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ON STANDARD RIEMANNIAN SPACE FORMS AND THEIR $\square$-BICONSERVATIVE HYPERSURFACES

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Abstract. According to a variational problem, the tensor of stress-energy, as specified by Hilbert (1924), is a bicovariant symmetric tensor with null divergence. This property is named the conservativeness of stress-energy tensor. In this literature, the stress-energy tensor associated to the bi-energy function with null divergence is said to be biconservative. In differential geometric point of view, a hypersurface $\xi : M^n \to \mathbb{M}^{n+1}(c)$ of a Riemannian space form is called biconservative if $\Delta^2\xi$ has null tangential component, where Δ is the Laplace operator on M^n. It is proved that such a hypersurface has constant mean curvature. We consider the hypersurfaces satisfying a progressive version of biconservativity condition. The $\square$-biconservativity condition is obtained by substituting the Cheng-Yau operator $\square$ instead of Δ. We prove that $\square$-biconservative hypersurfaces of Riemannian $(n+1)$-space forms (with some additional conditions) have constant scalar curvature.

Keywords: Cheng-Yau operator , $\square$-biconservative, scalar curvature.
MSC (2020): 53C40, 53C42, 58G25.

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1. Introduction

The subject of biconservative submanifolds is an interesting research topic in physics and mathematics, which has been started by Eells and Sampson ([6]) and followed by some other researchers (see for instance [10, 15]). From the physical points of view, this subject deals with the bienergy functional and its critical points arisen form the tension field. In geometric context, the subject of biconservative submanifolds has received much attentions. In 2015, Turgay studied the H-hypersurfaces of Euclidean spaces with at most three distinct principal curvatures ([16]). Also, the biconservative surfaces in spherical and hyperbolical cylinders have been studied in [8]. Recently, some researchers have studied biconservative hypersurfaces of Riemannian space forms of dimension 4 or 5 ([9, 17]). In this manuscript we study the $(n+1)$-dimensional Riemannian space forms and their hypersurfaces satisfying the $\square$-biconservativeness condition. The $\square$-biconservativity condition is obtained by substituting the Cheng-Yau operator $\square$ instead of Δ. The operator $\square$ denotes the linearized operator

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arisen from the first variation of the second mean curvature vector field ([5]). In general, The operator L_k denotes the linearized operator arisen from the first variation of the $(k+1)$th mean curvature vector field. In fact, $L_0 = \Delta$ and $L_1 = \Box$ are special cases ([1, 2, 3, 4, 11, 12, 14]).

By definition, a hypersurface in a Riemannian space form is called $\Box$-biconservative if its first and second mean curvatures H_1 and H_2 satisfy the equation

$$\mathcal{N}_2(\nabla H_2) - c\mathcal{N}_1(\nabla H_1) = \frac{3}{4}n(n-1)H_2\nabla H_2, \tag{1.1}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the first and second Newton transformations. We show that this condition gives the constancy of scalar curvature in some cases.

2. PREREQUISITES

Some preliminary concepts from [1, 2, 12, 13, 19] are the opening discussions. The main study will be done on the hypersurfaces of standard Riemannian space forms of dimension $(n+1)$ and curvature c denoted by $\mathbb{M}^{n+1}(c)$ for $c = 0, \pm 1$. Clearly, $\mathbb{M}^{n+1}(c)$ is $\mathbb{E}^{n+1}$ when $c = 0$, $\mathbb{S}^{n+1}$ when $c = 1$ and $\mathbb{H}^{n+1}$ when $c = -1$. The sphere of radius $\rho > 0$ is defined by

$$\mathbb{S}^{n+1}(\rho) = \{z \in \mathbb{E}^{n+2} | \langle z, z\rangle = \rho^2\}.$$

For convenience, we put $\mathbb{S}^{n+1} := \mathbb{S}^{n+1}(1)$. Similarly, $\mathbb{M}^{n+1}(-1) = \mathbb{H}^{n+1}$ is the hyperbolic space of dimension $(n+1)$ and radius (-1). In general, hyperbolic m-space of radius $(-\rho)$, as a hyperquadric in the Lorentz-Minkowski space $\mathbb{L}^{m+1} = \mathbb{E}_1^{m+1}$, is defined by $\mathbb{H}^m(-\rho) = \{z \in \mathbb{L}^{m+1} | \langle z, z\rangle = -\rho^2, z_1 > 0\}$.

From now on, we consider a Riemannian hypersurface defined by an isometric immersion ξ from M^n into $\mathbb{M}^{n+1}(c)$ (for $c = \pm 1, 0$). The main tool of our work is the shape operator S of M^n associated to a chosen (local) orthonormal tangent basis $\mathcal{P} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ and a unit normal vector field $\mathbf{n}$ on M^n. The eigenvalues of S are denoted by the real functions $\kappa_1, \ldots, \kappa_n$ on M^n. Using them, the jth mean curvature is defined by $\binom{n}{j}H_j = s_j$, where

$$s_j := \sum_{1 \le i_1 < \ldots < i_j \le n} \kappa_{i_1} \ldots \kappa_{i_j}$$

is the jth elementary symmetric function (for instance, see [1] and [2]).

The hypersurface M^n is named *j-minimal* if $H_{j+1} \equiv 0$. In special cases, H_1 is the ordinary mean curvature and $H_2 := (n-1)n(1-R)$, where R is the normalized scalar curvature function.

The Newton maps on M^n are defined as

$$\mathcal{N}_0 = I,\ \mathcal{N}_1 = -s_1 I + S,\ \mathcal{N}_2 = s_2 I - s_1 S + S^2,$$

where I denotes the identity map on the tangent bundle of M.

The Cheng-Yau operator $\Box : \mathcal{C}^\infty(M^n) \to \mathcal{C}^\infty(M^n)$ is defined by rule

$$\Box(f) = trace(\mathcal{N}_1 \circ \nabla^2 f),$$

where $\langle \nabla^2 f(\mathrm{V}), \mathrm{W}\rangle = Hess^f(\mathrm{V},\mathrm{W})$ for $f \in \mathcal{C}^\infty(M^n)$ and $\mathrm{V},\mathrm{W} \in \chi(M^n)$. In other words, $\Box(f)$ is given by $\Box(f) = \sum_{i=1}^n \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. So, we get

$$\Box\mathbf{x} = \alpha_n[H_2\mathbf{n} - cH_1\mathbf{x}],$$

and

$$\begin{aligned}\Box^2\mathbf{x} &= 2\alpha_n[\mathcal{N}_2(\nabla H_2) - c\mathcal{N}_1(\nabla H_1)] - \frac{3}{2}\alpha_n^2 H_2\nabla H_2 \\ &+ \left[\alpha_n\Box(H_2) - \alpha_n(tr(S^2\circ\mathcal{N}_1) - \alpha_n H_1)H_2\right]\mathbf{n} \\ &+ c\left[\alpha_n\Box(H_1) - \alpha_n^2 H_2^2 + \alpha_n^2 c H_1^2\right]\mathbf{x},\end{aligned} \tag{2.2}$$

where $\alpha_n = n(n-1)$. By definition, M^n is called $\Box$-*biconservative* if $\mathbf{x}$ satisfies $(\Box^2\mathbf{x})^\top = 0$ (i.e the condition (1.1)).

According to (local) orthonormal tangent basis $\{\mathbf{w}_m\}_{m=1}^{n+1}$ and its co-frame $\{\omega_m\}_{1\le m\le n+1}$ on $\mathbb{M}^{n+1}(c)$, where $\mathbf{w}_{n+1}$ is positively normal to M^n, by a Cartan Lemma, we have

$$\omega_{n+1,i} = \sum_{j=1}^n h_{ij}\omega_j, \tag{2.3}$$

where $h_{ij} = h_{ji}$ for $i,j = 1,\ldots,n$. Therefore, the structure equations of $\mathbb{M}^{n+1}(c)$ (as may be seen in [19]) are

$$d\omega_m = \sum_{k=1}^{n+1}\omega_{mk}\wedge\omega_k,\ \ \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{n+1}\omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k\wedge\omega_l.$$

Of course, we have $\omega_{n+1} = 0$ and then $d\omega_{n+1} = \sum\limits_{k=1}^{n}\omega_{n+1,k}\wedge\omega_k = 0$ on M. Also, we have the Gauss equation $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} stand for the components of the tensor of Riemannian curvature on M^n. Finally, we have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{2.4}$$

where h_{ijk} is the covariant derivative of h_{ij}. Thus, by exterior differentiation of (2.3), we obtain the Codazzi equation $h_{ijk} = h_{ikj}$. One can choose $w_1,\ldots,w_n$ such that $h_{ij} = \kappa_i\delta_{ij}$. On the other hand, the Levi-Civita connection of M^n satisfies $\nabla_{w_i}w_j = \sum_k\omega_{jk}(w_i)w_k$, and then $w_i(\kappa_j) = \omega_{ij}(w_j)(\kappa_i - \kappa_j)$ and $\omega_{ij}(w_l)(\kappa_i - \kappa_j) = \omega_{il}(w_j)(\kappa_i - \kappa_l)$ whenever i,j,l are distinct.

3. EXAMPLES

We see several samples of $\Box$-biconservative hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ with constant first and second mean curvatures (see [2, 14]).

Example 3.1. Suppose that $0 < \varrho < 1$ and $\Sigma_0 = \mathbb{S}^n(\varrho) \subset \mathbb{S}^{n+1}$ defined as

$$\Sigma_0 = \{\mathbf{y} = (y_1,\ldots,y_{n+2}) \in \mathbb{S}^{n+1} | y_1^2 + \ldots + y_{n+1}^2 = \varrho^2, y_{n+2} = \sqrt{1-\varrho^2}\}.$$

Its Gauss map is $\mathbf{n}(\mathbf{y}) = \frac{-\sqrt{1-\varrho^2}}{\varrho}(y_1, \ldots, y_{n+1}, 0) + \frac{\varrho}{\sqrt{1-\varrho^2}}(0, \ldots, 0, y_{n+2})$. It has a constant principal curvature $\kappa = \frac{\sqrt{1-\varrho^2}}{\varrho}$ of multiplicity n. One can see that Σ_0 is $\Box$-biconservative and its mean and scalar curvatures are constant.

Example 3.2. Let $1 \leq k \leq n-1$, $0 < \varrho < 1$ and $\Sigma_1 = \mathbb{S}^k(\varrho) \times \mathbb{S}^{n-k}(\sqrt{1-\varrho^2}) \subset \mathbb{S}^{n+1}$ defined as

$$\Sigma_1 = \{\mathbf{y} = (y_1, \ldots, y_{n+2}) \in \mathbb{S}^{n+1} | y_1^2 + \ldots + y_{k+1}^2 = \varrho^2, y_{k+2}^2 + \ldots + y_{n+2}^2 = 1 - \varrho^2\},$$

whose Gauss map is $\mathbf{n}(\mathbf{y}) = \frac{-\sqrt{1-\varrho^2}}{\varrho}(y_1, \ldots, y_{k+1}, 0, \ldots, 0) + \frac{\varrho}{\sqrt{1-\varrho^2}}(0, \ldots, 0, y_{k+2}, \ldots, y_{n+2})$. Its principal curvatures are $\kappa_1 = \frac{\sqrt{1-\varrho^2}}{\varrho}$ of multiplicity k and $\kappa_2 = \frac{-\varrho}{\sqrt{1-\varrho^2}}$ of multiplicity $n-k$. One can see that Σ_1 is $\Box$-biconservative and its mean and scalar curvatures are constant.

Example 3.3. Let $1 \leq k \leq n-1$, $\varrho > 0$ and $\Sigma_2 = \mathbb{H}^k(-\sqrt{\varrho^2+1}) \times \mathbb{S}^{n-k}(\varrho) \subset \mathbb{H}^{n+1}$ defined as

$$\Sigma_2 = \{\mathbf{y} = (y_1, \ldots, y_{n+2}) \in \mathbb{H}^{n+1} | -y_1^2 + y_2^2 + \ldots + y_{k+1}^2 = -1 - \varrho^2, y_{k+2}^2 + \ldots + y_{n+2}^2 = \varrho^2\},$$

with the Gauss map $\mathbf{n}(\mathbf{y}) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, \ldots, y_{k+1}, 0, \ldots, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, \ldots, 0, y_{k+2}, \ldots, y_{n+2})$ and two distinct constant principal curvatures $\kappa_1 = \frac{-\varrho}{\sqrt{1+\varrho^2}}$ of multiplicity k and $\kappa_2 = \frac{-\sqrt{1+\varrho^2}}{\varrho}$ of multiplicity $n-k$. So, Σ_2 is $\Box$-biconservative.

Example 3.4. Assume that $\varrho > 0$ and $\Sigma_3 = \mathbb{S}^n(\varrho) \subset \mathbb{H}^{n+1}$ is defined by

$$\Sigma_3 = \{\mathbf{y} = (y_1, \ldots, y_{n+2}) \in \mathbb{H}^{n+1} | y_1 = \sqrt{1+\varrho^2}, y_2^2 + \ldots + y_{n+2}^2 = \varrho^2\}.$$

Its Gauss map is $\mathbf{n}(y) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, 0, \ldots, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, y_2, \ldots, y_{n+2})$ and it has one constant principal curvature $\kappa = \frac{-\sqrt{1+\varrho^2}}{\varrho}$ of multiplicity n. So, Σ_3 is $\Box$-biconservative.

Example 3.5. Suppose that Σ_4 is the product $\mathbb{H}^n(-\sqrt{\varrho^2+1}) \subset \mathbb{H}^{n+1}$, where $\varrho \geq 0$. It can be represented as

$$\Sigma_4 = \{\mathbf{y} = (y_1, \ldots, y_{n+2}) \in \mathbb{H}^{n+1} | -y_1^2 + y_2^2 + \ldots + y_{n+1}^2 = -1 - \varrho^2, y_{n+2} = \varrho\}$$

with the Gauss map $\mathbf{n}(\mathbf{y}) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, \ldots, y_{n+1}, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, \ldots, 0, y_{n+2})$, only one constant principal curvature $\kappa = \frac{-\varrho}{\sqrt{1+\varrho^2}}$ of multiplicity n. Hence, Σ_4 is $\Box$-biconservative.

Example 3.6. For each unit constant vector (point) $\mathbf{q} \in \mathbb{E}^{n+2}$ and each real number $-1 < \eta < 1$, we consider the totally umbilical hypersurface

$$\Pi_{\mathbf{q},\eta} := \{\mathbf{z} \in \mathbb{S}^{n+1} : \langle \mathbf{z}, \mathbf{q} \rangle = \eta\} = \mathbb{S}^n(\sqrt{1-\eta^2}).$$

Its Gauss map is $\mathbf{n}(\mathbf{z}) = \frac{1}{\sqrt{1-\eta^2}}(\mathbf{q} - \eta \mathbf{z})$ and its shape operator is $\mathrm{S} = \frac{\eta}{\sqrt{1-\eta^2}} I$. In particular, its first two mean curvatures are $H_1 = \frac{\eta}{\sqrt{1-\eta^2}}$, $H_2 = \frac{\eta}{1-\eta^2}$. So, Σ_η is $\Box$-biconservative.

Example 3.7. Let $\mathbf{w} \in \mathbb{L}^{n+2}$ be a nonzero constant vector such that $\langle \mathbf{w}, \mathbf{w} \rangle \in \{0, \pm 1\}$. The subset

$$\Upsilon_\sigma := \{\mathbf{q} \in \mathbb{H}^{n+1} : \langle \mathbf{q}, \mathbf{w} \rangle = \sigma\}$$

is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ if $\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle > 0$. In fact

$$\Upsilon_\sigma = \begin{cases} \mathbb{S}^n(\sqrt{\sigma^2 - 1}) \subset \mathbb{E}^{n+2} & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = -1, |\sigma| > 1) \\ \mathbb{E}^n & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 0, \sigma \neq 0) \\ \mathbb{H}^n(-\sqrt{1 + \sigma^2}) & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 1). \end{cases}$$

Its first and second mean curvatures are constant given by $H_1 = \frac{-\sigma}{\sqrt{\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle}}$ and $H_2 = \frac{\sigma^2}{\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle}$. So, Υ_σ is □-biconservative.

4. Main results

We study the hypersurfaces of standard Riemannian space form $\mathbb{M}^{n+1}(c)$ for $c = 0, \pm 1$ that satisfies the condition (1.1). Some similar studies may be found for the ordinary biconservativeness condition in [7, 16, 18]. Let $\xi : M^n \to \mathbb{M}^{n+1}(c)$ be a □-biconservative hypersurface which has at most two distinct principal curvatures. We show that such a hypersurface in $\mathbb{M}^{n+1}(c)$ under some additional conditions has a constant scalar curvature. First, we see a lemma which has a similar proof to the Lemma A in [15].

Lemma 4.1. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is a hypersurface with principal curvatures of constant multiplicities. Then, the distribution generated by principal directions is completely integrable. Also, each principal curvature of multiplicity greater than 1 is fixed on each integral submanifold of its distribution.

Theorem 4.1. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ be a □-biconservative hypersurfaces having exactly one principal curvature of multiplicity n. Then, its scalar curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction. In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n$. Also, we have

$$\mu_{i;2} = \frac{1}{2}(n-2)(n-1)\lambda^2, \;\; H_2 = \lambda^2. \tag{4.5}$$

By condition (1.1) and constancy of H_1, we have $\mathcal{N}_2(\nabla H_2) = \frac{3}{4}n(n-1)H_2\nabla H_2$, which, by expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2 = \sum_{i=1}^n \langle \nabla H_2, \mathbf{w}_i \rangle \mathbf{w}_i$, gives $\langle \nabla H_2, \mathbf{w}_i \rangle (\mu_{i;2} - \frac{3}{4}n(n-1)H_2) = 0$ for every i.

If $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$ for an i, then we obtain $\mu_{i;2} = \frac{3}{4}n(n-1)H_2$, which using equalities (4.5), gives that $H_2 = 0$ on $\mathcal{U}$. So $\mathcal{U} = \emptyset$. This is a contradiction which gives the constancy of H_2 on M^n. □

Theorem 4.2. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is a hypersurface of a Riemannian space form satisfying the condition (1.1). If it has constant mean curvature and its shape operator has only two eigenvalues η and λ of multiplicities 1 and $n-1$ (respectively), then its scalar curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction.

In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n$. Also, we have

In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n-1$ and $\mathrm{S}\mathbf{w}_n = \eta \mathbf{w}_n$. Also, we have

$$\begin{aligned}
&\mu_{1,2} = \ldots = \mu_{n-1,2} = (n-2)\lambda\eta + \binom{n-2}{2}\lambda^2, \ \mu_{n,2} = \binom{n-1}{2}\lambda^2, \\
&nH_1 = \eta + (n-1)\lambda, \ \binom{n}{2}H_2 = \binom{n-1}{2}\lambda^2 + (n-1)\lambda\eta, \\
&\binom{n}{3}H_3 = \binom{n-1}{3}\lambda^3 + \binom{n-1}{2}\lambda^2\eta.
\end{aligned} \tag{4.6}$$

By expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2 = \sum_{i=1}^{n} \langle \nabla H_2, \mathbf{w}_i \rangle \mathbf{w}_i$, from equality (1.1) we have

$$\langle \nabla H_2, \mathbf{w}_i \rangle (\mu_{i;2} - \frac{3(n-1)n}{4} H_2) = 0$$

for every i. On the other hand, we have $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$ on $\mathcal{V}$ for some i. Therefore, we have

$$\mu_{i,2} = \frac{3}{4} n(n-1) H_2, \tag{4.7}$$

for some i. Two different situations can arise.

Situation 1. $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$, for at least one $i \leq n-1$. Equality (4.7) (applying (4.6)) gives

$$(n-9)(n-2)\lambda^2 = 4(n+l)\lambda\eta.$$

If $\lambda \neq 0$, then we have $\eta = -\frac{(n+3)(n-2)}{2(n+1)}\lambda$, which gives $\lambda = \frac{2(n+1)n}{n^2-n+4} H_1$ and $H_2 = -\frac{8n(n-2)(n+1)}{(n^2-n+4)^2} H_1^2$. If $\lambda = 0$ then $H_2 = 0$. In each case, the constancy of H_2 is guaranteed.

Situation 2. $\langle \nabla H_2, \mathbf{w}_n \rangle \neq 0$ and $\langle \nabla H_2, \mathbf{w}_i \rangle = 0$, for each $i \leq n-1$. The equality (4.7) (using identities (4.6)) gives $\lambda = 0$ or $\eta = \frac{2-n}{6}\lambda$.

Assuming $\lambda \neq 0$, we get $\lambda = \frac{6n}{5n-4} H_1$ and $H_2 = \frac{24n(n-2)}{(5n-4)^2} H_1^2$. If $\lambda = 0$ then we get $H_2 = 0$. Again, in each case, the constancy of H_2 is guaranteed.

Therefore, we get $\mathcal{U} = \emptyset$, which is a contradiction. So H_2 is constant on M^n and then the scalar curvature of M^n is constant. □

Theorem 4.3. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is C-biconservative hypersurface of Riemannian space form and suppose that η and λ are its only principal curvatures of multiplicities ℓ and $n - \ell$ (respectively), where $2 \leq \ell \leq n-2$. Then, it has constant scalar curvature if its mean curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction. In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \eta \mathbf{w}_i$ for

$i = 1, \dots, \ell$ and $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = \ell+1, \dots, n$. Also, we have

$$\begin{aligned}
&nH_1 = \ell\lambda + (n-\ell)\eta, \;\; n(n-1)H_2 = (\ell-1)\ell\lambda^2 + (n-\ell)(n-\ell-1)\eta^2 + 2(n-\ell)\ell\lambda\eta,\\
&\mu_{i;2} = \sum_{s=0}^{2} \binom{\ell-1}{s}\binom{n-\ell}{2-s}\lambda^{2-s}\eta^s, \;\; (i = 1, \dots, \ell),\\
&\mu_{i;2} = \sum_{s=0}^{2} \binom{\ell}{s}\binom{n-\ell-1}{2-s}\lambda^{2-s}\eta^s, \;\; (i = \ell+1, \dots, n).
\end{aligned} \tag{4.8}$$

By expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2 = \sum_{i=1}^{n} \langle \nabla H_2, \mathbf{w}_i\rangle \mathbf{w}_i$, from equality (1.1) we obtain

$$\langle \nabla H_2, \mathbf{w}_i\rangle(\mu_{i,2} - \frac{3}{4}n(n-1)H_2) = 0,$$

for $i = 1, \dots, n$. On the other hand, we have $\langle \nabla H_2, \mathbf{w}_i\rangle \neq 0$ on $\mathcal{U}$ for at least one i and so

$$\mu_{i,2} = \frac{3}{4}n(n-1)H_2. \tag{4.9}$$

Two different situations can arise.

Situation 1. $\langle \nabla H_2, \mathbf{w}_i\rangle \neq 0$, for at least one $i \in \{1, \dots, \ell\}$. Equality (4.9) (using (4.8)) gives $(\ell-1)(\ell+4)\eta^2 + (n-\ell-1)(n-\ell)\lambda^2 + 2(n-\ell)(\ell+2)\eta\lambda = 0$. Hence, $\lambda = d_0\eta$, where

$$d_0 = -\left(\frac{n-\ell+2}{n-\ell-1} \pm \frac{\sqrt{n(\ell+3)(n-\ell) + (n-\ell)(5n-5\ell-4)}}{(n-\ell-1)(n-\ell)}\right).$$

Hence, we get $\eta = \frac{n}{n-(n-\ell)(1-d_0)}H_1$ and $\lambda = \frac{nd_0}{n-(n-\ell)(1-d_0)}H_1$, which give $H_2 = d_1 H_1^2$ for a constant d_1. Hence, the constancy of H_2 is guaranteed.

Situation 2. Suppose that for each $i \leq \ell$ we have $\langle \nabla H_2, \mathbf{w}_i\rangle = 0$ and for at least one $i \geq \ell+1$, $\langle \nabla H_2, \mathbf{w}_i\rangle \neq 0$. From equality (4.9), we get

$$[3n(n-1-2\ell) + \ell(5+\ell)]\eta^2 + 2\ell(n+2-\ell)\lambda\eta - [2n(n-3-2\ell) + \ell(9-\ell)]\lambda^2 = 0,$$

which gives $\ell\eta(6\eta + (n-2)\eta) = 0$. If $\eta = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = -\frac{n-2}{6}\eta$, which gives $\eta = \frac{6n}{(6-n+\ell)n-4(n-\ell)}H_1$ and $\lambda = -\frac{n(n-2)}{(6-n+\ell)n-4(n-\ell)}H_1$ and then $H_2 = d_2 H_1^2$ for a constant d_2. So, in each case, the constancy of H_2 is guaranteed.

Therefore, $\mathcal{U} = \emptyset$, then H_2 and the scalar curvature are constant on M^n. □

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