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AN OPTIMUM PARAMETER METHOD TO OBTAIN NUMERICAL SOLUTIONS OF THE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The main purpose of this article is to use a method with a free parameter which is named optimum asymptotic homotopy method (*OHAM*) in order to obtain the solution of differential equations, partial differential equations and the system of coupled partial differential equations featuring fractional derivative. This method is preferable to others since it has faster convergence toward homotopy perturbation method as well as the convergence rate can be set as controlled area. Various examples are given to better understand the use of this method. The approximate solutions are compared with exact solutions as well.

1. INTRODUCTION

Fractional arithmetic and fractional differential equations appeared in many disciplines, including medicine [1], economics [2], dynamical problems [3, 4], chemistry [5], mathematical physics [6], traffic model [7] and fluid flow [8] and so on.

Scholars and researchers are invited to study books that have been written to better understand the concept of fractional arithmetic [9, 10, 11].

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In order to find the approximate solution for partial differential equations with fractional derivative that we explored in this paper is presented as follows:

$$D_t^\alpha u(x, t) + \mathfrak{A}(x, t, u(x, t), u_x(x, t), u_{xx}(x, t), \dots) = g(x, t), \quad (1.1)$$

in which \mathfrak{A} is the partial differential operator, $D^\alpha u(x, t)$ is the fractional Caputo derivative, $k - 1 < \alpha \leq k$ and $k \in \mathbb{N}$.

A number of articles that can be found to express modeling, deploying and extent of differential equation (DEs), partial differential equation (PDEs) and fractional partial differential equations (FPDEs) are in [12, 13].

It is necessary to announce that there are no accurate analytical solutions for most DEs, PDEs and FPDEs thus; a relatively large number of approximate solution expressed by the scholars are not possible if they find the accurate analytical solutions with the existing procedures for the DEs, PDEs and FPDEs. Accordingly, for such differential equations, we have to employ some direct and iterative methods. Some of these techniques which can be used by scholars include discrete element method and finite difference method [14, 15, 16, 17, 18], homotopy perturbation method (HPM) [19], differential transform method (DTM) [20], Adomian's decomposition method (ADM) [21], optimal homotopy asymptotic method (OHAM) [22], homotopy analysis method (HAM) [23], variational iteration method (VIM) [19], new homotopy asymptotic method (NHAM) [24] and so on [25, 26, 27].

The OHAM was presented and developed by Marinca et al. [28, 29, 30] and it can be shown that HPM is a special case of OHAM. The goal is achieved here by using auxiliary functions, auxiliary convergence controlling parameters, and a homotopy in a particular way to make OHAM simple and effective. The accuracy is also improved with increase in the number of auxiliary parameters in the auxiliary function. Several authors have proved the effectiveness, generalization and reliability of this method. The advantage of OHAM is built in convergence criteria, which is controllable. In OHAM, the control and adjustment of the convergence region are provided in a convenient way. Numerical results show that OHAM is found the best in giving better and more accurate results. It consists of few steps and converges to almost exact solution. The applied method is simple in learning and easy to apply.

This paper is organized as follows: in Section 2, definition and some proposition of the Caputo fractional derivative are introduced. In Section 3, description of OHAM is given. In Section 4, we have expressed the convergence of OHAM. In Section 5, the application

of OHAM to the Eq. 1.1 are illustrated, and some numerical examples are presented. And conclusions are drawn in Section 6.

2. FRACTIONAL CALCULUS

Definition 2.1. A real function $f(x)$, $x > 0$, is considered to be in the space C_ν , ($\nu \in \mathbb{R}$), if there exists a real number $n(> \nu)$, so that $f(x) = x^n f_1(x)$, where $f_1(x) \in C[0, \infty)$, it is said to be in the space C_ν^k if and only if $f^{(k)} \in C_\nu$, $k \in \mathbb{N}$ [10, 11].

Definition 2.2. [10, 11] The Riemann-Liouville fractional integral operator of order of $\alpha > 0$, of a function $f \in C_\nu$, $\nu \geq -1$, is given by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r) dr.$$

$$I^\alpha f(x) = I_0^\alpha f(x), \quad I^0 f(x) = f(x).$$

Definition 2.3. [10, 11] The Caputo's fractional derivative of f is defined as

$$D^\alpha f(x) = I^{k-\alpha} D^k f(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r) dr, \quad x > 0.$$

where, $f \in C_{-1}^k$, $k-1 < \alpha \leq k$ and $k \in \mathbb{N}$.

Proposition 2.1. For $k-1 < \alpha \leq k$, $k \in \mathbb{N}$, $f \in C_\nu^k$, $\nu \geq -1$ and $x > 0$, the following properties satisfy

- i) $D_a^\alpha I_a^\alpha f(x) = f(x)$
- ii) $I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{j=0}^{k-1} f^{(j)}(a^+) \frac{(x-a)^j}{j!}$.

The Caputo fractional derivative of order α for $u(x, t)$ is defined as:

$$D_t^\alpha u(x, t) = \frac{1}{\Gamma(k+1-\alpha)} \int_0^t (t-s)^{k-\alpha} u^{(k+1)}(x, s) ds, \quad k < \alpha \leq k+1, \quad k \in \mathbb{Z}^+. \quad (2.2)$$

3. DESCRIPTION OF OHAM

The overall dimensions of the proposed approach [31] in this section is given and represented in the following differential equation

$$L\left(u(x, t)\right) + N\left(u(x, t), u(\eta_0(x), \varsigma_0(t)), u_x(\eta_1(x), \varsigma_1(t)), \dots, \underbrace{u_x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t))\right) + g(x, t) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n, \quad t > 0 \quad (3.3)$$

featuring the boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \quad t \in \Gamma, \quad (3.4)$$

in which $L = D_t^\alpha$ is linear operator and N is nonlinear operator may consist of the space derivatives of integer order with respect to x along with delay functions, $u(x, t)$ is unknown function, $g(x, t)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω . Also, $\eta_j(x)$ and $\varsigma_j(t)$ are delay functions. In this work, we consider $\eta_j(x) = p_j x$ and $\varsigma_j(t) = q_j t$, for $j = 0, 1, \dots, n$.

According to OHAM, we concoct structural homotopy $v(x, t; p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which fulfills the conditions in the following equation

$$(1-p)L(v(x, t; p) - u_0(x, t)) = H(p)\left(L(v(x, t; p)) + g(x, t) + N(u(x, t), u(\eta_0(x), \varsigma_0(t)), u_x(\eta_1(x), \varsigma_1(t)), \dots, \underbrace{u_x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t)))\right), \quad (3.5)$$

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a non zero auxiliary function for $p \neq 0$ and $H(0) = 0$. When $p = 0$ and $p = 1$, we have $v(x, t; 0) = u_0(x, t)$ and $v(x, t; 1) = u(x, t)$ respectively.

Thus, when p provides from 0 to 1, the solution $v(x, t; p)$ approaches from the initial guess $u_0(x, t)$ to exact solution $u(x, t)$. In which $u_0(x, t)$ obtained from 3.4 to 3.5 with $p = 0$ giving

$$L(u_0(x, t; 0)) + g(x, t) = 0. \quad (3.6)$$

The auxiliary function $H(p)$ is elected in the following display:

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \dots, \quad (3.7)$$

in which c_1, c_2, c_3, \dots are convergence control parameters which are unfamiliar and can be calculated. Another demonstration form $H(p)$ offered by Herişanu and his associate in [31]. To compute the approximate solution, we expand $v(x, t; p, c_i)$, in Taylor series around p which is as follows:

$$v(x, t; p, c_i) = u_0(x, t) + \sum_{k=1}^{\infty} u_k(x, t; c_i) p^k, \quad i = 1, 2, \dots \quad (3.8)$$

Defining the vectors

$$\vec{c}_l = \{c_1, c_2, \dots, c_l\}, \quad (3.9)$$

and

$$\begin{aligned} \vec{u}_s = & \{u_0(x, t), u_1(x, t; \vec{c}_1), \dots, u_s(x, t; \vec{c}_s), \\ & (u_0)_x(\eta_1(x), \varsigma_1(t)), (u_1)_x(\eta_1(x), \varsigma_1(t); \vec{c}_1), \dots, (u_s)_x(\eta_1(x), \varsigma_1(t); \vec{c}_s) \\ & \vdots \\ & (u_0)_{\underbrace{x \dots x}_{n \text{ order}}}(\eta_n(x), \varsigma_n(t)), (u_1)_{\underbrace{x \dots x}_{n \text{ order}}}(\eta_n(x), \varsigma_n(t); \vec{c}_1), \dots, (u_s)_{\underbrace{x \dots x}_{n \text{ order}}}(\eta_n(x), \varsigma_n(t); \vec{c}_s)\}. \end{aligned}$$

The Zero-order problem by (3.6), and the first-order equation by

$$L(u_1(x, t)) = c_1 N_0(\vec{u}_0) + g(x, t) \tag{3.10}$$

and second-order equation by

$$L(u_2(x, t)) - L(u_1(x, t)) = c_2 N_0(\vec{u}_0) + c_1 \left(L(u_1(x, t)) + N_1(\vec{u}_1) \right). \tag{3.11}$$

are considered. The equations in the general case $u_k(x, t)$, are

$$\begin{aligned} L(u_k(x, t)) - L(u_{k-1}(x, t)) = & \tag{3.12} \\ & c_k N_0(u_0(x, t)) + \sum_{m=1}^{k-1} c_m \left(L(u_{k-m}(x, t)) + N_{k-m}(\vec{u}_{k-1}) \right), \end{aligned}$$

in which $k = 2, 3, \dots$ and N_m is the coefficient of "p^m", in the development of $N(v(x, t; p))$, about the embedding parameter "p" and we have

$$N(v(x, t; p, c_i)) = N_0(u_0(x, t)) + \sum_{m=1}^{\infty} N_m(\vec{u}_m) p^m. \tag{3.13}$$

It can be seen that, convergence series (3.8) is dependent on the constants c_1, c_2, \dots . If it is convergent at $p = 1$, one has

$$\tilde{v}(x, t; c_i) = u_0(x, t) + \sum_{k=1}^m u_k(x, t; c_i), \quad i = 1, 2, \dots, m. \tag{3.14}$$

The following residual is the result obtained as a result of embedding (3.14) in (3.3):

$$\begin{aligned} R(x, t; c_i) = & L(\tilde{v}(x, t; p, c_i)) + \\ & g(x, t) + N(\tilde{v}(x, t; p, c_i)), \quad i = 1, 2, \dots, m. \end{aligned} \tag{3.15}$$

If $R = 0$, then \tilde{v} will be the exact solution 3.3.

Using the method of least squares and knowing the exact solution of the problem, we can minimize the L^2 -norm of the error $E v_m(c_1, c_2, c_3, \dots, c_m)$. The L^2 -norm of the error is signified as

$$\|E \tilde{v}_m(c_1, \dots, c_m)\|_2 = \left(\int_{\Omega} \int_{\Gamma} \tilde{v}_m^2(x, t) dt dx \right)^{\frac{1}{2}},$$

in which $E \tilde{v}_m(x, t) = |\tilde{v}_{exact}(x, t) - \tilde{v}_m(x, t; c_1, \dots, c_m)|$.

4. CONVERGENCE OF OHAM

Topics in this section are provided for convergence of the OHAM.

Theorem 4.1. [32] *Let the solution components u_0, u_1, u_2, \dots , be defined as given in Eqs.(3.10)-(3.12). The series solution $\sum_{k=0}^{m-1} u_k(x, t)$ defined in 3.14 converges, if $\exists 0 < \rho < 1$ such that $\|u_{k+1}\| \leq \rho \|u_k\| \forall k \geq k_0$ for some $k_0 \in \mathbb{N}$.*

Proof. Under consideration

$$T_0 = u_0$$

$$T_1 = u_0 + u_1$$

$$T_2 = u_0 + u_1 + u_2$$

...

$$T_n = u_0 + u_1 + u_2 + \dots + u_n,$$

as the sequence $\{T_n\}_{n=0}^{\infty}$. Evidence is sufficient to show that the sequence $\{T_n\}_{n=0}^{\infty}$ in the Hilbert space \mathbb{R} is a Cauchy sequence. To achieve this, consider

$$\begin{aligned} \|T_{n+1} - T_n\| &= \|u_{n+1}\| \\ &\leq \rho \|u_n\| \\ &\leq \rho^2 \|u_{n-1}\| \\ &\vdots \\ &\leq \rho^{n-k_0+1} \|u_{k_0}\|. \end{aligned}$$

Assuming that $n \geq m > k_0$ and for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \|T_n - T_m\| &= \|(T_n - T_{n-1}) + (T_{n-1} - T_{n-2}) + \dots + (T_m - T_{m-1})\| \\ &\leq \|(T_n - T_{n-1})\| + \|(T_{n-1} - T_{n-2})\| + \dots + \|(T_m - T_{m-1})\| \\ &\leq \rho^{n-k_0} \|u_{k_0}\| + \rho^{n-k_0-1} \|u_{k_0}\| + \dots + \rho^{m-k_0+1} \|u_{k_0}\| \\ &= \left(\frac{1 - \rho^{n-m}}{1 - \rho} \right) \rho^{m-k_0+1} \|u_{k_0}\|. \end{aligned}$$

According to the $0 < \rho < 1$, it results that $\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} \|T_n - T_m\| = 0$. Thereupon, in the Hilbert space \mathbb{R} , sequence $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence and this implies that series solution converges to series $\sum_{k=0}^{\infty} u_k(x, t)$.

5. TEST EXAMPLES

Now that it is easier to understand OHAM, various examples will be described in this section and then will be calculated. These examples include solutions of nonlinear partial differential equation featuring fractional derivative. In all these examples, mathematical software *Mathematica* is used for calculations and graphs.

Example 5.1. *For the first example, we propose the time-fractional advection differential equation:*

$$D_t^\alpha u(x, t) + u(x, t) u_x(x, t) = x(1 + t^2), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \quad (5.16)$$

with the precise solution $u(x, t) = x t$ for $\alpha = 1$ and the primary condition:

$$u(x, 0) = 0. \quad (5.17)$$

Following the OHAM, according to what was formulated and presented in Section 3 for Eqs.(5.16)-(5.17), we get:

$$\begin{aligned}
 u_0(x, t) &= \frac{xt^\alpha (\alpha^2 + 3\alpha + 2t^2 + 2)}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)}, \\
 u_1(x, t) &= -\frac{2c_1xt^{\alpha+2}}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} + \frac{2c_1xt^{\alpha+2}}{(\alpha^3 + 3\alpha^2 + 2\alpha) \Gamma(\alpha)} + \frac{2c_1x\Gamma(2\alpha + 1)t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha)} + \\
 &\quad \frac{8c_1x\Gamma(2\alpha)t^{3\alpha}}{(\alpha^2 + 3\alpha + 2) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 1)} + \frac{13c_1x\Gamma(2\alpha + 1)t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 &\quad \frac{12c_1x\Gamma(2\alpha + 1)t^{3\alpha}}{\alpha (\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{\alpha^2c_1x\Gamma(2\alpha + 1)t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 &\quad \frac{24c_1x\Gamma(2\alpha + 2)t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 4)} + \frac{36c_1x\Gamma(2\alpha + 3)t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \\
 &\quad \frac{24c_1x\Gamma(2\alpha + 3)t^{3\alpha+2}}{\alpha (\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \frac{8c_1x\Gamma(2\alpha + 4)t^{3\alpha+4}}{\alpha (\alpha + 1) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 5)} \\
 &\dots
 \end{aligned}$$

Thereupon, considering the first two sentences as estimates of solution for Eq.(5.16):

TABLE 1. A comparison between approximate solutions with some methods for test example 5.1.

t	x	u_{VIM}	v_{ADM}	u_{HPM}	u_{VHPIM}	u_{Oq-HAM}	u_{OHAM}	u_{Exact}
0.2	0.25	0.050309	0.050000	0.0499876	0.0499876	0.050318	0.050214	0.050000
	0.50	0.100619	0.100000	0.099978	0.0999746	0.091040	0.100428	0.100000
	0.75	0.150928	0.150001	0.149968	0.149962	0.150025	0.150642	0.150000
	1.0	0.201237	0.200001	0.199957	0.199951	0.20100	0.150642	0.200000
0.4	0.25	0.101894	0.100023	0.099645	0.0995290	0.09609	0.101537	0.100000
	0.50	0.203787	0.200046	0.199290	0.199059	0.20370	0.203074	0.200000
	0.75	0.305681	0.300069	0.298935	0.298588	0.300009	0.304611	0.300000
	1.0	0.407575	0.400092	0.398580	0.398118	0.400001	0.304611	0.400000
0.6	0.25	0.153094	0.150411	0.147158	0.145690	0.153001	0.154166	0.150000
	0.50	0.306188	0.300823	0.294317	0.291380	0.300088	0.308331	0.300000
	0.75	0.459282	0.451234	0.441475	0.437070	0.450207	0.462497	0.450000
	1.0	0.612376	0.601646	0.588634	0.582759	0.600633	0.462497	0.600000

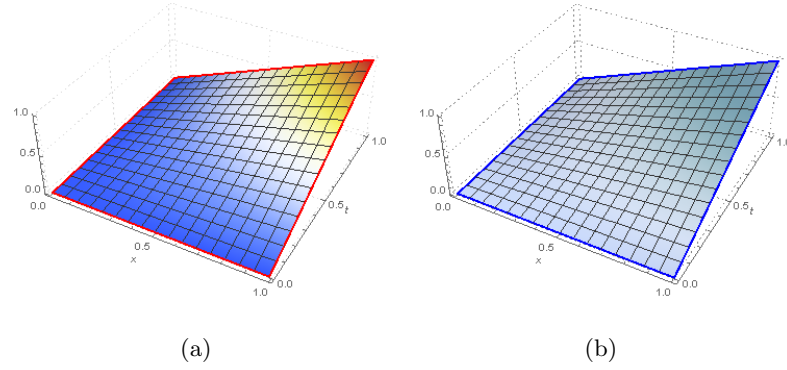


FIGURE 1. (a) The accurate solution (b) The estimate solution in the case $\alpha = 1.0$.

$$\begin{aligned}
 u(x, t) \approx & \frac{xt^\alpha (\alpha^2 + 3\alpha + 2t^2 + 2)}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} - \frac{2c_1 xt^{\alpha+2}}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} + \frac{2c_1 xt^{\alpha+2}}{(\alpha^3 + 3\alpha^2 + 2\alpha) \Gamma(\alpha)} + \\
 & \frac{2c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha)} + \frac{8c_1 x \Gamma(2\alpha) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 1)} + \\
 & \frac{13c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{12c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{\alpha (\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 & \frac{\alpha^2 c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{24c_1 x \Gamma(2\alpha + 2) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 4)} + \\
 & \frac{36c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \frac{24c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{\alpha (\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \\
 & \frac{8c_1 x \Gamma(2\alpha + 4) t^{3\alpha+4}}{\alpha (\alpha + 1) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 5)}.
 \end{aligned} \tag{5.18}$$

According to least square method for the calculations of the constants c_1 and c_2 , we can gain $c_1 = 0$, $c_2 = -0.668223$.

In Table 1, we can see the estimated solutions toward $\alpha = 1$, which is derived for various values of x applying OHAM and a comparison between ADM, VIM, HPM, VHPIM and Oq-HAM [7].

In figure 1, we can view the precise and approximate answers featuring $\alpha = 1$.

Table 2 shows comparison between the exact and the approximation solution (5.16) with OHAM of test example 5.1 for different values of α , x and t .

Comparison of exact and approximate solution can be seen for test example 5.1 with different values of α , x and t , in Figure 2.

TABLE 2. The exact and approximate result of test example 5.1 featuring various values of α .

x	t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1.0$	u_{Exact}
0.25	0.2	0.114114	0.079887	0.050214	0.05
	0.4	0.148258	0.131658	0.101537	0.1
	0.6	0.164999	0.173966	0.154166	0.15
	0.8	0.162959	0.205729	0.206301	0.2
0.50	0.2	0.228229	0.159774	0.100428	0.1
	0.4	0.296516	0.263317	0.203074	0.2
	0.6	0.329999	0.347933	0.308331	0.3
	0.8	0.325918	0.411458	0.412602	0.4

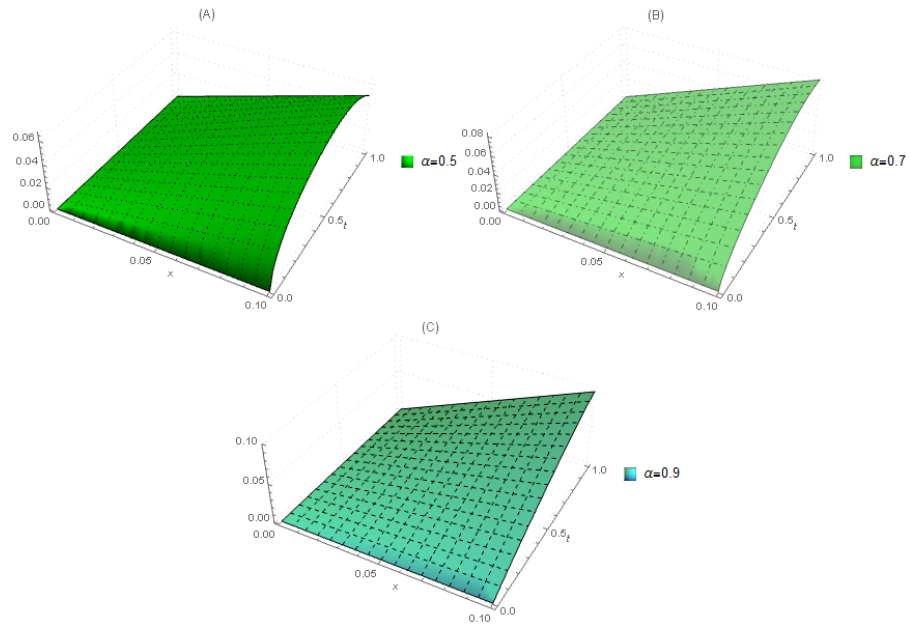


FIGURE 2. Comparison between the exact and the approximation solution with OHAM of test example 5.1 for different values of α , x and t .

Example 5.2. For the second example, we propound the time-fractional Klein-Gordon differential equation:

$$D_t^\alpha u(x, t) - u_{xx}(x, t) + u(x, t) = t^2 + x^2, \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \alpha \leq 2, \quad (5.19)$$

given that the primary condition

$$u(x, 0) = x^2 - \exp(x), \quad u_t(x, 0) = 0. \quad (5.20)$$

With the help of the OHAM, according to what was formulated and presented in section 3 for Eqs. (5.19)-(5.20), we get:

$$\begin{aligned}
 u_0(x, t) &= x^2 - e^x + \frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} + \frac{x^2t^\alpha}{\alpha\Gamma(\alpha)}, \\
 u_1(x, t) &= -\frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{x^2t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{x^2t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \\
 &\quad \frac{2t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{c_1x^2t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \\
 &\quad \frac{c_1x^2t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \frac{2c_1t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \\
 &\quad \frac{2c_1t^\alpha}{\alpha\Gamma(\alpha)} - \frac{2c_1t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{\sqrt{\pi}2^{1-2\alpha}c_1t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})} + \\
 &\quad \frac{2c_1t^{2\alpha+2}}{\Gamma(2\alpha + 3)} + \frac{\sqrt{\pi}4^{-\alpha}c_1x^2t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})}, \\
 &\dots
 \end{aligned}$$

Then, assuming the first two sentences as estimates of solution for Eq.(5.19)

$$\begin{aligned}
 u(x, t) \approx &x^2 - e^x + \frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} + \frac{x^2t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{x^2t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \\
 &\frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \\
 &\frac{c_1x^2t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{2c_1t^\alpha}{\alpha\Gamma(\alpha)} - \frac{c_1x^2t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \frac{2c_1t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1t^{2\alpha+2}}{\Gamma(2\alpha + 3)} + \\
 &\frac{2c_1t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{\sqrt{\pi}2^{1-2\alpha}c_1t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})} + \\
 &\frac{\sqrt{\pi}4^{-\alpha}c_1x^2t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})}.
 \end{aligned} \tag{5.21}$$

For the calculations of the constants c_1, c_2 using the method of least squares, we have computed that

$$c_1 = -0.942868, \quad c_2 = 0.00777353.$$

In Table 3 and in figure 3, we can view the precise and approximate answers featuring $\alpha = 2$ through applying OHAM. With the knowledge that $\alpha = 2$, the approximate solution obtained by the proposed method corresponds to the precise solution $u(x, t) = t^2 + x^2 - e^x$.

Example 5.3. For the third example, we offer the time-fractional partial differential equation:

$$D_t^\alpha u(x, t) - u_{xx}(x, t) - u(x, t) = 3t, \quad t > 0, \quad x \in \mathbb{R}, \quad 2 < \alpha \leq 3, \tag{5.22}$$

TABLE 3. Approximate result of test example 5.2.

t	x	u_{OHAM}	$Exact$	$Absolute\ error$
0.0	0.0	-1.	-1.	0.0
0.1	0.5	-0.84694	-0.845171	0.00176917
0.2	0.4	-1.02272	-1.0214	0.00131616
0.3	0.3	-1.17067	-1.16986	0.000814838
0.4	0.2	-1.2922	-1.29182	0.000379601
0.5	0.1	-1.38882	-1.38872	0.0000949325

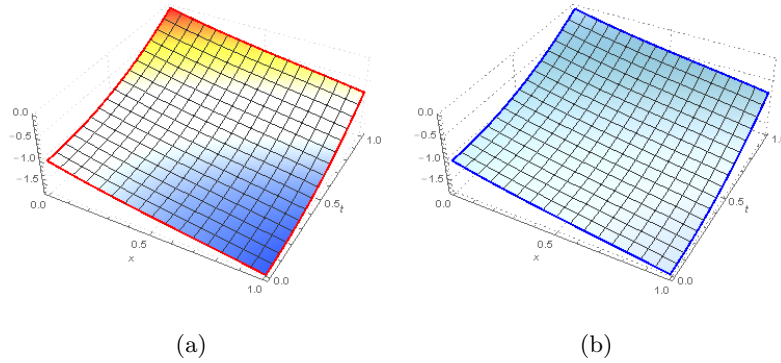


FIGURE 3. (a) The accurate solution (b) The estimate solution in the case $\alpha = 2.0$.

including the primary condition

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin(x) - 3, \quad u_{tt}(x, 0) = 0. \quad (5.23)$$

With due attention to the OHAM, according to section 3 for Eqs.(5.22)-(5.23), we get:

$$\begin{aligned}
 u_0(x, t) &= \frac{3t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + t(\sin(x) - 3), \\
 u_1(x, t) &= \frac{3c_1t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{6c_1\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)}, \\
 u_2(x, t) &= \frac{3c_1t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \frac{3c_1^2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1^2t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \\
 &\quad \frac{3c_2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + \frac{6c_1^2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} - \frac{6c_2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2t^{3\alpha+1}}{\Gamma(3\alpha + 2)}. \\
 \dots
 \end{aligned}$$

Hence, supposing the first two sentences as estimates of solution for Eq.(5.22):

$$\begin{aligned}
 u(x, t) \approx & \frac{3t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + t(\sin(x) - 3) + \frac{6c_1t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \\
 & \frac{6c_1\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1^2t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \frac{6c_1^2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \\
 & \frac{3c_2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{6c_2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2t^{3\alpha+1}}{\Gamma(3\alpha + 2)}.
 \end{aligned}
 \tag{5.24}$$

Using the method of least squares, to obtain the constants c_1 and c_2 , we will have

$$c_1 = 0, \quad c_2 = 1.02134.$$

It can be seen in Table 4 and Figure 4 that solving equations with approximate expression is calculated and displayed for $\alpha = 3$ and various values of x and t . Toward $\alpha = 3$, the

TABLE 4. Approximate result of test example 5.3.

t	x	u_{OHAM}	Exact	Absolute error
0.0	0.0	0.0	0.0	0.0
0.1	0.5	-1.45008	-1.45008	0.000161988
0.2	0.4	-1.1206	-1.12053	0.0000672998
0.3	0.3	-0.811365	-0.811344	0.0000214763
0.4	0.2	-0.522121	-0.522116	4.26071×10^{-6}
0.5	0.1	-0.252058	-0.252057	2.6672×10^{-7}

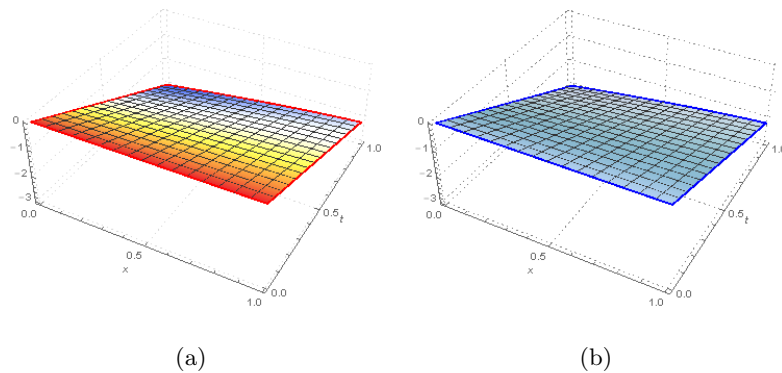


FIGURE 4. (a) The accurate solution (b) The estimate solution in the case $\alpha = 3.0$.

solution that we have gained is in accordance with the precise solution $u(x, t) = t \sin(x) - 3t$.

6. CONCLUSION

We have successfully applied OHAM to obtain approximate solution of the non linear partial differential equations featuring fractional derivative. The result indicate that a few iteration of OHAM will results in some useful solutions.

Finally, it should be added that the suggested technique has the potentials to be practical in solving other similar nonlinear and linear problems in partial differential equations featuring fractional derivative.

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