} - a\right) = \frac{1}{3(cx)^{3/2}} + O\left(\frac{b^2}{(cx)^2}\right).$$

Proof. Taking into account that $F(t) = \frac{1}{3t^3} + O\left(\frac{b^2}{t^4}\right)$, it follows that

$$F\left(\frac{x}{K+1} - a\right) = \frac{1 + O\left(\sqrt{\frac{a}{cx}}\right)}{3(cx)^{3/2}} + O\left(\frac{b^2}{(cx)^2}\right).$$

□

We have also

Lemma 2.4. *Let $x \geq \frac{9b^2}{c}$. Then,*

$$\begin{aligned} \sum_{0 < k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) &= -\frac{1}{5c^{3/2}x^{3/2}} \\ &\quad + x^{-4} \sum_{0 < k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) \\ &\quad + O\left(\frac{(1+b)^2}{c^3x^2}\right). \end{aligned}$$

Proof. Thanks to the estimate (2.12) and by using the inequality (2.13), we conclude that $\frac{x}{k} - a \geq \frac{\sqrt{ax}}{6}$ and $\frac{x}{k} - b \geq \frac{\sqrt{bx}}{6}$. Consequently, we see that

$$\begin{aligned} \sum_{0 < k \leq K} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) &= \sum_{0 \leq k \leq K} \left(-\frac{ck^4}{x^4} + O\left(\frac{b^2k^5}{x^5}\right) \right) \\ &= -\sum_{0 \leq k \leq K} \frac{ck^4}{x^4} + O\left(\frac{b^2K^6}{x^5}\right), \end{aligned}$$

where

$$\sum_{N < k \leq M} k^4 = \frac{1}{5}(M^5 - N^5) + \frac{1}{2}(M^4 - N^4) + \frac{1}{3}(M^3 - N^3) - \frac{1}{30}(M - N).$$

A particular case when $N = 0$ and $M = K$ gives

$$\sum_{0 < k \leq K} k^4 = \frac{1}{5}K^5 + \frac{1}{2}K^4 + \frac{1}{3}K^3 - \frac{1}{30}K.$$

Now, using the fact that $K = \sqrt{\frac{x}{c}} + O(1)$, we obtain

$$\sum_{0 < k \leq K} k^4 = \frac{x^{5/2}}{5c^{5/2}} + O\left(\frac{x^2}{c^2}\right).$$

Therefore,

$$\sum_{0 < k \leq K} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = -\frac{1}{5c^{3/2}x^{3/2}} + O\left(\frac{1}{cx^2}\right).$$

On the other hand, one has

$$\begin{aligned} & \sum_{0 < k \leq K} \left(\frac{1}{\left(\frac{x}{k} - a\right)^4} \left(\left\{ \frac{x}{k} \right\} - a - 2(a+b) - \frac{1}{2} \right) - \frac{1}{\left(\frac{x}{k} - b\right)^4} \left(\left\{ \frac{x}{k} \right\} - b - 2(a+b) - \frac{1}{2} \right) \right) \\ &= x^{-4} \sum_{0 < k \leq K} k^4 \left(\left\{ \frac{x}{k} \right\} - a - \left\{ \frac{x}{k} \right\} + b \right) + O\left(\frac{b^2}{c^3 x^2}\right). \end{aligned}$$

This allows us to say that

$$\sum_{0 < k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \ll (1+b)^2 \sum_{0 < k \leq K} \frac{k^5}{x^5} \leq (1+b)^2 \frac{K^6}{x^5} \leq \frac{(1+b)^2}{c^3 x^2}.$$

□

2.3. Study of the sum $\sum_{d>D} Q_d(x)$.

Proposition 2.2. *Let D be a sufficiently large real number. Then,*

$$\sum_{d>D} Q_d(x) \leq \sqrt{\frac{cx}{D-1}} + 1. \quad (2.14)$$

Proof. If $d = \left\lfloor \frac{x}{n+a} \right\rfloor - \left\lfloor \frac{x}{n+b} \right\rfloor > D$, it follows that

$$\frac{cx}{n^2} > \frac{cx}{(n+a)(n+b)} = d + \left\{ \frac{x}{n+a} \right\} - \left\{ \frac{x}{n+b} \right\} > D-1,$$

which means that $n < \sqrt{\frac{cx}{D-1}}$. From this, we deduce that

$$\sum_{d>D} Q_d(x) < \sum_{0 \leq n < \sqrt{\frac{cx}{D-1}}} 1 < \sqrt{\frac{cx}{D-1}} + 1.$$

□

2.4. Study of the sum $\sum_{0 < d \leq D} Q_d(x)$. We start this section by the following useful remarks

Remark 2.1. *A direct computation justifies the fact that*

$$\frac{x}{k+1} - a \leq \frac{x}{h+1} - b \implies \frac{x}{k} - a \leq \frac{x}{h} - b$$

and

$$\frac{x}{k} - a > \frac{x}{h} - b \implies \frac{x}{k+1} - a > \frac{x}{h+1} - b.$$

As a consequence,

$$E_d = E_{d,1} \cup E_{d,2} \cup E_{d,3},$$

with

$$E_{d,1} = \left\{ (k-d, k) : d \leq k \leq \frac{x}{a}, (k-d+1)(k+1) \leq \frac{xd}{c} \right\}, \quad (2.15)$$

$$\frac{x}{k-d+1} - b < n \leq \frac{x}{k} - a,$$

$$E_{d,2} = \left\{ (k-d, k) : d \leq k \leq \frac{x}{a}, (k-d)k > \frac{xd}{c} \right\}, \quad (2.16)$$

$$\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b,$$

$$E_{d,3} = \{(k-d, k) : d \leq k \leq \frac{x}{a}, (k-d)k \leq \frac{xd}{c} < (k-d+1)(k+1)\}, \quad (2.17)$$

$$\frac{x}{k+1} - a < n \leq \frac{x}{k} - a.$$

As a result, we can rearrange Q_d as follows:

$$Q_d = Q_{d,1} + Q_{d,2} + Q_{d,3},$$

with

$$Q_{d,i} = \sum_{(h,k) \in E_{d,i}} \sum_{n \in I_{(h,k)}} \left(\frac{cx}{(n+a)(n+b)} - k + h \right)^2, \quad (1 \leq i \leq 3).$$

At this level, it is suitable to upper bound the quadratic conditions with respect to k . For this purpose, let us define

$$K_d(t) = \left\lfloor \frac{d + \sqrt{d^2 - 4dt}}{2} \right\rfloor, \quad d \in \mathbb{N}, t > 0. \quad (2.18)$$

It is easy to see that $K_d(t)$ is viewed as the largest integer $k \geq d$ such that $(k-d)k \leq dt$, with $K_0(t) = 0$.

Remark 2.2. As a direct consequence, we can upper bound the integer k appearing in the definitions of the sets $E_{d,i}$, $i = 1, 2, 3$. More precisely, one has

(1) For $E_{d,1}$,

$$k \leq K_d(x/c) - 1, \quad (2.19)$$

(2) For $E_{d,2}$,

$$k > K_d(x/c), \quad (2.20)$$

(3) For $E_{d,3}$,

$$k = K_d(x/c). \quad (2.21)$$

There is another remark regarding the integer k that appears in the sets $E_{d,i}$, $i = 1, 2, 3$.

Remark 2.3. The condition $k \leq x/a$ is redundant for $i = 1, 3$ provided that $d \leq cx/ab$. Indeed, the relation $d \leq \frac{cx}{ab}$ can be rewritten as:

$$(K_d(x/c) - d)K_d(x/c) \leq dx/c; \quad d \leq cx/ab \leq x/a,$$

and the fact that $t \mapsto (t-d)t$ is strictly increasing for $t \geq d$, leads to the inequality:

$$K_d(x/c) \leq \frac{x}{a}.$$

2.4.1. Calculation of $Q_{d,1}(x)$. In the sequel, for simplicity, we will write K_d instead of $K_d(x/c)$. In this section, we focus on the quantity $Q_{d,1}(x)$. To this end, we assume that $0 < d \leq \frac{cx}{ab}$. Let us introduce the function G , defined as follows

$$G(t) = \frac{1}{c} \sum_{n>t} \left(\frac{1}{n+a} - \frac{1}{n+b} \right). \quad (2.22)$$

Note that according to conditions (2.15) and (2.19), we get

$$Q_{d,1} = \sum_{d < k \leq K_d - 1} \sum_{\frac{x}{k-d+1} - b < n \leq \frac{x}{k} - a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2.$$

Furthermore, the sum

$$\sum_{\frac{x}{k-d+1}-b < n \leq \frac{x}{k}-a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2$$

is well defined if

$$\frac{x}{k-d+1} - b < \frac{x}{k} - a.$$

Immediately, we can conclude that if $k > K_{d-1}$ (recall that $K_0 = 0$), one has

$$Q_{d,1} = \sum_{K_{d-1} < k \leq K_d-1} \sum_{(x/(k-d+1))-b < n \leq (x/k)-a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2, \quad (2.23)$$

which simplifies to

$$\begin{aligned} Q_{d,1} &= d^2 \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right) \\ &\quad - 2cdx \sum_{K_{d-1} < k \leq K_d-1} \left(G \left(\frac{x}{k-d+1} - b \right) - G \left(\frac{x}{k} - a \right) \right) \\ &\quad + c^2 x^2 \sum_{K_{d-1} < k \leq K_d-1} \left(F \left(\frac{x}{k-d+1} - b \right) - F \left(\frac{x}{k} - a \right) \right). \end{aligned} \quad (2.24)$$

For the first sum of (2.24), we obtain

$$\begin{aligned} &\sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right) \\ &= \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - a \right\rfloor \right) \\ &\quad + \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k-d+1} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right). \end{aligned}$$

At this level, we use the following well-known “r-telescopic” identity:

$$\sum_{\alpha < k \leq \beta} (u_k - u_{k-r}) = \sum_{\beta-r < k \leq \beta} u_k - \sum_{\alpha-r < k \leq \alpha} u_k. \quad (2.25)$$

This allows us to write

$$\begin{aligned} \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - a \right\rfloor \right) &= \sum_{K_{d-1}-d+1 < k \leq K_d-1} \left\lfloor \frac{x}{k} - a \right\rfloor \\ &\quad - \sum_{K_{d-1} < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned}$$

Using a similar argument for the second and third sums of (2.24), we obtain

$$\begin{aligned}
Q_{d,1} = & 2cdx \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\
& - c^2x^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\
& + 2cdx \sum_{K_d-d < k \leq K_{d-1}} G\left(\frac{x}{k} - a\right) - 2cdx \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} G\left(\frac{x}{k} - a\right) \\
& - c^2x^2 \sum_{K_d-d < k \leq K_{d-1}} F\left(\frac{x}{k} - a\right) + c^2x^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} F\left(\frac{x}{k} - a\right) \\
& + d^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\
& + d^2 \sum_{K_d-d < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor - d^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \tag{2.26}
\end{aligned}$$

2.4.2. Calculation of $Q_{d,2}(x)$. Taking into account (2.16) and (2.20) and the condition $0 < d \leq \frac{cx}{ab}$, we get

$$Q_{d,2} = \sum_{K_d < k \leq x} \sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b} \left(\frac{cx}{(n+a)(n+b)} - d \right)^2.$$

It is important to note that the sum

$$\sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b} \left(\frac{cx}{(n+a)(n+b)} - d \right)$$

is well defined only if the lower bound does not exceed the upper bound, that is, $\frac{x}{k-d} - b \geq \frac{x}{k+1} - a$. This condition can be rewritten equivalently as $k \leq K_{d+1} - 1$, which provides a natural upper bound for the summation index. Then, we deduce that

$$\begin{aligned}
Q_{d,2} = & d^2 \sum_{K_d < k \leq K_{d+1}-1} \left(\left\lfloor \frac{x}{k-d} - b \right\rfloor - \left\lfloor \frac{x}{k+1} - a \right\rfloor \right) \\
& - 2cdx \sum_{K_d < k \leq K_{d+1}-1} \left(G\left(\frac{x}{k+1} - a\right) - G\left(\frac{x}{k-d} - b\right) \right) \\
& + c^2x^2 \sum_{K_d < k \leq K_{d+1}-1} \left(F\left(\frac{x}{k+1} - a\right) - F\left(\frac{x}{k-d} - b\right) \right). \tag{2.27}
\end{aligned}$$

The decomposition in (2.27) follows from expanding the square and expressing the resulting sums in terms of the auxiliary functions F and G previously defined.

To proceed further, we focus on the first sum of (2.27). By introducing an intermediate term, we can split this expression into two more manageable parts

$$\begin{aligned} & \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - b \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) = \\ & \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - b \right\rfloor - \left\lfloor \frac{x}{k-d-1} - a \right\rfloor \right) \\ & + \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - a \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right). \end{aligned}$$

This decomposition is strategic: the first term on the right-hand side captures the difference between the parameters a and b , while the second term exhibits a telescopic structure.

The use of the telescopic identity, introduced in (2.25), gives us

$$\begin{aligned} \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - a \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) = & - \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor \\ & + \sum_{K_d-d < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned}$$

This transformation is crucial as it converts the original sum into boundary terms that will be easier to estimate in subsequent sections.

A similar manipulation applies to the second and third sums of (2.27), where we apply the same telescopic technique to the functions G and F . After collecting all terms and reorganizing the summation indices, we obtain

$$\begin{aligned} Q_{d,2} = & c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\ & - 2cdx \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\ & - 2cdx \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} G\left(\frac{x}{k} - a\right) + 2cdx \sum_{K_d-d < k \leq K_{d+1}} G\left(\frac{x}{k} - a\right) \\ & + c^2 x^2 \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} F\left(\frac{x}{k} - a\right) - c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}} F\left(\frac{x}{k} - a\right) \\ & + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - b \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) \\ & - d^2 \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor + d^2 \sum_{K_d-d < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned} \tag{2.28}$$

The final expression (2.28) separates the main terms (involving differences $F(\cdot - a) - F(\cdot - b)$ and $G(\cdot - a) - G(\cdot - b)$) from the boundary terms, which will be shown to contribute to the error estimate.

2.4.3. **Calculation of $Q_{d,3}(x)$.** For $0 < d \leq \frac{cx}{ab}$, and from (2.17) and (2.21), we get

$$Q_{d,3} = d^2 \left(\left\lfloor \frac{x}{K_d} - a \right\rfloor - \left\lfloor \frac{x}{K_d+1} - a \right\rfloor \right) - 2cdx \left(G \left(\frac{x}{K_d+1} - a \right) - G \left(\frac{x}{K_d} - a \right) \right) + c^2x^2 \left(F \left(\frac{x}{K_d+1} - a \right) - F \left(\frac{x}{K_d} - a \right) \right). \quad (2.29)$$

2.4.4. **Calculation of $Q_d(x)$.** Keeping in mind (2.26), (2.28), (2.29) and the fact that for $0 < d \leq \frac{cx}{ab}$, we write

$$\begin{aligned} Q_d(x) = & d^2 \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ & + 2cdx \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(G \left(\frac{x}{k} - a \right) - G \left(\frac{x}{k} - b \right) \right) \\ & - c^2x^2 \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(F \left(\frac{x}{k} - a \right) - F \left(\frac{x}{k} - b \right) \right) \\ & + d^2 \left(\sum_{K_{d+1}-d-1 < k \leq K_{d+1}} - \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \right) \left\lfloor \frac{x}{k} - a \right\rfloor \\ & + 2cdx \left(2 \sum_{K_d-d < k \leq K_d} - \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} - \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \right) G \left(\frac{x}{k} - a \right) \\ & + c^2x^2 \left(\sum_{K_{d+1}-d-1 < k \leq K_{d+1}} + \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} - 2 \sum_{K_d-d < k \leq K_d} \right) F \left(\frac{x}{k} - a \right). \quad (2.30) \end{aligned}$$

2.4.5. **Estimate of $Q_d(x)$ for $d \geq 1$.**

Proposition 2.3. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$Q_d = \vartheta(d)\sqrt{cx} + \delta(d) + O \left(\frac{(1+b)^4 d^3}{c} \right),$$

with

$$\vartheta(d) = \frac{32d^2}{5}\sqrt{d} - \frac{48d^2 + 16d - 2}{15}\sqrt{d+1} - \frac{48d^2 - 16d - 2}{15}\sqrt{d-1},$$

and

$$\delta(d) = \left(\sum_{K_d-d < k \leq K_{d+1}-d-1} - \sum_{K_{d-1}-d+1 < k \leq K_d-d} \right) \left(\frac{c^2k^4}{x^2} - \frac{2cd}{x} + d^2 \right) \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right). \quad (2.31)$$

Remark 2.4. In this section, it is necessary to recall some useful technical results established in [3, p. 12], [4, p. 421], and [2, p. 4-6].

(1) For $0 < d \leq x/c$ and $x > c$, one has:

$$K_{d+1} - K_d = \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{\frac{x}{c}} + O(1). \quad (2.32)$$

$$\sum_{K_d < k \leq K_{d+1}} k^2 = \frac{(d+1)\sqrt{d+1} - d\sqrt{d}}{3} \left(\frac{x}{c}\right)^{3/2} + O\left((d+1)\frac{x}{c}\right). \quad (2.33)$$

$$K_{d+1} - K_d \ll \left(\frac{x}{cd}\right)^{1/2} \quad (0 < d \leq x/c). \quad (2.34)$$

$$\sqrt{\frac{dx}{c}} + \frac{d}{2} - 1 < K_d \leq \sqrt{\frac{dx}{c}} + d. \quad (2.35)$$

(2) For $0 \leq d \leq x/c$ and $x \geq c$, one has:

$$K_d = \sqrt{\frac{dx}{c}} + \frac{d}{2} + O\left(d^{3/2}c^{1/2}x^{-1/2}\right) + O(1). \quad (2.36)$$

$$\sum_{K_d - d < k \leq K_d} k = d\sqrt{\frac{dx}{c}} + O\left(d^{5/2}x^{-1/2}c^{1/2}\right) + O(d). \quad (2.37)$$

$$\sum_{K_d - d < k \leq K_d} k^3 = \frac{x}{c} \left(d^2 \sqrt{\frac{dx}{c}} + O\left(d^{7/2}x^{-1/2}c^{1/2}\right) + O(d^2) \right). \quad (2.38)$$

$$\sum_{K_d - d < k \leq K_{d+1} - (d+1)} k^4 = \frac{(d+1)^2 \sqrt{d+1} - d^2 \sqrt{d}}{5} \left(\frac{x}{c}\right)^{5/2} + O\left((d+1)^2 c^{-2} x^2\right). \quad (2.39)$$

Lemma 2.5. For $t > 0$ and $b > a > 0$, we have

$$G(t) = \frac{1}{t} + \frac{\{t\} - \frac{a+b+1}{2}}{t^2} + O\left(\frac{(1+b)^2}{t^3}\right). \quad (2.40)$$

Proof. For $t > 0$, $n \in \mathbb{N}^*$ and $b > a > 0$, we have according to (2.22)

$$G(t) = \sum_{n>t} \frac{1}{(n+a)(n+b)},$$

Then

$$\begin{aligned} \sum_{n>t} \frac{1}{n^2} &= \frac{1}{t} + \frac{\{t\} - \frac{1}{2}}{t^2} + O(t^{-3}), \quad \sum_{n>t} \frac{1}{n^3} = \frac{1}{2t^2} + O(t^{-3}), \\ \sum_{n>t} \frac{1}{n^4} &= O(t^{-3}), \end{aligned}$$

which allows us to conclude that

$$G(t) = \frac{1}{t} + \frac{\{t\} - \frac{a+b+1}{2}}{t^2} + O\left(\frac{1+b^2}{t^3}\right).$$

□

Lemma 2.6. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$\begin{aligned} & d^2 \sum_{K_d - d < k \leq K_{d+1} - d - 1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ &= d^2 \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{cx} \\ &+ d^2 \sum_{K_d - d < k \leq K_{d+1} - d - 1} \left(\left\{ \frac{x}{k} - b \right\} - \left\{ \frac{x}{k} - a \right\} \right) \\ &+ O(cd^2). \end{aligned} \quad (2.41)$$

Proof. By a direct computation, we obtain

$$d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) = cd^2 (K_{d+1} - K_d - 1) \\ + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\{ \frac{x}{k} - b \right\} - \left\{ \frac{x}{k} - a \right\} \right).$$

Then we use the result noted (2.32), and get

$$d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ = d^2 \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{cx} + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left(\frac{x}{k} - b \right) - \left(\frac{x}{k} - a \right) \right) + O(cd^2).$$

□

Lemma 2.7. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$2cdx \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\ = 2d \left(\frac{d\sqrt{d} - (d+1)\sqrt{d+1}}{3} \right) \sqrt{cx} \\ + 2d \sum \left(\frac{ck^2}{x} \right) \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{(1+b)^2 d^2}{c} \right). \quad (2.42)$$

Proof. For practical reasons, we set

$$S = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right).$$

According to (2.34), the contribution to the sum S of the error term of (2.40) is expressed as follows:

$$S \ll (1+b)^2 (K_{d+1} - K_d) \left(\frac{d}{cx} \right)^{3/2} \leq (1+b)^2 \sqrt{\frac{x}{c(d+1)}} \left(\frac{d}{cx} \right)^{3/2} \leq \frac{(1+b)^2 d}{xc^2}.$$

According to formulas (2.33), (2.34), and (2.35), the contribution to the sum S of the $1/t$ term of (2.40) is

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{\frac{x}{k} - a} - \frac{1}{\frac{x}{k} - b} \right) \\ = \frac{d\sqrt{d} - (d+1)\sqrt{d+1}}{3} x^{-1/2} c^{-1/2} + O\left(\frac{d+1}{x} \right) + O\left(\frac{abd}{xc} \right).$$

Finally, the contribution to the sum S of the term $\frac{\{t\} - \frac{a+b+1}{2}}{t^2}$ of (2.40) is expressed as follows:

$$\begin{aligned} & \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{1}{\left(\frac{x}{k} - a\right)^2} \left(\left\{ \frac{x}{k} - a \right\} - \frac{a+b+1}{2} \right) \\ & - \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{1}{\left(\frac{x}{k} - b\right)^2} \left(\left\{ \frac{x}{k} - b \right\} - \frac{a+b+1}{2} \right) \\ & = \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{k^2}{x^2} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{(b+1)^2 d}{c^2 x}\right). \end{aligned}$$

We obtain relation (2.42) by collecting these estimates. $\square$

Lemma 2.8. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, one has

$$\begin{aligned} & c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\ & = \frac{d^2 \sqrt{d} - (d+1)^2 \sqrt{d+1}}{5} \sqrt{cx} \\ & + \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{c^2 k^4}{x^2} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) \\ & + O\left(cb^2 d^2\right). \end{aligned} \tag{2.43}$$

Proof. For practical reasons, we set

$$S' = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k}\right) - F\left(\frac{x}{k} - a\right) \right).$$

According to (2.34), the contribution of the error term of (2.12) is estimated as follows:

$$\ll (1+b)^2 (K_{d+1} - K_d) \left(\frac{d}{cx}\right)^{5/2} \leq (1+b)^2 \sqrt{\frac{x}{c(d+1)}} \left(\frac{d}{cx}\right)^{5/2} \leq \frac{(1+b)^2 d^2}{c^3 x^2}.$$

The contribution of the term $\frac{1}{3t^3}$ of (2.12) is given by

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(-\frac{ck^4}{x^4} + O\left(\frac{b^2 k^5}{x^5}\right) \right),$$

then from (2.34) and (2.39), it follows that

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = \frac{d^2 \sqrt{d} - (d+1)^2 \sqrt{d+1}}{5} (cx)^{-3/2} + O\left(\frac{b^2 d^2}{c^3 x^2}\right).$$

Finally, according to (2.34) and (2.39), the contribution of the term $\frac{\{t\} - \frac{1+4(a+b)}{2}}{t^4}$ of (2.12) is expressed as follows:

$$\begin{aligned} & \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{\left(\frac{x}{k} - a\right)^4} \left(\left\{ \frac{x}{k} - a \right\} - \frac{1+4(a+b)}{2} \right) \right. \\ & \quad \left. - \frac{1}{\left(\frac{x}{k} - b\right)^4} \left(\left\{ \frac{x}{k} - b \right\} - \frac{1+4(a+b)}{2} \right) \right) \\ &= \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{b^2 d^2}{c^3 x^2}\right). \end{aligned}$$

Summing up, we obtain the desired result, namely the relation (2.43). $\square$

Lemma 2.9. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, and $K_d - d < k \leq K_d$, we have

$$\left\lfloor \frac{x}{k} - a \right\rfloor = \sqrt{\frac{cx}{d}} + O(a). \quad (2.44)$$

Proof. Since $k - \sqrt{\frac{dx}{c}} \ll d$ according to (2.35), one has

$$\left\lfloor \frac{x}{k} - a \right\rfloor = \frac{x}{k} + O(a) = \sqrt{\frac{cx}{d}} + O(a).$$

$\square$

Lemma 2.10. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, one has

$$2cdx \sum_{K_d-d < k \leq K_d} G\left(\frac{x}{k} - a\right) = 2d^2 \sqrt{cdx} + O(d^3) + O\left(\frac{c^{3/2}(1+b)^2 d^{7/2}}{x^{1/2}}\right). \quad (2.45)$$

Proof. According to (2.40), we see that

$$\sum_{K_d-d < k \leq K_d} G\left(\frac{x}{k} - a\right) = \sum_{K_d-d < k \leq K_d} \left(\frac{k}{x} + \frac{k^2}{x^2} \left(\frac{x}{k} - a - \frac{a+b+1}{2} \right) + O\left(\frac{(1+b)^2 k^2}{x^2}\right) \right).$$

Using (2.37), we obtain

$$\sum_{K_d-d < k \leq K_d} \frac{k}{x} = \frac{d}{x} \sqrt{\frac{dx}{c}} + O\left(\frac{d}{x}\right) + O\left(\frac{c^{1/2} d^{5/2}}{x^{3/2}}\right).$$

From (2.35), we can see that

$$\begin{aligned} \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \left(\frac{x}{k} - a \right) &\ll \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \leq \frac{dK_d^2}{x^2} \leq \frac{d^2}{cx}, \\ \frac{a+b+1}{2} \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} &\ll \frac{(a+b+1)d^2}{cx}, (1+b)^2 \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \ll \frac{(1+b)^2 d^2}{cx}. \end{aligned}$$

Then, the lemma's result is deduced. $\square$

Lemma 2.11. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$c^2 x^2 \sum_{K_d-d < k \leq K_d} F\left(\frac{x}{k} - a\right) = \frac{d^2}{3} \sqrt{dcx} + O(b^2 d^3) + O\left(c^{3/2} d^{7/2} x^{-1/2}\right). \quad (2.46)$$

Proof. According to (2.12), one has

$$\sum_{K_d-d < k \leq K_d} F\left(\frac{x}{k} - a\right) = \sum_{K_d-d < k \leq K_d} \left(\frac{k^3}{3x^3} + \frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - 2(a+b) - \frac{1}{2} \right) + O\left(\frac{b^2 k^4}{x^4}\right) \right).$$

Using (2.38), we get

$$\sum_{K_d-d < k \leq K_d} \frac{k^3}{3x^3} = \frac{1}{3x^2} \left(\frac{d^2}{c} \sqrt{\frac{dx}{c}} + O\left(\frac{d^3 \sqrt{\frac{d}{cx}}}{1}\right) + O\left(\frac{d^2}{c}\right) \right).$$

By using the fact that $K_d \ll \sqrt{\frac{dx}{c}}$, it follows that

$$\sum_{K_d-d < k \leq K_d} \left(\frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - 2(a+b) - \frac{1}{2} \right) + O\left(\frac{b^2 k^4}{x^4}\right) \right) \leq \frac{b^2 d^3}{c^2 x^2}.$$

□

The combination of formulas (2.41) through (2.46) and (2.30) leads to the proposition 2.3

2.5. The estimate of the sum $Q(x)$.

2.5.1. **About the main term in $Q(x)$.** In order to appear the main term and the error term involving the parameter D , we give another formulation of $Q(x)$.

Proposition 2.4. For two real numbers x and D such that $3 \leq D \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + R(D) + O\left(\frac{(1+b)^4 D^4}{c}\right) + O\left(\sqrt{\frac{cx}{D}}\right),$$

with

$$R(D) = \delta_0 + \sum_{1 \leq d \leq D} \delta(d).$$

Here, the quantities δ_0 and $\delta(d)$ are defined by (2.9) for $d = 0$ (resp. by (2.31) for $d > 0$).

Proof. According to the propositions 2.1, 2.2 and 2.3, the use of the estimate $\vartheta(d) \ll d^{-3/2}$, and the following equality (see [2, p. 11] and the equations (12)–(13)), we conclude that $\sum_{d \leq D} \vartheta(d) = -\frac{2}{15} + \frac{\zeta(3/2)}{\pi} + O(D^{-1/2})$, and we get

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + \delta_0 + \sum_{1 \leq d \leq D} \delta(d) + O\left(\frac{(1+b)^4 D^4}{c}\right) + O\left(\sqrt{\frac{cx}{D}}\right).$$

□

2.5.2. **The estimate of the sum $R(D)$.** To bound the sum $R(D)$, we will exploit the following result due to van der Corput.

Lemma 2.12. Let a and b be real numbers such that $b - a \geq 1$, and let $f : [a, b] \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a monotonic second derivative of constant sign. Then we have

$$\sum_{a < n \leq b} \left(\{f(n)\} - \frac{1}{2} \right) \ll \int_a^b |f''(t)|^{1/3} dt + \frac{1}{\sqrt{|f''(a)|}} + \frac{1}{\sqrt{|f''(b)|}}. \quad (2.47)$$

Proof. See [10, p. 32].

□

By partial summation, we derive the following estimate from (2.47).

Lemma 2.13. *Let a and b be real numbers such that $b - a > 0$, and let $f : [a, b] \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a monotonic second derivative of constant sign. Let $(v_n)_{a < n \leq b}$ be a sequence of real numbers. Then we have*

$$\sum_{a < n \leq b} v_n \left(\{f(n)\} - \frac{1}{2} \right) \ll U \int_a^b |f''(t)|^{1/3} dt + \frac{U}{\sqrt{|f''(a)|}} + \frac{U}{\sqrt{|f''(b)|}}, \quad (2.48)$$

where

$$U = \max |v_n| + \sum_{a < n \leq b} |v_{n+1} - v_n|.$$

And one has

Proposition 2.5. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq D \leq (x/c)^{1/21}$, one has*

$$R(D) \ll x^{1/3} \ln(x/c) + c^{-2/21} x^{3/7} D^{7/4}.$$

The proof of this proposition is based on the following lemmas.

Lemma 2.14. *For $0 < d \leq \frac{x}{c}$ and $K_d - d < k \leq K_{d+1} - (d+1)$, one has*

$$-\frac{2cd}{x} \leq \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2 \leq \frac{64d^3 c}{x} + 65d^2.$$

Proof. It suffices to show that

$$d^2 \left(\frac{c^2 k^2}{x^2} - 8 \frac{ck}{x} + 17 \right) - \frac{2cd}{x} \leq \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2 \leq (d+1)^2 \left(\frac{c^2 k^2}{x^2} - 8 \frac{ck}{x} + 16 \right) - \frac{2cd}{x} + d^2.$$

In fact, we choose $K_d - d$ as the largest integer verifying $k^2 + dk - \frac{4dx}{c} \leq 0$, which allows us to say that $k^2 \in \left[-dk + \frac{4dx}{c}, -(d+1)k + \frac{4(d+1)x}{c} \right]$. $\square$

Lemma 2.15. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq d \leq (x/c)^{1/21}$, one has*

$$\delta(d) \ll c^{-2/21} d^{-1} x^{3/7} + c^{-71/84} d^{3/4} x^{29/84}.$$

Proof. A simple computation shows that if $x \geq \frac{100(1+b)^{21/10}}{c}$, then $\left(\frac{x}{c}\right)^{1/21} \leq \frac{cx}{9(b+1)^2} - 1$.

Let us define $v_k = \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2$. Since the sequence (v_k) is increasing, Lemma 2.14 implies that

$$\max |v_k| + \sum_{K_d - d < k \leq K_{d+1} - (d+1)} |v_{k+1} - v_k| \ll \left(\frac{x}{c}\right)^{2/21}.$$

Applying (2.48) to the two functions $f(t) = \frac{x}{t} - a$ and $f(t) = \frac{x}{t} - b$, and taking into account (2.34) and (2.35), we obtain

$$\sum_{K_d - d < k \leq K_{d+1} - (d+1)} v_k \left(\{f(n)\} - \frac{1}{2} \right) \ll c^{-2/21} d^{-1} x^{3/7} + c^{-71/84} d^{3/4} x^{29/84}, \quad (2.49)$$

In the case where $K_d - d < k \leq K_{d+1} - d - 1$, we obtain the same result by following all the same arguments. $\square$

Lemma 2.16. *For $x \geq 1 + c$, one has*

$$\delta_0 \ll x^{1/3} \ln(2x/c) + c^{-3/4} x^{1/4}.$$

Proof. We have $\delta_0 = \frac{c^2}{x^2} \sum_{1 \leq k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right)$, with $K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor$. Using the same reasoning as in the proof of Lemma 2.15, we deduce that

$$\delta_0 \ll x^{1/3} \ln \left(\frac{2x}{c} \right) + c^{-3/4} x^{1/4}. \quad (2.50)$$

According to Proposition 2.4 as well as Lemmas 2.15 and 2.16, we get

$$R(D) = \delta_0 + \sum_{1 \leq d \leq D} \delta(d) \ll x^{1/3} \ln \left(\frac{x}{c} \right) + c^{-2/21} x^{3/7} D^{7/4}.$$

□

2.5.3. *The estimate of the sum $Q(x)$.* Combining the results obtained in propositions 2.4 and 2.5, we give the following result.

Proposition 2.6. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq d \leq (x/c)^{1/21}$, we have*

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O \left(\sqrt{\frac{cx}{D}} \right) + O \left(c^{-2/21} x^{3/7} D^{7/4} \right) + O \left(\frac{(1+b)^4 D^4}{c} \right) + O \left(x^{1/3} \ln \left(\frac{x}{c} \right) \right).$$

Proof of theorem 1.1. We are now ready to assemble all the previous estimates to establish our main result. For

$$x \geq 100(1+b)^{21/10} c^{-1}, \quad (2.51)$$

and $1 \leq d \leq (x/c)^{1/21}$, the combination of the results from propositions 2.5 and 2.6 indicates that

$$Q(x) - \frac{\zeta(3/2)}{\pi} \sqrt{cx} \ll \sqrt{\frac{cx}{D}} + c^{-2/21} x^{3/7} D^{7/4} + \frac{(1+b)^4 D^4}{c} + x^{1/3} \ln \left(\frac{x}{c} \right).$$

This inequality represents the total error bound as a sum of four distinct terms, each arising from different stages of our analysis.

To optimize this bound, we must choose the parameter D carefully. We choose $D = c^{50/189} x^{2/63}$ to balance the first two error terms. This strategic choice ensures that neither term dominates the other, leading to the sharpest possible estimate. This quantity satisfies the bounds $1 \leq D \leq (x/c)^{1/21}$ if

$$x \geq \max(c^{20}, c^{-25/3}). \quad (2.52)$$

We now verify that all remaining error terms are dominated by the main error term. On the other hand, under the assumptions (2.51) and (2.52), it follows that

$$x^{1/3} \ln \left(\frac{x}{c} \right) \ll x^{1/3} \left(\frac{x}{c} \right)^{19/126} \ll c^{139/378} x^{61/126},$$

which shows that the logarithmic term is absorbed into the main error bound.

It remains to control the term involving the parameter b . Keeping in mind and that

$$x \geq (1+b)^{14} c^{-13/15}, \quad (2.53)$$

we conclude that

$$\frac{(1+b)^4 D^4}{c} \asymp (1+b)^4 c^{11/89} x^{8/63} \ll c^{139/378} x^{61/126}.$$

This demonstrates that the dependence on b is also properly controlled under our assumptions.

Finally, we consolidate all conditions into a single requirement. Further, observe that if $x \geq 100c^{-25/3}(1+b)^{29}$, the three conditions (2.51), (2.52), and (2.53) are satisfied, and the result is demonstrated. This completes the proof of Theorem 1.1. $\square$

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