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ON GEOMETRY OF THE PARALLEL SURFACE OF THE TUBE SURFACE GIVEN BY THE FLC FRAME IN EUCLIDEAN 3-SPACE

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Abstract. In this study, first, the parallel surfaces of the tube surfaces given with the Flc frame are defined. By calculating the Gaussian and mean curvatures of these parallel surfaces, it was found the conditions developable and minimal. Afterwards, the conditions for parameter curves on the parallel surface to be asymptotic, geodesic and curvature lines were investigated. It has been proven that the tube and parallel tube surface preserve the Gaussian transform. Finally, examples of these surfaces are given.

Keywords: Parallel surface, Tubular surface, Flc frame, Gaussian transform.

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1. Introduction

It is known that two surfaces with a common normal are called parallel surfaces. Parallel surfaces have various uses in the design field and in the modeling of forging casting molds [25]. It has been one of the surfaces that has been the focus of attention of many mathematicians from past to present,[22, 8, 9, 10, 1, 11]. A large number of papers and books have been published in the literature which deal with parallel surfaces in both Minkowski space and Euclidean space. Kılıç showed that if a parallel transformation on E^n is a connection-preserving transformation, the fundamental curvatures of the underlying surface are constant [14]. Taleshian used Euler's theorem to examine the orthogonal curvatures of parallel hypersurfaces and stated that if the parallel transformation preserves the second fundamental form,

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the fundamental hypersurface defines a hyperplane [23]. Fukui and Hasegawa studied the singularities of parallel surfaces [12]. Önder and Kızıltuğ gave the relations between Bertrand and Mannheim partner D-curves on parallel surfaces in 3-dimensional Minkowski space [19]. Dede, Ekici and Çöken first defined parallel surfaces in Galilean space and examined the relationship between them, and then obtained the first, second fundamental forms and Gaussian, mean curvatures of the parallel surface depending on the first, second fundamental forms and Gaussian, mean curvatures of the main surface [5]. Savcı studied the relationship between the Darboux frame, geodesic curvatures, normal curvatures, and geodesic torsions of the curves lying on the parallel surface pair, showed that the parallel surface of a non-developable ruled surface is not a ruled surface, and obtained that the parallel surface of a Weingarten ruled surface is also a ruled Weingarten surface [20].

Craig worked on parallel surfaces of the ellipsoid [2]. Eisenhart wrote a section on parallel surfaces in his work "A treatise on the differential geometry of curves and surfaces" [7]. Nizamoğlu stated that the parallel ruled surface is a curve that depends on a parameter and gave some geometrical properties of such a surface [18]. Hacısalihoğlu and Tarakcı defined surfaces with constant ridge distance and showed that a parallel surface is a special case of a surface with constant ridge distance [24]. Again, Hacısalihoğlu and Yaşar studied the parallel surface of a hypersurface in Lorentz space and obtained new characterizations [27]. Çöken, Çiftçi and Ekici worked on parallel surfaces of timelike ruled surfaces [3]. Dae Won Yoon studied parallel Weingarten surfaces in Euclidean space and showed that for a surface to be a Weingarten surface, it is necessary and sufficient that its parallel surface is also a Weingarten surface [28]. In recent years, Kızıltuğ has taken a curve on a surface and obtained the image of this curve on a parallel surface and examined the characteristic features of this curve on the parallel surface [15, 16, 17]. Ünlütürk and Özüsağlam showed that the image of a curve that is geodesic on M by normal transformation in Minkowski 3-space on the parallel surface M_r is also a geodesic [26].

Given any curve in three-dimensional Euclidean space, an orthonormal vector system called the Frenet frame can be established at every point of this curve. The Frenet frame defines the curvature and torsion functions of the curve that characterize the curve. However, the disadvantage of this frame is that the Frenet frame cannot be established at points where the second derivative of the curve is zero. With the Flc frame defined by Dede in 2019, the singular points occurring in the second derivative of the curve were eliminated and a new frame was established. This shows that the Flc frame can be established along the curve, including the points where the Frenet frame cannot be established. Thus, the deformation on the surfaces created by taking this frame as a reference was also minimized, [4].

In this study, the parallel surfaces of tube surfaces defined using the Flc frame are first introduced. The Gaussian and mean curvatures of these parallel surfaces are calculated to determine the conditions under which they are developable or minimal. Next, the criteria for the parameter curves on the parallel surfaces to be asymptotic, geodesic, or curvature lines are analyzed. It is also demonstrated that both the tube surface and its parallel surfaces preserve the Gaussian transform. Finally, examples of these surfaces are provided.

2. Preliminaries

In this section, we remind some basic concepts that will be used throughout the paper. Let $\lambda = \lambda(t)$ be a regular space curve satisfying non-degenerate condition $\lambda^{'}(t) \wedge \lambda^{''}(t) \neq 0$. Then, the orthonormal vector system called Frenet frame is defined by

$$T(t) = \frac{\lambda^{'}(t)}{\|\lambda^{'}(t)\|}, \;\; B(t) = \frac{\lambda^{'}(t) \wedge \lambda^{''}(t)}{\|\lambda^{'}(t) \wedge \lambda^{''}(t)\|}, \;\; N(t) = B(t) \wedge T(t)$$

where T is tangent, N is principal normal, and B is binormal vector field. The Frenet formulas are given by

$$T' = \kappa\eta N, \;\; N' = -\kappa\eta T + \tau\eta B, \;\; B' = -\tau\eta N \;, \;\; \left\|\lambda'\right\| = \eta$$

where the curvature κ and torsion τ of the curve are, [4]

$$\kappa = \frac{\left\|\lambda^{'}(t) \wedge \lambda^{''}(t)\right\|}{\|\lambda^{'}(t)\|^{3}}, \qquad \tau = \frac{\left\langle \lambda^{'}(t) \wedge \lambda^{''}(t), \lambda^{'''}(t)\right\rangle}{\|\lambda^{'}(t) \wedge \lambda^{''}(t)\|^{2}}.$$

The n^{th} degree polynomial with parameter t is defined as

$$P(t) = \lambda_n t^n + \lambda_{n-1} t^{n-1} + ... + \lambda_1 t^1 + \lambda_0, \qquad \lambda_n \neq 0$$

where $n \in \mathbb{N}_0$, $\lambda_i \in \mathbb{R}$, $(0 \leq i \leq n)$, [4]. Now let us define a curve such that, $\lambda : [a,b] \to E^n$, $\lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$. If each $\lambda_i(t)$ are polynomials for $1 \leq i \leq n$, then $\lambda_t \in \mathbb{R}[s]$ is defined to be an $n-$dimensional polynomial curve [4]. The degree of such a polynomial curve as $\lambda(t)$ is given by

$$\deg \lambda(t) = max \left\{\deg\left(\lambda_1(t)\right), \deg\left(\lambda_2(t)\right), ..., \deg\left(\lambda_n(t)\right)\right\}.$$

The definition of the Flc frame of a polynomial space curve $\lambda = \lambda(t)$ given by Dede in [4] is as follows

$$T(t) = \frac{\lambda'(t)}{\|\lambda'(t)\|}, \ D_1(t) = \frac{\lambda'(t) \wedge \lambda^{(n)}(t)}{\left\|\lambda'(t) \wedge \lambda^{(n)}(t)\right\|}, \ D_2(t) = D_1(t) \wedge T(t)$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The new vectors D_1 and D_2 are called binormal-like vector and normal-like vector, respectively. The curvatures of the Flc-frame d_1, d_2, and d_3 are given by

$$d_1 = \frac{\langle T', D_2 \rangle}{\eta}, \ d_2 = \frac{\langle T', D_1 \rangle}{\eta}, \ d_3 = \frac{\langle {D_2}', D_1 \rangle}{\eta}$$

where $\|\lambda'\| = \eta$. The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} T' \\ {D_2}' \\ {D_1}' \end{bmatrix} = \eta \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}.$$

The relationship between the Frenet and Frenet like frame (Flc) is given by

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

and the relations between the curvatures of two frames are

$$d_1 = \kappa cos\theta, \qquad d_2 = -\kappa sin\theta, \qquad \theta = arctan\left(-\frac{d_2}{d_1}\right), \qquad d_3 = \frac{d\theta}{\eta} + \tau$$

where $\theta = \sphericalangle(N, D_2)$. Let E^3 be a 3-dimensional Euclidean space provided with the metric given by

$$< X, X >= dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{< X, X >}$, [13]. The parametric equation of a parallel surface is given as: Let M_1 and M_2 be two surfaces in 3-dimensional Euclidean space and the unit normal vector field of M_1 be Z. If there is a function f defined as

$$f : M_1 \longrightarrow M_2, f(P) = P + rZ_p$$

where r is a constant number, then the surfaces M_1 and M_2 are called parallel surfaces. Given the surface M,

$$M_r = \{P + rZ_p : P \in M, r \in \mathbf{R} \text{ and r=constant } \}$$

the set M_r given by the equation is a surface parallel to M. The normal vector field of the surface M_r is computed as

$$N_{M_r}(t,\theta) = \frac{M_{r_t} \wedge M_{r_\theta}}{\|M_{r_t} \wedge M_{r_\theta}\|}.$$

In addition, the first and second fundamental forms of the surface M_r are given by

$$I = Edt^2 + 2Fdtd\theta + Gd\theta^2,$$

$$II = Ldt^2 + 2Mdtd\theta + Nd\theta^2$$

while the Gaussian and mean curvatures are

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}$$

where the coefficients are found by following:

$$E =< M_{r_t}, M_{r_t} >, \quad F =< M_{r_t}, M_{r_\theta} >, \quad G =< M_{r_\theta}, M_{r_\theta} >,$$

$$L =< M_{r_{tt}}, N_{M_r} >, \quad N =< M_{r_{t\theta}}, N_{M_r} >, \quad M =< M_{r_{\theta\theta}}, N_{M_r} > .$$

Concerning the Gaussian and mean curvatures, the following definitions exist

- A surface is said to be developable and has parabolic points if the Gaussian curvature vanishes,
- A surface is said to have hyperbolic (resp. elliptic) points, if it has a negative (resp. positive) Gaussian curvature,
- A surface is said to be minimal if the mean curvature vanishes, [6].

3. On geometry of the parallel surface of the tube surface given by the Flc frame in Euclidean 3-space

Let $M(t)$ be a polynomial space curve of degree n. We can parametrize a tubular surface generated by an Flc-frame as follows

$$K(t,\theta) = M(t) + r\left[\cos\theta D_2(t) + \sin\theta D_1(t)\right] \tag{3.1}$$

where $\theta \in [0, 2\pi)$, $r \in \mathbb{R}$ is the radius of the tubular surface and the curve $M(t)$ is the center curve of the tubular surface, [4]. The derivatives according to parameters t and θ of the tubular surface $K(t, \theta)$ are, respectively,

$$K_t = \nu(1 - r(\cos\theta d_1 + \sin\theta d_2))T - \nu r \sin\theta d_3 D_2 + \nu r \cos\theta d_3 D_1,$$
$$K_\theta = -r\sin\theta D_2 + r\cos\theta D_1.$$

The normal vector field of the tubular surface $K(t, \theta)$ is obtained as

$$N(t, \theta) = cos\theta D_2 + sin\theta D_1. \tag{3.2}$$

If the parallel surface of the tube surface $K(t, \theta)$ is represented by $K_P(t, \theta)$, the equation of this surface is defined as

$$K_P(t, \theta) = K(t, \theta) + \varepsilon N(t, \theta).$$

If the expressions (3.1) and (3.2) are written here, the expression of the parallel surface $K_P(t, \theta)$ with respect to the Flc frame becomes,

$$\begin{aligned} K_p(t, \theta) &= K(t, \theta) + \varepsilon N(t, \theta) \\ &= M(t) + (r + \varepsilon)\left[\cos\theta D_2(t) + \sin\theta D_1(t)\right]. \end{aligned}$$

If the first order partial derivatives of the surface $K_p(t, \theta)$ are taken with respect to the parameters t and θ

$$\begin{aligned} K_{p_t} =& \nu(1 - (r + \varepsilon)(\cos\theta d_1 + \sin\theta d_2))T - \nu(r + \varepsilon)\sin\theta d_3 D_2 \\ &+ \nu(r + \varepsilon)\cos\theta d_3 D_1, \\ K_{p_\theta} =& -(r + \varepsilon)\sin\theta D_2 + (r + \varepsilon)\cos\theta D_1, \end{aligned}$$

is found. Here the unit normal vector of the surface is

$$N_p(t, \theta) = \frac{K_{p_t} \wedge K_{p_\theta}}{\|K_{p_t} \wedge K_{p_\theta}\|} = cos\theta D_2 + sin\theta D_1.$$

The coefficients of the first fundamental form of the surface are as follows

$$\begin{aligned} E_p &=< K_{p_t}, K_{p_t} >= \nu^2\left[1 - (r + \varepsilon)(\cos\theta d_1 + \sin\theta d_2)\right]^2 + \nu^2(r + \varepsilon)^2 {d_3}^2, \\ F_p &< K_{p_t}, K_{p_\theta} >= \nu(r + \varepsilon)^2 d_3, \\ G_p &=< K_{p_\theta}, K_{p_\theta} >= (r + \varepsilon)^2. \end{aligned} \tag{3.3}$$

The second-order partial derivatives of the surface $K_p(t,\theta)$ are as follows:

$$\begin{aligned} K_{p_{tt}} =&[\nu^2(r+\varepsilon)d_3(sin\theta d_1 - cos\theta d_2) - \nu(r+\varepsilon)(cos\theta d_1' + sin\theta d_2') \\ &- \nu'(r+\varepsilon)(cos\theta d_1 + sin\theta d_2) + \nu']T \\ &- [\nu^2(r+\varepsilon)cos\theta(d_1^2+d_3^2) + \nu^2(r+\varepsilon)d_1 d_2 sin\theta + (r+\varepsilon)sin\theta(vd_3)' - \nu^2 d_1]D_2 \\ &- [\nu^2(r+\varepsilon)sin\theta(d_2^2+d_3^2) + \nu^2(r+\varepsilon)d_1 d_2 cos\theta + (r+\varepsilon)cos\theta(vd_3)' - \nu^2 d_2]D_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} K_{p_{t\theta}} =&\nu(r+\varepsilon)(sin\theta d_1 - cos\theta d_2)T - \nu(r+\varepsilon)cos\theta d_3 D_2 - \nu(r+\varepsilon)sin\theta d_3 D_1, \\ K_{p_{\theta\theta}} =&-(r+\varepsilon)cos\theta D_2 - (r+\varepsilon)sin\theta D_1. \end{aligned} \tag{3.5}$$

The coefficients of the first fundamental form of the surface are written as follows

$$\begin{aligned} e_p =< K_{p_{tt}}, N_p > =& \nu^2(d_1 cos\theta + d_2 sin\theta) - \nu^2(r+\varepsilon)(d_1 cos\theta + d_2 sin\theta)^2 \\ &- \nu^2(r+\varepsilon)d_3^2, \end{aligned} \tag{3.6}$$

$$f_p =< K_{p_{t\theta}}, N_p >= -\nu(r+\varepsilon)d_3, \tag{3.7}$$

$$g_p =< K_{p_{\theta\theta}}, N_p >= -(r+\varepsilon). \tag{3.8}$$

With the help of these expressions, the Gaussian curvature $\mathbb{K}_p$ and the mean curvature $\mathbb{H}_p$ of the parallel surface $K_p(t,\theta)$ are written as follows, respectively:

$$\mathbb{K}_p = \frac{-cos\theta d_1 - sin\theta d_2}{(r+\varepsilon)[1-(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)]},$$

$$\mathbb{H}_p = \frac{1-2(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)}{2(r+\varepsilon)[1-(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)]}.$$

Theorem 3.1. *Singular points of the parallel surface $K_p(t,\theta)$ satisfy the equation*

$$cos\theta_0 d_1 + sin\theta_0 d_2 = \frac{1}{r+\varepsilon}.$$

Proof. For the parallel surface $K_p(t,\theta)$ to have singular points at the point (t_0,θ_0) ,

$$\|K_{p_t} \wedge K_{p_\theta}\|(t_0,\theta_0) = 0.$$

If the necessary operations are carried out from here, the following is obtained:

$$\|K_{p_t} \wedge K_{p_\theta}\|(t_0,\theta_0) = 0 \Rightarrow \nu(r+\varepsilon)\left(\cos\theta\, d_1(r+\varepsilon) + \sin\theta\, d_2(r+\varepsilon) - 1\right) = 0$$

$$\Rightarrow (r+\varepsilon)cos\theta_0 d_1 + (r+\varepsilon)sin\theta_0 d_2 = 1$$

$$\Rightarrow cos\theta_0 d_1 + sin\theta_0 d_2 = \frac{1}{r+\varepsilon}.$$

□

Corollary 3.1. *In particular, if* $\theta_0 = 0$ *is taken, then* $d_1 = \frac{1}{r+\varepsilon}$. *In this case, the locus of singular points of the surface is a curve of the form*

$$K_p(t,0) = M(t) + (r+\varepsilon)D_2(t).$$

Corollary 3.2. *If* $\theta_0 = \frac{\pi}{2}$ *or* $\theta_0 = \frac{3\pi}{2}$ *is taken, then* $d_2 = \frac{1}{r+\varepsilon}$. *In this case, the geometric locus of the singular points of the surface is a curve of the form*

$$K_p(t,\frac{\pi}{2}) = M(t) + (r+\varepsilon)D_1(t),$$

$$K_p(t,\frac{3\pi}{2}) = M(t) - (r+\varepsilon)D_1(t).$$

Theorem 3.2. *For* $K_p(t,\theta)$ *parallel surface:*

(i) *t parametric curves are asymptotic if and only if*

$$(r+\varepsilon)d_3^2 + (r+\varepsilon)(cos\theta d_1 + sin\theta d_2)^2 = cos\theta d_1 + sin\theta d_2.$$

(ii) *The parameter curves* θ *are not asymptotic curves.*

Proof. (i) For the parameter curves of the parallel surface $K_p(t,\theta)$ to be asymptotic curves, it is necessary and sufficient that $e_p = 0$. From the equation (3.6) we find:

$$e_p = 0 \Rightarrow \nu^2(d_1cos\theta + d_2sin\theta) - \nu^2(r+\varepsilon)(d_1cos\theta + d_2sin\theta)^2 - \nu^2(r+\varepsilon)d_3^2 = 0$$

$$\Rightarrow (r+\varepsilon)d_3^2 + (r+\varepsilon)(d_1cos\theta + d_2sin\theta)^2 = d_1cos\theta + d_2sin\theta.$$

(ii) For the θ parameter curves of the parallel surface $K_p(t,\theta)$ to be asymptotic curves, the necessary and sufficient condition is that $g_p = 0$. From the equation (3.8), since $g_p = -r-\varepsilon$ and $r,\varepsilon \neq 0$, the θ parameter curves cannot be asymptotic.

□

Theorem 3.3. *For* $K_p(t,\theta)$ *parallel surface:*

(i) *t parametric curves are geodesic if and only if*

$$v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$

$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)' = 0,$$

$$\big(cos\theta + sin\theta\big)\Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$

$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0.$$

(ii) *The θ parameter curves are always geodesic.*

Proof. (i) The necessary and sufficient condition for the parameter curves t of the parallel surface $K_p(t,\theta)$ to be geodesic curves is that $N_p \wedge K_{p_{tt}} = 0$. From the equations (3.2) and (3.4), the vector $N_p \wedge K_{p_{tt}}$ is given by

$$N_p \wedge K_{p_{tt}} = \Big(v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$

$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)'\Big) T(t)$$

$$+ sin\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$

$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) D_2(t)$$

$$- cos\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$

$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) D_1(t).$$

For $N_p \wedge K_{p_{tt}} = 0$ the coefficients must be zero. Therefore,

$$v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$

$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)' = 0,$$

$$sin\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$

$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0,$$

$$cos\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$

$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0.$$

If these equations are arranged, the desired result is obtained.

(ii) The necessary and sufficient condition for the parameter curves of the parallel surface $K_p(t,\theta)$ to be geodesic curves is that $N_p \wedge K_{p_{\theta\theta}} = 0$. Using the equations (3.2) and (3.5), $N_p \wedge K_{p_{\theta\theta}} = 0$ means that the θ parameter curves are always geodesic.

□

Theorem 3.4. *Let the parallel surface $K_p(t,\theta)$ be given. In order for the parameter curves on the surface to be lines of curvature, the necessary and sufficient condition is that $d_3 = 0$.*

Proof. For the parameter curves of the parallel surface $K_p(t,\theta)$ to be lines of curvature, it is necessary and sufficient that $F_p = f_p = 0$. From the equations (3.3) and (3.7) we write

$$\nu(r+\varepsilon)^2 d_3 = 0 \qquad \text{and} \qquad -\nu(r+\varepsilon)d_3 = 0.$$

Here $d_3 = 0$ since $\nu, r \neq 0$. □

In this case, the following result can be given:

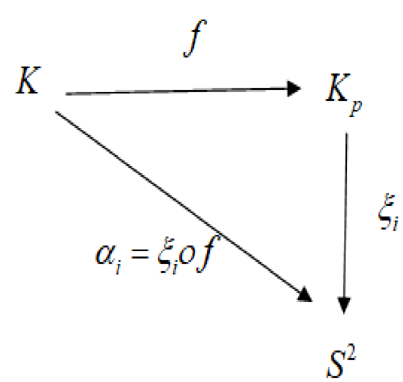
Corollary 3.3. *If the parameter curves t and θ on the parallel surface $K_p(t,\theta)$ are planar, these curves are the curvature lines of the surface.*

Theorem 3.5. *Let (K, K_p) be the pair of parallel surfaces in $\mathbb{E}^3$. There is a relation between the Gaussian transformations,*

$$\eta = \eta_p$$

where the unit normal vectors of the surfaces K and K_p are N and N_p, respectively.

Proof. Let the coordinates of the unit normal vectors K and K_p be $\alpha_i = (\alpha_1, \alpha_2, \alpha_3)$ and $\xi_i = (\xi_1, \xi_2, \xi_3)$, respectively.



$$\eta : K \longrightarrow S^2$$
$$X \longrightarrow \eta(X) = \sum_{i=1}^{3} \alpha_i(X) \frac{\partial}{\partial Y_i|_X}$$

is the Gaussian transform of the surface K. On the other hand, the Gaussian transform of the surface K_p is as follows, where $f : K \longrightarrow K_p$ is the parallel transform:

$$\begin{aligned}
&\eta_p : K_p \longrightarrow S^2\\
f(X) \longrightarrow \eta_p(f(X)) &= \sum_{i=1}^{n} \xi_i(f(X))\frac{\partial}{\partial Y_i}|_{f(X)}\\
&= \sum_{i=1}^{n} (\xi_i \circ f(X))\frac{\partial}{\partial Y_i}|_{f(X)}\\
&= \sum_{i=1}^{n} \alpha_i(X)\frac{\partial}{\partial Y_i}|_{f(X)}\\
&= \eta_p(X)
\end{aligned}$$

Since the Gaussian transformation will be provided for $\forall X \in K$, $\eta = \eta_p$ is obtained. □

Example 3.1. *Let $M : I \longrightarrow \mathbb{R}^3$ be a polynomial curve with center curve $M(t) = (t, t^8, t^9)$. If the derivatives of the curve $M(t)$ are calculated, it is as follows:*

$$\begin{aligned}
M'(t) &= (1, 8t^7, 9t^8),\\
M''(t) &= (0, 56t^6, 72t^7),\\
M^{(9)}(t) &= (0, 0, 362880).
\end{aligned}$$

The Flc frame vectors of the polynomial curve $M(t)$ are found as follows, respectively:

$$\begin{aligned}
T(t) &= \frac{M'(t)}{\|M'(t)\|} = \Big(\frac{1}{\sqrt{81t^{16}+64t^{14}+1}}, \frac{8t^7}{\sqrt{81t^{16}+64t^{14}+1}}, \frac{9t^8}{\sqrt{81t^{16}+64t^{14}+1}}\Big),\\
D_1(t) &= \frac{M'(t)\times M^{(9)}(t)}{\|M'(t)\times M^{(9)}(t)\|} = \Big(\frac{8t^7}{\sqrt{64t^{14}+1}}, -\frac{1}{\sqrt{64t^{14}+1}}, 0\Big),\\
D_2(t) &= T(t)\times D_1(t)\\
&= \Big(-\frac{9t^8}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}}, -\frac{72t^{15}}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}},\\
&\qquad \frac{\sqrt{64t^{14}+1}}{\sqrt{81t^{16}+64t^{14}+1}}\Big).
\end{aligned}$$

On the other hand, Flc curvatures are as follows:

$$d_1(t) = \frac{\langle T'(t), D_2(t)\rangle}{\|M'(t)\|} = \frac{72t^7\left(8t^{14}+1\right)}{\sqrt{64\,s^{14}+1}\,(81t^{16}+64t^{14}+1)^{3/2}},$$

$$d_2(t) = \frac{\langle T'(t), D_1(t)\rangle}{\|M'(t)\|} = -\frac{56t^6}{\sqrt{64\,t^{14}+1}\,(81\,t^{16}+64\,t^{14}+1)},$$

$$d_3(t) = \frac{\left\langle D_2(t)', D_1(t)\right\rangle}{\|M'(t)\|} = \frac{504t^{14}}{(64\,t^{14}+1)\,(81\,t^{16}+64\,t^{14}+1)}.$$

If the radius $r = 0.25$ *is taken, the parametric equation of the tube surface* $K(t,\theta)$ *is as follows:*$(-1 \leq t \leq 1, \ \ -\pi \leq \theta \leq \pi)$

$$K(t,\theta) = \Big(t - \frac{9t^8\cos\theta}{4\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} + \frac{2t^7\sin\theta}{\sqrt{64t^{14}+1}},$$

$$t^8 - \frac{18t^{15}\cos\theta}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} - \frac{\sin\theta}{4\sqrt{64t^{14}+1}},$$

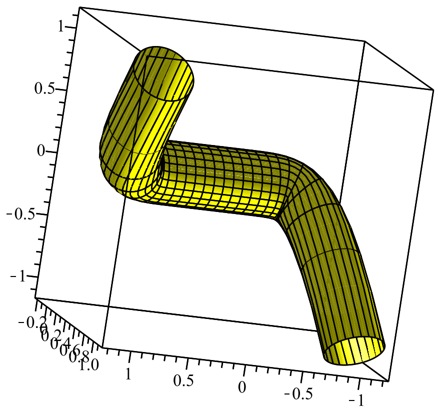
$$t^9 + \frac{\cos\theta\sqrt{64t^{14}+1}}{4\sqrt{81t^{16}+64t^{14}+1}}\Big).$$

If $\epsilon = 0.5$ *is taken, the equation of the parallel surface* $K_p(t,\theta)$ *is as follows:* $(-1 \leq t \leq 1, \ \ -\pi \leq \theta \leq \pi)$

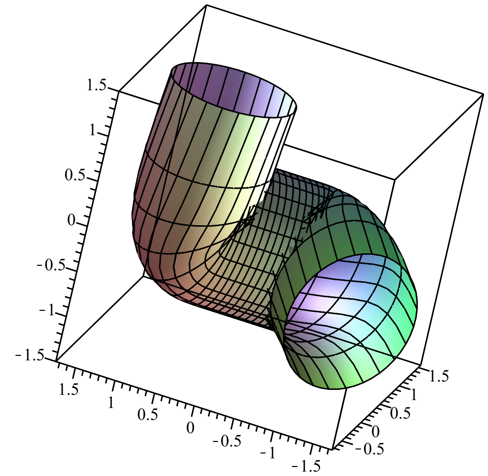
$$K_p(t,\theta) = \Big(t - \frac{27t^8\cos\theta}{4\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} + \frac{6t^7\sin\theta}{\sqrt{64t^{14}+1}},$$

$$t^8 - \frac{54t^{15}\cos\theta}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} - \frac{3\sin\theta}{4\sqrt{64t^{14}+1}},$$

$$t^9 + \frac{3\cos\theta\sqrt{64t^{14}+1}}{4\sqrt{81t^{16}+64t^{14}+1}}\Big).$$



(A) $K(t,\theta)$ tube surface



(B) $K_p(t,\theta)$ parallel surface

4. Conclusion

In this study, first of all, the parallel surfaces of the tube surface given with the Flc frame were defined. It was seen that the surface created by investigating the geometric features of this parallel surface was developable and minimal. The parameter curves of the parallel surface were examine. Subsequently, the tube surface and parallel surface were shown to preserve the Gaussian transform. Finally, the tube surface, which accepts a polynomial curve as its center curve, and the parallel surface of this tube surface, are given as an example, and are shown. This work can be studied in various spaces such as Minkowski space and Galilean space, and can also be repeated for higher-dimensional curves.

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References

[1] Akyiğit M., Eren K., & Kösal H.H. (2021). Tubular Surfaces with Modified Orthogonal Frame, Honam Mathematical Journal, 43(3), 453-463.

[2] Craig, T. (1883). Note on parallel surfaces. Journal Für Die Reine und Angewandte Mathematik (Crelle's journal), 94, 162-170.

[3] Çöken, A.C., Çiftci, U., & Ekici, C. (2008). On parallel timelike ruled surfaces with timelike rulings. Kuwait Journal of Science and Engineering, 35, 21-31.

[4] Dede, M. (2019). A new representation of tubular surfaces. Houston J. Math., 45, 707-720.

[5] Dede, M., Ekici, C., & Çöken, A. C. (2013). On the parallel surfaces in Galilean Space. Hacettepe Journal of Mathematics on Statistics, 42(6), 605-615.

[6] Do-Carmo, P. (1976). Differential geometry of curves and surfaces, IMPA.

[7] Eisenhart, L.P. (1909). A Treatise On The Differential Geometry Of Curves and Surfaces. Ginn and Company, Boston:New York.

[8] Eren, K., Ayvacı K. H., & S. Şenyurt.(2022). On Characterizations of Spherical Curves Using Frenet Like Curve Frame, Honam Mathematical Journal, 44, no.3 (September 1,): 391–401.

[9] Eren, K., Ayvacı K. H., & S. Senyurt.(2023). On Ruled Surfaces Constructed by the Evolution of a Polynomial Space Curve, J. of Science and Arts, 23(1),77-90.

[10] Eren K., Yıldız, Ö.G., & Akyiğit, M. (2022). Tubular surfaces associated with framed base curves in Euclidean 3-space, Math Meth App Sci., 45(18), 12110-12118.

[11] Eren K. (2021). On the Harmonic Evolute Surfaces of Tubular Surfaces in Euclidean 3-Space, Journal of Science and Arts, 2(55),449-460.

[12] Fukui, T., & Hasegawa, M. (2012). Singularities of parallel surfaces, arXix:1203.3715v1 [math.DG].

[13] Goetz, A. (1968). Introduction to Differential Geometry. Addison-Wesley Publishing Company, Canada.

[14] Kılıç, A. (1993). On the principal curvatures of parallel hypersurfaces in E^n. Erciyes Üniversitesi Fen Bilimleri Dergisi, 10, 1-4.

[15] Kızıltuğ, S. (2013). Paralel Yüzeyler ve Eğriler. Doktora tezi, Atatürk Ünviversitesi Fen Bilimleri Enstitüsü, Erzurum.

[16] Kızıltuğ, S., & Yaylı, Y. (2012). Timelike Curves on Timelike Parallel Surfaces in Minkowski 3-space. Mathematica Aeterna, 2(8), 675-687.

[17] Kızıltuğ, S., & Yaylı, Y. (2013). Spacelike Curves on Spacelike Parallel Surfaces in Minkowski 3-space. International Journal of Mathematics and Computation, 19, 0974-5718.

[18] Nizamoğlu, S. (1986). Surfaces regles parallles. Ege Univ. Fen Fak. Derg., 9 (Ser. A), 37-48.

[19] Önder, M., & Kızıltuğ, S. (2012). Bertrand and Mannheim partner D-curves on parallel surfaces in Minkowski 3-space. International Journal of Geometry, 1(2), 34-45.

[20] Savcı, Ü. Z. (2011). 3-boyutlu Öklid uzayında Paralel Regle Weingarten Yüzeyler üzerine. Doktora Tezi. Eskişehir Osmangazi Üniversitesi Fen Bilimleri Ensitüsü, 71s.

[21] Şenyurt, S., & Ayvacı, K. H. (2022). On geometry of focal surfaces due to Flc frame in Euclidean 3-space. Authorea. https://doi.org/10.22541/au.166816461.17575964/v1

[22] Şenyurt S., Eren K., & Ayvacı K. H.(2022). A Study on Inextensible Flows of Polynomial Curves with Flc Frame, Applications and Applied Mathematics: An International Journal (AAM), 17(1),123-133,.

[23] Taleshian, A. (2007). Ortogonal curvature of parallel hypersurfaces. Int. J. Contemp. Math. Sciences, 2(26), 1257-1262.

[24] Tarakcı, Ö. (2002). Sabit Sırt Uzaklıklı Hiperyüzeyler Doktora tezi Ankara Üniv. Fen Bilimleri Ens., Ankara.

[25] Ünlütürk, Y., & Ekici, C. (2013). On parallel surfaces of timelike ruled weingarten surfaces. Balkan Journal of Mathematics, 1, 72-91.

[26] Ünlütürk, Y., & Özüsağlam, E. (2012). On Parallel Surfaces in Minkowski 3-Space. TWMS. J. App. Eng. Math., 3(2), 214-22.

[27] Yaşar, A. (2011). Lorentz Uzayında Bir Hiperyüzeyin Yüksek Mertebeden Gausian Eğrilikleri. Yüksek lisans tezi. Ankara Üniversitesi Fen Bilimleri Enstitüsü, Ankara.

[28] Yoon, D.W. (2008). Some Properties of Parallel Surfaces in Euclidean 3-Spaces. Honam Mathematical J., 30(4), 637-644.

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