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η-RICCI-BOURGUIGNON SOLITONS ON K-PARACONTACT AND PARACONTACT METRIC $(\kappa \neq -1,\ \mu)$-MANIFOLDS

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Abstract. In this study, we focus on $(2n+1)$-dimensional K-paracontact manifolds admitting η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons. We then completely present the classification of a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1,\ \mu)$-manifold that admits a gradient η-Ricci-Bourguignon soliton. Finally, we construct examples that provide our results.

Keywords: K-paracontact manifolds, Paracontact (κ, μ)-manifolds, η-Ricci-Bourguignon solitons, Gradient η-Ricci-Bourguignon solitons.

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1. Introduction and Motivations

Geometric flows represent a powerful tool for the topological classification of manifolds, providing profound insights into their structural intricacies through the study of metric evolution over time. In this process, questions concerning the short- and long-term behavior of metrics such as whether they smooth out or develop singularities come to the forefront. Moreover, geometric flows have significant applications in physical theories, including general relativity and quantum gravity, particularly in modeling the dynamics of the universe's geometric structure. In this context, self-similar solutions to the flow, known as solitons (e.g., Ricci solitons), play a critical role in understanding the long-term behavior of the flow and

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contribute to the identification of stable or special geometric structures. This is particularly evident in the case of the Poincaré conjecture, a century-old problem that was resolved in the early 2000's by Perelman through the use of Ricci flows, [18], [19], [20]. Ricci solitons were instrumental in resolving the Poincaré conjecture, a problem that had been debated for more than a hundred years. Thus, given a geometric flow, it is natural to study the solitons associated to that flow. As a result of this, in 1981, Hamilton [11] introduced *Ricci flow* by

$$\frac{\partial}{\partial t}g\left(t\right)=-2Rc\left(t\right),$$

where Rc represents Ricci tensor of type $(0,2)$ and g is the time dependent metric of the space evolving under the flow.

Hamilton [12] also defined *Yamabe flow* as follows.

$$\frac{\partial}{\partial t}g\left(t\right)=-r\left(t\right)g\left(t\right),$$

where $r(t)$ represents the scalar curvature of the metric $g\left(t\right)$.

In 1981, a new geometric flow, named *Ricci-Bourguignon flow*, was introduced and extended the Ricci flow notation by Bourguignon [3] as follows:

$$\frac{\partial}{\partial t}g\left(t\right)=-2\left(Rc\left(t\right)-\rho r\left(t\right)g\left(t\right)\right), \tag{1.1}$$

where $\rho\in\mathbb{R}$.

Einstein flow [6] is given by

$$\frac{\partial}{\partial t}g\left(t\right)=-2(Rc\left(t\right)-\frac{r\left(t\right)}{2}g\left(t\right)).$$

Moreover, Ricci-Bourguignon flow is known as a generalization of Einstein flow. Depending on the choice of ρ, the Ricci-Bourguignon flow may turn to certain geometric flows, namely, for $\rho=\frac{1}{2}$ this flow turn to be Einstein flow, for $\rho=\frac{1}{2}(n-1)$ it will turn to the Schouten flow and for $\rho=0$ it will turn to the famous Ricci flow.

The solutions of (1.1) are called *Ricci-Bourguignon solitons (RB-solitons) or ρ-Einstein solitons* which are given in [9] by the following

$$\mathcal{L}_{\mathcal{W}}g+2\left(Rc-\rho rg\right)=2\lambda g, \tag{1.2}$$

where λ is a constant and $\mathcal{L}$ denotes the Lie derivative. λ and $\mathcal{W}$ are called soliton constant and potential vector field, respectively. If λ is a smooth function, then it is called *almost Ricci-Bourguignon soliton* [9]. The Ricci-Bourguignon soliton, a prominent concept in Riemannian geometry, arises as a solution to the Einstein field equations in the context of general relativity. These solitons, which have garnered considerable attention in recent years, play a crucial role in Riemannian geometry. Interestingly, Ricci Bourguignon solitons are critical points of the Ricci flow, and studying the flow's behavior near a soliton provides important insights into the global geometry of the manifold. Ricci-Bourguignon soliton is called *trivial* if $\mathcal{W}$ is zero or a Killing vector field (i.e. $\mathcal{L}_{\mathcal{W}}g = 0$). If $\rho = 0$ in (1.2), a *Ricci soliton* (a solution of the Ricci flow) is obtained. Theoretical physicists are fascinated by Ricci solitons because of their link to string theory and the fact that the soliton equation represents a particular instance of the Einstein field equations. A Ricci soliton extends the concept of an Einstein metric when there is a smooth, non-zero vector field $\mathcal{W}$ and a constant λ. Recently, numerous researchers have examined Ricci solitons and gradient Ricci solitons on certain types of three-dimensional almost contact metric manifolds. For instance, the study of Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds is investigated in [8]. Additionally, a comprehensive classification of Ricci solitons on three-dimensional Kenmotsu manifolds is provided in [7] and [10].

The solutions of the Einstein flow are *Einstein solitons* and Einstein solitons are given by

$$\mathcal{L}_{\mathcal{W}}g + 2(Rc - \frac{1}{2}rg) = 2\lambda g.$$

A generalization of Einstein soliton is *RB soliton (or ρ-Einstein soliton).* Also a generalization of Ricci-Bourguignon flow is *η-Ricci-Bourguignon flow* which is given by

$$\frac{\partial}{\partial t}g\left(t\right) = -2\left(Rc\left(t\right) - \rho r\left(t\right)g\left(t\right) - \sigma\eta\left(t\right)\otimes\eta\left(t\right)\right), \tag{1.3}$$

where σ and ρ are real numbers.

An essential aspect of studying any geometric flow is analyzing its associated solitons, which produce self-similar solutions to the flow and frequently serve as models for singularities. Motivated by the concept of Ricci solitons, it is intriguing to explore special solutions of the flow (1.3) which is known as a generalization of Ricci-Bourguignon soliton is *η-Ricci-Bourguignon soliton (η-RB soliton)* and is given by

$$\mathcal{L}_{\mathcal{W}}g + 2\left(Rc - \rho rg - \sigma\eta\otimes\eta\right) = 2\lambda g, \tag{1.4}$$

where σ and ρ are real numbers, if λ and σ are smooth functions, it is called an *almost η-Ricci-Bourguignon soliton* [2]. For $\rho = \frac{1}{2}$, the soliton reduces to *η-Einstein soliton* and for $\rho = 0$, it is *η-Ricci-soliton.*

The soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively.

If the potential vector field $\mathcal{W}$ is the gradient of a smooth function f, denoted by ∇f, then (1.4) can be written

$$Hessf + (Rc - \rho r g - \sigma \eta \otimes \eta) = \lambda g, \tag{1.5}$$

where $Hessf$ is the Hessian of f. (1.5) is called a *gradient η-Ricci-Bourguignon soliton*.

A significant amount of work has been contributed by various researchers to explore the geometric properties of Ricci-Bourguignon solitons. For instance, in [5], Catino et al. investigated the Ricci-Bourguignon solitons, where they discussed important rigidity results. In recent year, in [22] Shaikh et al. demonstrated that a compact gradient Ricci-Bourguignon soliton with constant scalar curvature is isometric to the Euclidean sphere. A similar result was established for a gradient Ricci-Bourguignon soliton with a vector field of bounded norm, subject to additional conditions. [21].

Recently, it is worth to mention that in [15] Mandal et al. studied η-Ricci-Bourguignon solitons on K-contact and contact (κ,μ)-manifolds. Also, in [16], Mandal et al. investigated η-Ricci-Bourguignon solitons on three-dimensional almost coKaehler manifolds. Blaga and Ozgur [1] worked on submanifolds as almost η-Ricci Bourguignon solitons.

As far as our knowledge goes, η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons on K-paracontact manifolds and paracontact $(\kappa \neq -1, \mu)$-manifolds are not studied by the researchers. This manuscript will fill these gaps.

This paper is structured as follows: In Section 2, we review some concepts essential for the discussion. Section 3 focuses on $(2n+1)$-dimensional K-paracontact manifolds which admit η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons. We proved that if a $(2n+1)$-dimensional K-paracontact manifold admits an η-Ricci-Bourguignon soliton whose potential vector field being collinear with ξ, we showed that the manifold is η-Einstein and then the scalar curvature $r = -2n\,(2n+1+\sigma)$ is constant. Also we proved that if a $(2n+1)$-dimensional K-paracontact manifold admits a gradient η-Ricci-Bourguignon soliton, then the scalar curvature is constant and the manifold is η-Einstein. In Section 4, we completely give the classification of a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold that admits a gradient η-Ricci-Bourguignon soliton.

Finally, we construct examples which verifies our results.

2. Preliminaries

In this section, we review various concepts and results that will be essential for the rest of the paper.

A smooth manifold M^{2n+1}has an *almost paracontact structure* (ϕ,ξ,η) if it possesses a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η that satisfy the compatibility conditions listed below.

$i) \phi(\xi)=0, \eta\circ\phi=0,$

$ii) \eta(\xi)=1, \phi^2=id-\eta\otimes\xi,$

$iii)$the tensor field ϕ gives rise to an almost paracomplex structure on each fibre of the horizontal distribution $\mathcal{D}=Ker\eta$ [13]

A differentiable manifold $M^{\,2n+1}$ equipped with an almost paracontact structure is referred to as an *almost paracontact manifold.*

A direct implication of the definition of an almost paracontact structure is that the endomorphism ϕ has rank $2n$.

If a manifold $M^{\,2n+1}$ endowed with (ϕ,ξ,η)-structure possesses a pseudo-Riemannian metric g such that

$$g(\phi\zeta_1,\phi\zeta_2)=-g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2), \tag{2.6}$$

for all vector fields $\zeta_1,\zeta_2\in\Gamma(M^{2n+1})$, then we say that $M^{\,2n+1}$ has an *almost paracontact metric structure* and g is called *compatible metric.* The differentiable manifold $M^{\,2n+1}$ given by the almost paracontact metric structure is called an *almost paracontact metric manifold.* Any metric g that is compatible with a given almost paracontact structure must have a signature of $(n+1,n)$.

Within the framework of almost paracontact manifolds, the tensor $N^{(1)}$ of type $(1,2)$ can be introduced by

$$N^{(1)}(\zeta_1,\zeta_2)=[\phi,\phi](\zeta_1,\zeta_2)-2d\eta(\zeta_1,\zeta_2)\xi$$

where

$$[\phi,\phi](\zeta_1,\zeta_2)=\phi^2[\zeta_1,\zeta_2]+[\phi\zeta_1,\phi\zeta_2]-\phi[\phi\zeta_1,\zeta_2]-\phi[\zeta_1,\phi\zeta_2]$$

is the Nijenhuis torsion of ϕ. The almost paracontact manifold is designated as *normal*, when $N^{(1)}=0$ [23].

Setting $\zeta_2=\xi$, we have $g(\zeta_1,\xi)=\eta(\zeta_1)$. From here and (2.6) follows

$$g(\phi\zeta_1,\zeta_2)=-g(\zeta_1,\phi\zeta_2).$$

for all vector fields $\zeta_1,\zeta_2\in\Gamma(M^{2n+1})$. In an almost paracontact metric manifold, an orthogonal basis always exists. $\{\zeta_{11},...,\zeta_{1n},\zeta_{21},...,\zeta_{2n},\xi\}$, namely ϕ-basis, such that $g(\zeta_{1i},\zeta_{1j})=-g(\zeta_{2i},\zeta_{2j})=\delta_{ij}$ and $\phi\zeta_{1i}=\zeta_{2i}$, for any $i,j\in\{1,...,n\}$.

The fundamental 2-form is defined by

$$\Phi(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2),$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

If $d\eta(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2)$ (where $d\eta(\zeta_1, \zeta_2) = \frac{1}{2}(\zeta_1\eta(\zeta_2) - \zeta_2\eta(\zeta_1) - \eta[\zeta_1, \zeta_2])$), then η is a paracontact form and the almost paracontact metric manifold is said to be *paracontact metric manifold*.

Lemma 2.1. [23] *On a paracontact metric manifold M^{2n+1}, $h = \frac{1}{2}\mathcal{L}_\xi\phi$ is a symmetric operator and satisfy the followings:*

$$\begin{aligned} trh &= tr\phi h = 0,\ h\xi = 0,\ h\phi + \phi h = 0, \\ \nabla_{\zeta_1}\xi &= -\phi\zeta_1 + \phi h\zeta_1, \\ Rc(\xi, \xi) &= -2n + trh^2, \end{aligned} \tag{2.7}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$, tr is the trace operator.

It is important to note that h is equal to zero if and only if the vector field ξ is Killing. When ξ is Killing, the paracontact metric manifold is referred to as a *K-paracontact manifold*. A normal almost paracontact metric manifold is said to be *para-Sasakian manifold* if $\Phi = d\eta$. Furthermore, a para-Sasakian manifold is also K-paracontact, with the reverse holding true solely in a three-dimensional [23].

An almost paracontact metric manifold is called *η-Einstein* if its Ricci tensor Rc takes the form of

$$Rc = ag + b\eta \otimes \eta$$

where a and b are smooth functions on the manifold.

For a K-paracontact manifold M^{2n+1}, we have the following relations [23]

$$\nabla_{\zeta_1}\xi = -\phi\zeta_1, \tag{2.8}$$

$$R(\xi, \zeta_1)\zeta_2 = -g(\zeta_1, \zeta_2)\xi + \eta(\zeta_2)\zeta_1, \tag{2.9}$$

$$Rc(\zeta_1, \xi) = -2n\eta(\zeta_1), \tag{2.10}$$

$$R(\xi, \zeta_1)\zeta_2 = (\nabla_{\zeta_1}\phi)\zeta_2, \tag{2.11}$$

$$R(\zeta_1, \xi)\xi = -\zeta_1 + \eta(\zeta_1)\xi, \tag{2.12}$$

$$(\nabla_{\phi\zeta_1}\phi)\phi\zeta_2 - (\nabla_{\zeta_1}\phi)\zeta_2 = 2g(\zeta_1, \zeta_2)\xi - (\zeta_1 + \eta(\zeta_1)\xi)\eta(\zeta_2), \tag{2.13}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$, where Q is the *Ricci operator* defined by $g(Q\zeta_1, \zeta_2) = Rc(\zeta_1, \zeta_2)$.

Also followings hold on a $(2n+1)$-dimensional K-paracontact manifold [17],

$$(\nabla_{\zeta_1} Q)\,\xi = Q\phi\zeta_1 + 2n\phi\zeta_1 \tag{2.14}$$

and

$$(\nabla_{\xi} Q)\,\zeta_1 = Q\phi\zeta_1 - \phi Q\zeta_1 \tag{2.15}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$.

On a $(2n+1)$-dimensional paracontact metric manifold, the notion of (κ, μ)-*nullity distribution* is given by

$$N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \left[\begin{array}{c} \zeta_3 \in T_pM : R(\zeta_1, \zeta_2)\,\zeta_3 = \kappa\left(g(\zeta_2, \zeta_3)\,\zeta_1 - g(\zeta_1, \zeta_3)\,\zeta_2\right) \\ +\mu\left(g(\zeta_2, \zeta_3)\,h\zeta_1 - g(\zeta_1, \zeta_3)\,h\zeta_2\right), \end{array} \right]$$

for every vector fields $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(M^{2n+1})$ and $\kappa,\ \mu \in \mathbb{R}$. If ξ belongs to above distribution, namely,

$$R(\zeta_1, \zeta_2)\,\xi = \kappa\left(\eta(\zeta_2)\,\zeta_1 - \eta(\zeta_1)\,\zeta_2\right) + \mu\left(\eta(\zeta_2)\,h\zeta_1 - \eta(\zeta_1)\,h\zeta_2\right), \tag{2.16}$$

then the paracontact metric manifold is called a paracontact metric (κ, μ)-manifold. When $\mu = 0$, a paracontact metric (κ, μ)-manifold reduces to $N(\kappa)$-paracontact metric manifold [4].

Lemma 2.2. [4]*Let M^{2n+1} be a paracontact metric (κ, μ)-manifold, then the following identities hold:*

$$h^2\zeta_1 = (1+\kappa)\,\phi^2\zeta_1, \tag{2.17}$$

$$\begin{aligned} R(\xi, \zeta_1)\,\zeta_2 &= \kappa\left[g(\zeta_1, \zeta_2)\,\xi - \eta(\zeta_2)\,\zeta_1\right] \\ &\quad +\mu\left[g(h\zeta_1, \zeta_2)\,\xi - \eta(\zeta_2)\,h\zeta_1\right], \end{aligned} \tag{2.18}$$

$$(\nabla_{\zeta_1}\eta)\,\zeta_2 = g(\zeta_1 - h\zeta_1, \phi\zeta_2), \tag{2.19}$$

for all vector fields $\zeta_1,\ \zeta_2 \in \Gamma(M^{2n+1})$.

Lemma 2.3. [4]*Let M^{2n+1} be a paracontact metric $(\kappa \neq -1, \mu)$-manifold, then the following identities hold:*

$$(\nabla_{\zeta_1}\phi)\,\zeta_2 = -g(\zeta_1 - h\zeta_1, \zeta_2)\,\xi + \eta(\zeta_2)\,(\zeta_1 - h\zeta_1)\ \ , \tag{2.20}$$

$$Rc(\zeta_1, \xi) = 2n\kappa\eta(\zeta_1), \tag{2.21}$$

$$\begin{aligned} Rc(\zeta_1,\zeta_2) &= [2(1-n)+n\mu]\, g(\zeta_1,\zeta_2)+[2(n-1)+\mu]\, g(h\zeta_1,\zeta_2) \\ &\quad +[2(n-1)+n(2\kappa-\mu)]\,\eta(\zeta_1)\,\eta(\zeta_2), \end{aligned} \tag{2.22}$$

$$\begin{aligned} (\nabla_{\zeta_1} h)\,\zeta_2 &= -[(1+\kappa)\, g(\zeta_1,\phi\zeta_2)+g(\zeta_1,\phi h\zeta_2)]\,\xi \\ &\quad +\eta(\zeta_2)\,[(1+\kappa)\,\phi\zeta_1-\phi h\zeta_1]-\mu\eta(\zeta_1)\,\phi h\zeta_2, \end{aligned} \tag{2.23}$$

$$Q\xi = 2n\kappa\xi, \tag{2.24}$$

$$r = 2n\,[2(1-n)+\kappa+n\mu], \tag{2.25}$$

for all vector fields $\zeta_1,\ \zeta_2 \in \Gamma(M^{2n+1})$.

Theorem 2.1. [24] *Let* M^{2n+1} *be a paracontact metric manifold and suppose that* $R(\zeta_1,\zeta_2)\,\xi = 0$ *for all vector fields* ζ_1 *and* ζ_2*. Then locally* M^{2n+1} *is the product of a flat* $(n+1)$*-dimensional manifold and* n*-dimensional manifold of negative constant curvature equal to* -4*, for* $n > 1$ *and its locally flat for* $n = 1$*.*

Lemma 2.4. *On a paracontact metric* (κ,μ)*-manifold* M^{2n+1}*, we have*

$$(\nabla_\xi h)\,\zeta_1 = \mu h\phi\zeta_1, \tag{2.26}$$

$$(\nabla_\xi Q)\,\zeta_1 = \mu\,[2(n-1)+\mu]\, h\phi\zeta_1, \tag{2.27}$$

$$(\nabla_{\zeta_1} Q)\,\xi = Q(\phi\zeta_1-\phi h\zeta_1)-2n\kappa(\phi\zeta_1-\phi h\zeta_1),\ \kappa\neq -1 \tag{2.28}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$.

Proof. If we write $\zeta_1 = \xi$ in (2.23), we obtain (2.26).

From (2.22), we get

$$Q\zeta_1 = [2(1-n)+n\mu]\,\zeta_1+[2(n-1)+\mu]\, h\zeta_1+[2(n-1)+n(2\kappa-\mu)]\,\eta(\zeta_1)\,\xi. \tag{2.29}$$

If we take the covariant derivative of (2.29) along ξ and use (2.26), we have (2.27). If we take the covariant derivative of (2.24) along ζ_1 and use (2.7), we obtain (2.28). □

3. η-Ricci-Bourguignon and Gradient η-Ricci-Bourguignon Solitons on K-Paracontact Manifolds

In this section, we will investigate η-Ricci-Bourguignon and Gradient η-Ricci-Bourguignon solitons on K-paracontact manifolds.

Theorem 3.1. *Let M^{2n+1} be a K-paracontact manifold. If M^{2n+1} admits an η-Ricci-Bourguignon soliton whose potential vector field being collinear with ξ, the manifold is η-Einstein and the scalar curvature $r = -2n(2n+1+\sigma)$ is constant.*

Proof. Now assume that $\mathcal{W}$=$f\xi$, where f is a smooth function. Letting $\mathcal{W}$ by $f\xi$ and using (2.8) in (1.4), we get

$$Rc(\zeta_1, \zeta_2) + \frac{1}{2}(\zeta_1(f)\eta(\zeta_2) + \zeta_2(f)\eta(\zeta_1)) = (\lambda + \rho r)\, g(\zeta_1, \zeta_2) + \sigma\eta(\zeta_1)\,\eta(\zeta_2)\,. \tag{3.30}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

Putting ζ_2 by ξ in (3.30), we have

$$Rc(\zeta_1, \xi) + \frac{1}{2}(\zeta_1(f) + \xi(f)\eta(\zeta_1)) = (\lambda + \rho r)\,\eta(\zeta_1) + \sigma\eta(\zeta_1)\,. \tag{3.31}$$

Using (2.10) in (3.31), we get

$$gradf = (2(\lambda + \rho r) + 2\sigma - \xi(f) + 4n)\,\xi. \tag{3.32}$$

On the other hand putting $\zeta_1 = \zeta_2 = \xi$ and using again (2.10) in (3.30), we have

$$-2n + \xi(f) = \lambda + \rho r + \sigma. \tag{3.33}$$

If we use (3.33) in (3.32), we obtain

$$gradf = \xi(f)\xi. \tag{3.34}$$

If we take the covariant derivative of (3.34) along ζ_1 and using (2.8), we get

$$g(\nabla_{\zeta_1} gradf, \zeta_2) = \xi(f) g(\nabla_{\zeta_1}\xi, \zeta_2) + \zeta_1(\xi(f))\eta(\zeta_2) \tag{3.35}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

By using $g(\nabla_{\zeta_1} gradf, \zeta_2) = g(\nabla_{\zeta_2} gradf, \zeta_1)$ we have

$$\zeta_1(\xi(f))\eta(\zeta_2) - \zeta_2(\xi(f))\eta(\zeta_1) = -2\xi(f)d\eta(\zeta_1, \zeta_2) \tag{3.36}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$. Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (3.36), we obtain $\xi(f) = 0$, because of $d\eta \neq 0$. So from (3.34), we have $gradf = 0$, namely f is constant and so the manifold is η-Einstein.

Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis. Taking the summation over i for ζ_1 $= \zeta_2 = w_i$ in (3.30), we obtain

$$r = (\lambda + \rho r)(2n+1) + \sigma. \tag{3.37}$$

Using (3.33) in (3.37), we get

$$r = -2n(2n+1+\sigma)$$

which completes the proof. □

Theorem 3.2. *If a $(2n+1)$-dimensional K-paracontact manifold admits a gradient η-Ricci-Bourguignon soliton, then the scalar curvature is constant and the manifold is η-Einstein.*

Proof. By virtue of (1.5), we have

$$\nabla_{\zeta_1} gradf = -Q\zeta_1 + (\lambda + \rho r)\zeta_1 + \sigma\eta(\zeta_1)\xi. \tag{3.38}$$

Taking the covariant derivative of (3.38) with ζ_2 and using (2.8), we get

$$\nabla_{\zeta_2}\nabla_{\zeta_1} gradf = -\nabla_{\zeta_2} Q\zeta_1 + (\lambda + \rho r)\nabla_{\zeta_2}\zeta_1 + \rho\zeta_2(r)\zeta_1 + \sigma(\nabla_{\zeta_2}\eta(\zeta_1)\xi - \eta(\zeta_1)\phi\zeta_2). \tag{3.39}$$

Interchanging ζ_1 and ζ_2 in the last equation, we derive

$$\nabla_{\zeta_1}\nabla_{\zeta_2} gradf = -\nabla_{\zeta_1} Q\zeta_2 + (\lambda + \rho r)\nabla_{\zeta_1}\zeta_2 + \rho\zeta_1(r)\zeta_2 + \sigma(\nabla_{\zeta_1}\eta(\zeta_2)\xi - \eta(\zeta_2)\phi\zeta_1). \tag{3.40}$$

From (3.38), we obtain

$$\nabla_{[\zeta_1,\zeta_2]} gradf = -Q[\zeta_1,\zeta_2] + (\lambda + \rho r)[\zeta_1,\zeta_2] + \sigma\eta([\zeta_1,\zeta_2])\xi. \tag{3.41}$$

In the view of (3.39), (3.40) and (3.41), we can compute

$$\begin{aligned} R(\zeta_1,\zeta_2) gradf &= -(\nabla_{\zeta_1} Q)\zeta_2 + (\nabla_{\zeta_2} Q)\zeta_1 + \rho(\zeta_1(r)\zeta_2 - \zeta_2(r)\zeta_1) \\ &\quad +\sigma(-2g(\phi\zeta_1,\zeta_2)\xi + \eta(\zeta_1)\phi\zeta_2 - \eta(\zeta_2)\phi\zeta_1). \end{aligned} \tag{3.42}$$

Contracting the last equation over ζ_1 and using

$$divQ\zeta_2 = \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{w_i} Q)\zeta_2, w_i) = \frac{1}{2}\zeta_2(r).$$

We conclude that

$$Rc(\zeta_2, gradf) = \left(\frac{1}{2} - 2n\rho\right)\zeta_2(r). \tag{3.43}$$

By (2.10), we have

$$Rc(gradf, \xi) = -2n\xi(f). \tag{3.44}$$

Since ξ is Killing, $\xi(r) = 0$. Putting $\zeta_2 = \xi$ in (3.43) and using (3.44), we get $\xi(f) = 0$.

Taking the inner product of (3.42) with ξ and using equation (2.14) , we obtain

$$\begin{aligned} g\left(R\left(gradf,\xi\right)\zeta_1,\zeta_2\right) &= g\left(Q\phi\zeta_2,\zeta_1\right)-g\left(Q\phi\zeta_1,\zeta_2\right)-2\left(2n+\sigma\right)g\left(\phi\zeta_1,\zeta_2\right) \\ &\quad +\rho\left[\zeta_1\left(r\right)\eta\left(\zeta_2\right)-\zeta_2\left(r\right)\eta\left(\zeta_1\right)\right]. \end{aligned} \tag{3.45}$$

Replacing ζ_1 by ξ in (3.45) and using the fact that $\xi(r)=0$ and $\xi(f)=0$, equations (2.9) and (2.10), we have

$$\zeta_2\left(f-\rho r\right)=0,$$

this leads to the conclusion that $f-\rho r$ is a constant.

Substituting $\zeta_2=\xi$ in (3.42) and taking the inner product with ζ_2 and using (2.11), (2.14) and (2.15) we get

$$g((\nabla_{\zeta_1}\phi)\zeta_2,gradf)=-\left(2n+\sigma\right)g\left(\phi\zeta_1,\zeta_2\right)-g\left(\phi Q\zeta_1,\zeta_2\right)+\rho\zeta_1\left(r\right)\eta\left(\zeta_2\right). \tag{3.46}$$

First, if we replace ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (3.46) and then subtract (3.46) from the obtained equation, we obtain following equation

$$Q\phi\zeta_1+\phi Q\zeta_1=-2\left(2n+\sigma\right)\phi\zeta_1, \tag{3.47}$$

by using (2.13) and $\xi(f)=0$.

Let $\{w_i\}$ $(1\leq i\leq 2n+1)$ be an orthonormal basis, after writing $\zeta_1=w_i$ in (3.47), we have

$$Q\phi w_i+\phi Qw_i=-2\left(2n+\sigma\right)\phi w_i. \tag{3.48}$$

Moreover, we can calculate following

$$g\left(\phi Qw_i,\phi w_i\right)=-g\left(Qw_i,\phi^2w_i\right)=-g\left(Qw_i,w_i\right). \tag{3.49}$$

By virtue of (3.48) and (3.49), we get

$$\begin{aligned} r &= Rc\left(\xi,\xi\right)+\sum_{i=1}^{n}\left\{Rc\left(w_i,w_i\right)-Rc\left(\phi w_i,\phi w_i\right)\right\} \\ &= -2n+\sum_{i=1}^{n}\left\{-g\left(\phi Qw_i+Q\phi w_i,\phi w_i\right)\right\} \\ &= -2n\left(2n+1\right)-2n\sigma. \end{aligned}$$

constant, so from $f-\rho r$ is constant, we have f is constant. Hence from $\mathcal{W}=gradf$, $\mathcal{W}=0$. By (1.5), the manifold is η-Einstein. This concludes the proof. ☐

4. Gradient η-Ricci-Bourguignon Solitons on Paracontact $(\kappa \neq -1, \mu)$-Manifolds

In this section, we will investigate gradient η-Ricci-Bourguignon solitons on paracontact metric $(\kappa \neq -1, \mu)$-manifolds.

Lemma 4.1. *If a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold admits a gradient η-Ricci-Bourguignon soliton, then we have*

$$\kappa (2-\mu) = \mu (n+1) + \sigma. \tag{4.50}$$

Proof. By virtue of (1.5), we have

$$\nabla_{\zeta_1} gradf + Q\zeta_1 = (\lambda + \rho r)\, \zeta_1 + \sigma \eta (\zeta_1)\, \xi. \tag{4.51}$$

Taking the covariant derivative of (4.51) with ζ_2 and using (2.7), we get

$$\nabla_{\zeta_2} \nabla_{\zeta_1} gradf + \nabla_{\zeta_2} Q\zeta_1 = (\lambda + \rho r)\, \nabla_{\zeta_2} \zeta_1 + \sigma \left(\nabla_{\zeta_2} \eta (\zeta_1)\, \xi - \eta (\zeta_1)\, \phi\zeta_2 + \eta (\zeta_1)\, \phi h\zeta_2 \right). \tag{4.52}$$

Interchanging ζ_1 and ζ_2 in the last equation, we obtain

$$\nabla_{\zeta_1} \nabla_{\zeta_2} gradf + \nabla_{\zeta_1} Q\zeta_2 = (\lambda + \rho r)\, \nabla_{\zeta_1} \zeta_2 + \sigma \left(\nabla_{\zeta_1} \eta (\zeta_2)\, \xi - \eta (\zeta_2)\, \phi\zeta_1 + \eta (\zeta_2)\, \phi h\zeta_1 \right). \tag{4.53}$$

From (4.51), we have

$$\nabla_{[\zeta_1, \zeta_2]} gradf + Q\, [\zeta_1, \zeta_2] = (\lambda + \rho r)\, [\zeta_1, \zeta_2] + \sigma \eta \left([\zeta_1, \zeta_2] \right) \xi. \tag{4.54}$$

In the view of (4.52), (4.53) and (4.54), we can compute

$$\begin{aligned} R(\zeta_1, \zeta_2)\, gradf &= -\left(\nabla_{\zeta_1} Q \right) \zeta_2 + \left(\nabla_{\zeta_2} Q \right) \zeta_1 \\ &\quad + \sigma \left(2g (\zeta_1, \phi\zeta_2)\, \xi + \eta (\zeta_1)\, \phi\zeta_2 - \eta (\zeta_1)\, \phi h\zeta_2 - \eta (\zeta_2)\, \phi\zeta_1 + \eta (\zeta_2)\, \phi h\zeta_1 \right) \end{aligned} \tag{4.55}$$

Using (2.28) in (4.55), we obtain

$$\begin{aligned} g (R (\zeta_1, \zeta_2)\, gradf, \xi) &= g \left((Q\phi + \phi Q)\, \zeta_2, \zeta_1 \right) - g \left((Q\phi h + h\phi Q)\, \zeta_2, \zeta_1 \right) \\ &\quad - 4n\kappa g (\phi\zeta_2, \zeta_1) + 2\sigma g (\zeta_1, \phi\zeta_2)\, . \end{aligned} \tag{4.56}$$

Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (4.56) and using the fact that $R(\phi\zeta_1, \phi\zeta_2)\, \xi = 0$ from (2.16), we get

$$0 = \phi \left(-\left(Q\phi + \phi Q \right) \phi\zeta_1 + \left(Q\phi h + h\phi Q \right) \phi\zeta_1 + 4n\kappa\zeta_1 - 2\sigma\zeta_1 \right). \tag{4.57}$$

From (2.29), we can compute

$$\phi\left(Q\phi+\phi Q\right)\phi\zeta_1 = 2\left(2\left(1-n\right)+n\mu\right)\phi\zeta_1. \tag{4.58}$$

$$\phi\left(Q\phi h+h\phi Q\right)\phi\zeta_1 = -2\left(\kappa+1\right)\left(2\left(n-1\right)+\mu\right)\phi\zeta_1. \tag{4.59}$$

If we use (4.58) and (4.59) in (4.57), we get (4.50). □

Theorem 4.1. *If a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold admits a gradient η-Ricci-Bourguignon soliton, then either*

i) The manifold is η-Einstein, $\kappa = 0$, $\mu = 2(1-n)$, $r = 4n(1-n^2)$, or

ii) The manifold is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4 for $n > 1$ and its locally flat for $n = 1$, or

iii) The manifold is η-Einstein, $\kappa = \frac{1-n^2}{n} + \frac{\sigma}{2n}$, $\mu = 2(1-n)$, $r = 2(1-n^2)(1+2n)+\sigma$, or

iv) The manifold is paracontact metric $\left(\kappa > -1,\ \mu = \pm\frac{\kappa}{\sqrt{\kappa+1}}\right)$-manifold.

Proof. Substituting $\zeta_1 = \xi$ in (4.55) and then using (2.27) and (2.28), we get

$$R\left(\xi,\zeta_2\right)gradf = -\mu\left[2\left(n-1\right)+\mu\right]h\phi\zeta_2+Q\left(\phi\zeta_2-\phi h\zeta_2\right)-2n\kappa\left(\phi\zeta_2-\phi h\zeta_2\right)+\sigma\left(\phi\zeta_2-\phi h\zeta_2\right). \tag{4.60}$$

Putting $\zeta_1 = \zeta_2$, $\zeta_2 = gradf$ in (2.18), we obtain

$$R\left(\xi,\zeta_2\right)gradf = \kappa\left[\zeta_2\left(f\right)\xi-\xi\left(f\right)\zeta_2\right]+\mu\left[\left(h\zeta_2\right)\left(f\right)\xi-\xi\left(f\right)h\zeta_2\right]. \tag{4.61}$$

By equating the right-hand sides of equations (4.60) and (4.61) and subsequently taking the inner product of the resulting equation with ξ, we obtain

$$\kappa\left[\zeta_2\left(f\right)-\xi\left(f\right)\eta\left(\zeta_2\right)\right]+\mu\left[\left(h\zeta_2\right)\left(f\right)\right]=0. \tag{4.62}$$

If we substitute ζ_2 by $h\zeta_2$ in (4.62) and use (2.17), we get

$$\kappa\left(h\zeta_2\right)\left(f\right)+\mu\left(\kappa+1\right)\left[\zeta_2\left(f\right)-\eta\left(\zeta_2\right)\xi\left(f\right)\right]=0. \tag{4.63}$$

Combining (4.62) and (4.63), we obtain

$$\left[\zeta_2\left(f\right)-\xi\left(f\right)\eta\left(\zeta_2\right)\right]\left[\kappa^2-\mu^2\left(\kappa+1\right)\right]=0. \tag{4.64}$$

Contracting (4.55) over ζ_1 and using

$$divQ\zeta_2 = \sum_{i=1}^{2n+1}\varepsilon_i g((\nabla_{w_i}Q)\zeta_2, w_i) = \frac{1}{2}\zeta_2(r).$$

We conclude that

$$Rc\left(\zeta_2, gradf\right) = 0. \tag{4.65}$$

In the view of (2.22) and (4.65), we get

$$\begin{aligned} 0 &= [2(1-n)+n\mu]\, g(\zeta_1, gradf) + [2(n-1)+\mu]\, g(h\zeta_1, gradf) \\ &\quad + [2(n-1)+n(2\kappa-\mu)]\, \eta(\zeta_1)\, \eta(gradf). \end{aligned} \tag{4.66}$$

Substituting $\zeta_1 = \xi$ in (4.66), we have

$$2n\kappa\xi(f) = 0.$$

This gives either $\kappa = 0$, or $\xi(f) = 0$.

Case 1: Let $\kappa = 0$. From (4.64), we have

$$[gradf - \xi(f)\,\xi]\,\mu^2 = 0. \tag{4.67}$$

By (4.67), we have followings:

Case 1a: Let $\mu \neq 0$. So we obtain

$$gradf = \xi(f)\,\xi. \tag{4.68}$$

If we take the covariant derivative of (4.68) along ζ_1 and using (2.7), we get

$$g(\nabla_{\zeta_1} gradf, \zeta_2) = \xi(f) g(\nabla_{\zeta_1}\xi, \zeta_2) + \zeta_1(\xi(f))\eta(\zeta_2) \tag{4.69}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

By using $g(\nabla_{\zeta_1} gradf, \zeta_2) = g(\nabla_{\zeta_2} gradf, \zeta_1)$ we have

$$\zeta_1(\xi(f))\eta(\zeta_2) - \zeta_2(\xi(f))\eta(\zeta_1) = -2\xi(f)d\eta(\zeta_1, \zeta_2) \tag{4.70}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$. Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (4.70), we obtain $\xi(f) = 0$, because of $d\eta \neq 0$. So from (4.68), we have $gradf = 0$, namely f is constant and so the manifold is η-Einstein. So from (2.29), we obtain $\mu = 2(1-n)$. Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis. Taking the summation over i for $\zeta_1 = \zeta_2 = w_i$ in (2.29), we obtain $r = 4n\left(1-n^2\right)$. Note that in this subcase the scalar curvature can not be positive.

Case 1b: Let $\mu = 0$. So we can use Theorem 2.1.

Case 2: Let $\xi(f) = 0$. By (4.64) we have

$$gradf\left(\kappa^2 - \mu^2(\kappa+1)\right) = 0. \tag{4.71}$$

By (4.71), we have followings:

Case 2a: Let $gradf = 0$. Namely f is constant. So the manifold is η-Einstein. So from (2.29), we obtain $\mu = 2(1-n)$. Using this in (4.50), we get $\kappa = \frac{1-n^2}{n} + \frac{\sigma}{2n}$. Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis. Taking the summation over i for $\zeta_1 = \zeta_2 = w_i$ in (2.29), we obtain $r = 2(1-n^2)(1+2n) + \sigma$.

Case 2b: Let $\kappa^2 - \mu^2(\kappa+1) = 0$. We want to remind that $\kappa \neq -1$. It means that $\kappa > -1$ or $\kappa < -1$. Firstly let us suppose that $\kappa < -1$. In this case we say that $\kappa = 0$ and $\mu = 0$. But this case is contradiction with the assumption that $\kappa < -1$. Therefore, κ must be bigger than -1. Now, from $\kappa^2 - \mu^2(\kappa+1) = 0$, we obtain $\mu = \pm\frac{\kappa}{\sqrt{\kappa+1}}$ Namely the manifold is paracontact metric $\left(\kappa > -1,\, \mu = \pm\frac{\kappa}{\sqrt{\kappa+1}}\right)$-manifold. This concludes the proof. □

5. Examples

Example 5.1. *We consider the three-dimensional manifold M. Define the almost paracontact structure (ϕ, ξ, η) on M by*

$$\phi\xi = 0,\ \phi w_1 = w_2,\ \phi w_2 = w_1, \xi = w_3.$$

We have

$$[w_1, w_3] = 0, \quad [w_2, w_3] = 0, \quad [w_1, w_2] = -2\xi.$$

Let g be the semi-Riemannian metric defined by

$$g(w_2, w_2) = -1,\ g(w_1, w_1) = g(\xi, \xi) = 1,\ g(w_i, w_j) = 0,\ i \neq j$$

where $i, j = 1, 2, 3$. Let ∇ be the Levi-Civita connection with respect to g. Then by Koszul formula

$$\begin{aligned}
\nabla_{w_1} w_1 &= 0,\ \nabla_{w_2} w_1 = \xi,\ \nabla_{w_3} w_1 = -w_2, \\
\nabla_{w_1} w_2 &= -\xi,\ \nabla_{w_{w_2}} w_2 = 0,\ \nabla_{w_3} w_2 = -w_1, \\
\nabla_{w_1} w_3 &= -w_2,\ \nabla_{w_2} w_3 = -w_1, \nabla_{w_3} w_3 = 0.
\end{aligned}$$

It is easy to see that M is a K-paracontact manifold. The components of the curvature tensor are

$$\begin{aligned}
R(w_1, w_2)w_2 &= -3w_1, \quad R(w_1, w_2)w_3 = 0,\ R(w_3, w_2)w_2 = \xi, \\
R(w_1, w_3)w_3 &= -w_1,\ R(w_2, w_3)w_3 = -w_2, \quad R(w_1, w_3)w_2 = 0, \\
R(w_2, w_1)w_1 &= 3w_2,\ R(w_3, w_1)w_1 = -\xi, \quad R(w_2, w_3)w_1 = 0.
\end{aligned}$$

Using the components of the curvature tensor, we obtain

$$Rc(w_1, w_1) = 2,\ Rc(w_2, w_2) = -2,\ Rc(\xi, \xi) = -2$$

In view of above relations, we have $r = S(w_1, w_1) - S(w_2, w_2) + S(\xi, \xi) = 2$. *Using (1.4), we have*

$$Rc(w_1, w_1) = \lambda + \rho r = 2,\ Rc(w_2, w_2) = -(\lambda + \rho r) = -2,\ Rc(\xi, \xi) = \lambda + \rho r + \sigma = -2. \quad (5.72)$$

From (5.72), we get $\lambda + 2\rho = 2$ *and* $\sigma = -4$. *Hence we see that* M *admits an* η*-Ricci-Bourguignon soliton with* $\sigma = -4$, *for* $\mathcal{W}=f\xi$, f *constant.* M *is also* η*-Einstein manifold and verifies Theorem 3.1. Also the soliton is shrinking, steady or expanding according as* $2(1-\rho) > 0$, $2(1-\rho) = 0$ *and* $2(1-\rho) < 0$, *respectively.*

We used [14] while constructing following examples.

Example 5.2. *Let* M *be a three-dimensional manifold.* $w_1 = w$, $w_2 = \phi w$ *and* $w_3 = \xi$ *are vector fields such that*

$$[w, \xi] = (\tilde{\lambda} - 1)\phi w, \quad [\phi w, \xi] = -(\tilde{\lambda} + 1)w, \quad [w, \phi w] = 2\xi.$$

The semi-Riemannian metric g is defined by

$$g(w, w) = -1, \quad g(\phi w, \phi w) = g(\xi, \xi) = 1, \ g(w_i, w_j) = 0, \ i \neq j$$

where $i, j = 1, 2, 3$. *The 1-form* η *is defined by*

$$\eta(\zeta_1) = g(\zeta_1, \xi)$$

for all ζ_1 *on* M. *Let* ϕ *be the* $(1,1)$*-tensor field defined by*

$$\phi\xi = 0,\ \phi w_1 = w_2,\ \phi w_2 = w_1.$$

Then,

$$\begin{aligned} \eta(\xi) &= 1,\ \phi^2(\zeta_1) = \zeta_1 - \eta(\zeta_1)\xi \\ g(\phi\zeta_1, \phi\zeta_2) &= -g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2),\ d\eta(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2), \end{aligned}$$

for any vector fields ζ_1, ζ_2 *on* M. *Hence* (ϕ, ξ, η, g) *defines a paracontact structure. Let* ∇ *be the Levi-Civita connection on* M, *then by Koszul's formula, we obtain*

$$\begin{aligned}
\nabla_w w &= 0,\ \nabla_{\phi w} w = -(\tilde{\lambda}+1)\xi,\ \nabla_\xi w = 0,\\
\nabla_w \phi w &= (1-\tilde{\lambda})\xi,\ \nabla_{\phi w}\phi w = 0,\ \nabla_\xi \phi w = 0,\\
\nabla_w \xi &= (\tilde{\lambda}-1)\phi w,\quad \nabla_{\phi w}\xi = -(\tilde{\lambda}+1)w,\ \nabla_\xi \xi = 0.
\end{aligned}$$

Using the covariant derivatives, we compute the components of the Riemannian curvature tensor:

$$\begin{aligned}
R(w,\phi w)\phi w &= (1-\tilde{\lambda}^2)w,\quad R(\phi w,\xi)\xi = (\tilde{\lambda}^2-1)\phi w,\ R(w,\phi w)\xi = 0,\\
R(w,\xi)\xi &= (\tilde{\lambda}^2-1)w,\ R(\xi,w)w = (1-\tilde{\lambda}^2)\xi,\qquad R(w,\xi)\phi w = 0,\\
R(\phi w,w)w &= (\tilde{\lambda}^2-1)\phi e,\ R(\xi,\phi w)\phi w = (\tilde{\lambda}^2-1)\xi\ ,\ R(\phi w,\xi)w = 0.
\end{aligned}$$

Also, the followings are valid:

$$hw = \tilde{\lambda}w,\ \ h\phi w = -\tilde{\lambda}\phi w,\ \ h\xi = 0.$$

$$\begin{aligned}
Qw &= (1-\tilde{\lambda}^2+\frac{r}{2})w,\\
Q\phi w &= (1-\tilde{\lambda}^2+\frac{r}{2})\phi w,\\
Q\xi &= 2(\tilde{\lambda}^2-1)\xi. \qquad (5.73)
\end{aligned}$$

Thus, the manifold is a $(\kappa \neq -1{,}0)$*-paracontact metric manifold with* $\kappa = \tilde{\lambda}^2 - 1 > -1$.

From the components of the Riemannian curvature tensor, we derive $Rc(w,w) = 0$, $Rc(\phi w,\phi w) = 0$, $Rc(\xi,\xi) = 2\tilde{\lambda}^2 - 2$. *Hence, the scalar curvature* $r = 2(\tilde{\lambda}^2-1) = 2\kappa$. *Then, using this, (1.5) and (5.73) we get*

$$(-1+\tilde{\lambda}^2-\frac{r}{2}+\lambda+\rho r)w = 0,\ (-1+\tilde{\lambda}^2-\frac{r}{2}+\lambda+\rho r)\phi w = 0, (-2\tilde{\lambda}^2+2+\lambda+\rho r+\sigma)\xi = 0. \quad (5.74)$$

By (5.74), we get $\lambda+\rho r = 1-\tilde{\lambda}^2+\frac{r}{2}$ *and* $r = \sigma$. *If we use* $r = 2(\tilde{\lambda}^2-1)$ *in the last equation we have* $\lambda+\rho r = 0$. *Hence we see that* M *admits gradient* η*-Ricci-Bourguignon soliton with* $\sigma = 2(\tilde{\lambda}^2-1) = r$ *and constant* f. M *is also* η*-Einstein manifold and verifies Theorem 4.1.*

Example 5.3. *Let* M *be a three-dimensional manifold.* $w_1 = w$, $w_2 = \phi w$ *and* $w_3 = \xi$ *are vector fields such that*

$$[w,\xi] = 2w - \phi w,\quad [\phi w,\xi] = -w - 2\phi w,\quad [w,\phi w] = 2\xi.$$

The semi-Riemannian metric g is defined by

$$g(w,w)=-1,\ \ g(\phi w,\phi w)=g(\xi,\xi)=1,\ g(w_i,w_j)=0,\ i\neq j$$

where $i,j=1,2,3$. *The* 1*-form* η *is defined by*

$$\eta(\zeta_1)=g(\zeta_1,\xi)$$

for all ζ_1 *on* M. *Let* ϕ *be the* $(1,1)$*-tensor field defined by*

$$\phi\xi=0,\ \phi w_1=w_2,\ \phi w_2=w_1.$$

Then,

$$\begin{aligned}\eta(\xi) &= 1,\ \phi^2(\zeta_1)=\zeta_1-\eta(\zeta_1)\xi\\ g(\phi\zeta_1,\phi\zeta_2) &= -g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2),\ d\eta(\zeta_1,\zeta_2)=g(\zeta_1,\phi\zeta_2),\end{aligned}$$

for any vector fields ζ_1, ζ_2 *on* M. *Hence* (ϕ,ξ,η,g) *defines a paracontact structure. Let* ∇ *be the Levi-Civita connection on* M*, then by Koszul's formula, we obtain*

$$\begin{aligned}\nabla_w w &= 2\xi,\ \nabla_{\phi w}w=-\xi,\quad \nabla_\xi w=0,\\ \nabla_w\phi w &= \xi,\ \nabla_{\phi w}\phi w=2\xi,\ \nabla_\xi\phi w=0,\\ \nabla_w\xi &= -\phi w+2w,\quad \nabla_{\phi w}\xi=-w-2\phi w,\ \nabla_\xi\xi=0.\end{aligned}$$

Using the covariant derivatives, we compute the components of the Riemannian curvature tensor:

$$\begin{aligned}R(w,\phi w)\phi w &= 5w,\ R(\phi w,\xi)\xi=-5\phi w,\ R(w,\phi w)\xi=0,\\ R(w,\xi)\xi &= -5w,\quad R(\xi,w)w=5\xi,\ R(w,\xi)\phi w=0,\\ R(\phi w,w)w &= -5\phi w,\ R(\xi,\phi w)\phi w=-5\xi,\ R(\phi w,\xi)w=0.\end{aligned}$$

Also, the followings are valid:

$$hw=\tilde{\lambda}\phi w,\ \ h\phi w=-\tilde{\lambda}w,\ \ h\xi=0.$$

$$\begin{aligned}Qw &= (5+\frac{r}{2})w,\\ Q\phi w &= (5+\frac{r}{2})\phi w,\\ Q\xi &= -10\xi. \qquad (5.75)\end{aligned}$$

Thus, the manifold is a $(\kappa \neq -1{,}0)$*-paracontact metric manifold with* $\kappa = -5 < -1$.

From the components of the Riemannian curvature tensor, we derive $Rc(w,w) = 0$, $Rc(\phi w, \phi w) = 0$, $Rc(\xi,\xi) = -10$. *Hence, the scalar curvature* $r = -10 = 2\kappa$. *Then, using this, (1.5) and (5.75) we get*

$$(\lambda - 10\rho)w = 0,\ (\lambda - 10\rho)\phi w = 0,\ (10 + \lambda - 10\rho + \sigma)\xi = 0. \tag{5.76}$$

By (5.76), we get $\lambda - 10\rho = 0$ *and* $r = \sigma = -10$. *Hence we see that* M *admits a gradient* η*-Ricci-Bourguignon soliton with* $\sigma = -10$ *and constant* f. M *is also* η*-Einstein manifold and verifies Theorem 4.1.*

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