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SEMI-SYMMETRIC STATISTICAL MANIFOLDS

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Abstract. This paper studies semi-symmetric statistical manifolds (3S-manifolds for short) to generalise semi-Weyl manifolds. We prove that this class of manifolds is invariant under the conformal change of metrics. We show that every 3S-structure $(g,\omega,\omega^*,\nabla)$ on a Riemannian manifold (M,g) induces a statistical structure $(g,\widetilde{\nabla})$ on M and we find necessary and sufficient conditions for ∇ and $\widetilde{\nabla}$ to have the same sectional curvature. In addition, the analogue of the statistical Curvature is defined for 3S structures and its properties are investigated. We also give a method to construct 3S structures on a warped product manifold from 3S structures on the fiber and base manifolds.

Keywords: Semi-symmetric connection, Semi-Weyl structure, Dual connection, Semi-symmetric statistical manifold, 3S-manifold.

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1. Introduction

Statistical Manifolds were introduced by S.L. Lauritzen [14] and lie at the confluence of some research areas such as Information Geometry, Affine Geometry and Hessian Geometry (see [2, 11, 18] and references therein). Therefore, statistical manifolds have been intensively studied in several contexts where they are also associated with additional structures and lead to other concepts such as statistical holomorphic structures, statistical Sasakian structures, statistical submanifolds etc. (See [10, 9, 8, 16] and references therein.)

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There have been several attempts to generalize statistical structures which have led for instance to quasi-statistical structures, Weyl, semi-Weyl, quasi-semi-weyl structures and semi-symmetric non-metric statistical structures (see [1, 15, 6] and references therein). But, all these generalizations failed to satisfy some nice properties such as invariance under conformal changing of the Riemannian metric.

In this paper, we introduce and study Semi-Symmetric statistical manifolds (3S-manifolds for short), $(M, g, \omega, \omega^*, \nabla)$ which form a subclass of quasi-semi-Weyl manifolds, such that both ∇ and ∇^* are semi-symmetric connections. It appears that unlike the other generalizations of statistical manifolds [1, 15], the 3S-manifolds are invariant under the conformal changing of the metric. Other good results are obtained and compared, when possible, to the existing ones in a statistical setting.

The paper is organized as follows: The next Section is devoted to the preliminaries where we recall basic definitions and properties of statistical structures that we need in the sequel of the paper. Section 3 deals with 3S structures. We show that non-trivial statistical structures can be generated from 3S structures and vice-versa. And when the so-called 3S-mean vector field is torse-forming, the 3S connection is of constant sectional curvature if and only if its associated statistical connection is of the same constant sectional curvature. In the setting of this paper, α-connections associated to 3S-connections are introduced and studied and it is shown that surprisingly, the curvatures relations obtained for α-3S-connections are similar to those of the classical statistical setting. The section ends with the study of the analogue of the statistical curvature for the 3S connections and the condition for the statistical sectional curvature to be constant. In Section 4, we briefly show that a submanifold of a 3S-manifold is also a 3S-manifold. Finally, Section 5 deals with the warped product of 3S-manifolds. We give a way to construct a 3S structure on a warped product, starting with 3S structures on fiber and the base manifolds.

2. Preliminaries

In the present section, we give some basic definitions and fundamental formulae useful in the sequel.

In what follows, M denotes a smooth manifold, g a Riemannian metric on M, ∇^g the Levi-Civita connection of g, and ∇ an affine connection on M. Throughout the paper, we shall denote the tangent bundle of M by TM, its cotangent bundle by T^*M and the set of smooth sections of TM (respectively, of T^*M) by $\mathfrak{X}(M)$ (respectively, by $\Omega^1(M)$). We will also write T^{∇} to denote the torsion of ∇. From now on, for any $\omega \in \Omega^1(M)$, we denote by S_ω the tensor

$$S_\omega = \omega \otimes I - I \otimes \omega,$$

where $I : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the identity map.

The dual connection of ∇ with respect to g is the unique affine connection ∇^* on M such that:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(\nabla^*_X Z, Y), \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In this case the triplet (g, ∇, ∇^*) is called a dualistic structure on M.

Definition 2.1 ([12]). *A connection ∇ is said to be semi-symmetric affine connection if there exists a $1-$form ω such that $T^{\nabla} = S_w$.*

Definition 2.2 ([10, 6]). *The pair (g, ∇) is called a statistical structure on M (and (M, g, ∇) a statistical manifold) when ∇ and its dual ∇^* are torsion-free affine connections. This is equivalent to saying that ∇ is torsion-free and the cubic form $C = \nabla g$ is totally symmetric, that is*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) \tag{2.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.3 ([17]). *Let ∇ be a torsion-free affine connection and let ω a 1-form on M. The triplet (g, ω, ∇) is called a Weyl structure on M (and (M, g, ω, ∇) a Weyl manifold) if*

$$(\nabla_X g)(Y, Z) = -\omega(X)g(Y, Z), \tag{2.3}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.4 ([1]). *Let ∇ be a torsion-free affine connection and let ω be a 1-form on M. Then, (g, ω, ∇) is called a semi-Weyl structure on M (and (M, g, ω, ∇) a semi-Weyl manifold) if*

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = -g(S_\omega(X, Y), Z), \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In (2.3) and (2.4) when $\omega = 0$ one finds $\nabla = \nabla^g$. Hence statistical structure, Weyl structure and semi-Weyl structure may be regarded as generalizations of the Levi-Civita connection. A direct computation show that for any affine connection ∇ one has:

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = g(T^{\nabla^*}(X,Y) - T^{\nabla}(X,Y), Z). \tag{2.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Remark 2.1. *Observe that when (M, g, ω, ∇) is a semi-Weyl manifold, the dual connection ∇^* is rather semi-symmetric.*

Definition 2.5 ([1])**.** *Let ∇ be an affine connection on M with torsion tensor T^{∇} and let ω be a 1-form. Then, (g, ω, ∇) is called a quasi-semi- Weyl structure on M (and (M, g, ω, ∇) a quasi-semi-Weyl manifold) if*

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = -g(T^{\nabla}(X,Y) + S_\omega(X,Y), Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In the next section we are interested in quasi-semi-Weyl structures (g, ω, ∇) such that ∇ is semi-symmetric.

3. Semi-symmetric statistical manifolds

Semi-symmetric metric connections on statistical manifolds have been introduced and studied in [12, 13, 3]. Such a connection is also called statistical semi-symmetric connection in [13]. In this mentioned papers, the dual connection of a statistical semi-symmetric connection ∇ has the same torsion as ∇. We keep the same terminology to define something more general. In what follows, the torsion of the dual of ∇ may be different from the one of ∇.

Definition 3.1. *The pair (g, ∇) is called a semi-symmetric statistical structure on M (and (M, g, ∇) a semi-symmetric statistical manifold) if there are two 1-forms ω and ω^* on M such that*

$$T^{\nabla} = S_\omega \quad \text{and} \quad T^{\nabla^*} = S_{\omega^*}.$$

To be short, such a structure will be called a 3S-structure and the connection ∇ a 3S-connection.

Semi-symmetric statistical structure is a generalization of semi-symmetric metric connection studied in [12, 13]. When $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, we set

$$\omega^S = \frac{1}{2}(\omega + \omega^*), \tag{3.6}$$

and we call it the 3S-mean 1-form (for short the mean 1-form). The vector field V, g-associated to ω^S is called the mean vector field. The 3S-structure $(g, \omega, \omega^*, \nabla)$ is said to be closed when ω^S is closed.

Example 3.1. *We have the following simple examples of 3S-structures:*

1. *A statistical structure (g, ∇) is a 3S-structure and is called the trivial 3S-structure, where $\omega = \omega^* = 0$.*
2. *A semi-Weyl structure (g, ω, ∇) is a 3S-structure.*
3. *If (g, ∇) is a statistical structure then for all 1-form ω on M, $(g, \omega, -\omega, \overline{\nabla} =: \nabla + \omega \otimes I)$ is a 3S-structure. Just observe that the dual connection $\overline{\nabla}^*$ of $\overline{\nabla}$ with respect to g is given by $\overline{\nabla}^* = \nabla^* + \omega^* \otimes I$ with $\omega^* = -\omega$.*

It is proven in [6] that, if f is a function on M and ω_1, ω_2 are 1-forms g-associated to the vector fields ξ_1, ξ_2 on M respectively, then there exists a unique affine connection ∇ which satisfies the following equations:

$$T^{\nabla}(X, Y) = \omega_1(X)Y - \omega_1(Y)X, \tag{3.7}$$

$$(\nabla_X g)(Y, Z) = f\big(\omega_2(Y)g(X, Z) + \omega_2(Z)g(X, Y)\big). \tag{3.8}$$

Such a connection has been called semi-symmetric non-metric connection and is given by

$$\nabla_X Y = \nabla^g_X Y + \omega_1(Y)X - g(X, Y)\xi_1 - fg(X, Y)\xi_2.$$

Remark 3.1. *All semi-symmetric non-metric affine connection are 3S-connections. But a 3S-connection needs not to be semi-symmetric non-metric affine connection.*

Indeed, let ∇ be an affine connection which satisfies (3.7) and (3.8). Since X, Y, Z are arbitrary in (3.8), so we have

$$(\nabla_Y g)(X, Z) = f\big(\omega_2(X)g(Y, Z) + \omega_2(Z)g(X, Y)\big). \tag{3.9}$$

Using (3.8) and (3.9) we get

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = g(f\omega_2(Y)X - f\omega_2(X)Y, Z). \tag{3.10}$$

Moreover, using (3.7), (3.8) and (3.10) we have

$$T^{\nabla^*}(X,Y) = (f\omega_2 - \omega_1)(Y)X - (f\omega_2 - \omega_1)(X)Y.$$

Hence $T^{\nabla} = S_{\omega_1}$ and $T^{\nabla^*} = S_{\omega_1 - f\omega_2}$.

Conversely, take h a function on M. Then $(g, dh, -dh, \nabla^g + dh \otimes I)$ is 3S-structure on M but

$$(\nabla_X g)(Y,Z) = -2(dh)(X)g(Y,Z),$$

which shows that ∇ is not a semi-symmetric non-metric affine connection.

The following theorem shows that unlike statistical structures, 3S-structures are preserved under conformal changing of g.

Theorem 3.1. *Let $\widetilde{g}$ be a conformal metric of g such that for a function h on M we have, $\widetilde{g} = e^h g$. Then, $(g, \omega, \omega^*, \nabla)$ is 3S-structure on M if and only if $(\widetilde{g}, \omega, \omega^* + dh, \nabla)$ is 3S-structure on M.*

Proof. Let X, Y, Z be three vector fields on M, ∇^d the dual connection of ∇ with respect to $\widetilde{g}$ and ∇^* the dual connection of ∇ with respect to g. From the duality condition of ∇ with respect to $\widetilde{g}$ we have

$$X\widetilde{g}(Y,Z) = \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla^d_X Z),$$

that is

$$Xg(Y,Z) + X(h)g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^d_X Z). \tag{3.11}$$

As X, Y and Z are arbitrary, from (3.11) we have

$$Zg(X,Y) + Z(h)g(X,Y) = g(\nabla_Z Y, X) + g(Y, \nabla^d_Z X). \tag{3.12}$$

Take (3.11) and substract (3.12), then

$$\begin{aligned} &Xg(Y,Z) - Z(h)g(X,Y) + X(h)g(Y,Z) - Zg(X,Y) \\ &= g(\nabla_X Y, Z) - g(\nabla_Z Y, X) + g(T^{\nabla^d}(X,Z) + [X,Z], Y). \end{aligned}$$

That is

$$\begin{aligned} &g(Y, \nabla^*_X Z) - g(Y, \nabla^*_Z X) + g(Y, X(h)Z - Z(h)X) \\ &= g(T^{\nabla^d}(X,Z) + [X,Z], Y). \end{aligned}$$

Therefore, we have

$$T^{\nabla^d}(X,Z) = T^{\nabla^*}(X,Z) + S_{dh}(X,Z).$$

Thus $T^{\nabla^d} = S_{\omega^*+dh}$ if and only if $T^{\nabla^*} = S_{\omega^*}$. This ends the proof. □

One can easily prove the following:

Theorem 3.2. *Let (g, M) be a Riemannian manifold, ∇ a connection on M, η, ω and ω^* some 1-forms on M such that $2\omega^S = \omega + \omega^*$. Set V and ζ vector fields g-associated to ω^S and η respectively, i.e., $g(X, V) = \omega^S(X)$ and $g(X, \zeta) = \eta(X)$. The pair $(g, \widetilde{\nabla})$ where*

$$\widetilde{\nabla} = \nabla - \omega \otimes I + \eta \otimes I + I \otimes \eta + g(.,.)(\zeta - 2V), \tag{3.13}$$

is a statistical structure on M if and only if $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M. Moreover, the dual of $\widetilde{\nabla}$ with respect to g is given by

$$(\widetilde{\nabla})^* = \nabla^* + \omega \otimes I - \eta \otimes I - I \otimes \eta + 2I \otimes \omega^S - g(.,.)\zeta, \tag{3.14}$$

where ∇^* is the dual of ∇ with respect to g.

Taking $\eta = 0$ in (3.13), one obtains

$$\widetilde{\nabla} = \nabla - \omega \otimes I - 2g(.,.)V, \tag{3.15}$$

$$(\widetilde{\nabla})^* = \nabla^* + \omega \otimes I + 2I \otimes \omega^S. \tag{3.16}$$

$\widetilde{\nabla}$ is called the statistical connection with respect to ω and associated to ∇.
For a 3S-structure $(g, \omega, \omega^*, \nabla)$ on M, the mean vector field V is called torse-forming with respect to ω if

$$\nabla_X V = \omega(X)V,$$

for any X in $\mathfrak{X}(M)$.

Example 3.2. *We consider the manifold $M = \{(x, y) \in \mathbb{R}^2, x; y > 0\}$ equipped with its canonical metric $g_0 = dx^2 + dy^2$ and we set the function $h(x, y) = -\frac{1}{2}ln(x + y)$ on M. The gradient ∇h of h with respect to g_0 is given by*

$$\nabla h = -\frac{1}{2(x+y)}(\vec{i} + \vec{j}),$$

where $(\vec{i}, \vec{j}) = (\partial_x, \partial_y)$ is the canonical basis of $\mathfrak{X}(M)$. From Theorem 3.1, we see that $(M, \widetilde{g}, \omega, \omega^*, \nabla^{g_0})$ is a 3S-manifold, that is $(M, \widetilde{g}, \omega^*, \omega, (\nabla^{g_0})^*)$ is a 3S-manifold, where $\widetilde{g} = e^{2h}g_0$, $\omega = 0$, $\omega^* = 2dh$, ∇^{g_0} is a Levi-Civita connection with respect to g_0 and the dual $(\nabla^{g_0})^*$ of ∇^{g_0} with respect to $\widetilde{g}$ is given by

$$(\nabla^{g_0})^* = \nabla^{g_0} + \omega^* \otimes I. \tag{3.17}$$

Moreover, we have $\omega^S = dh$, *that is* $(\omega^S)^{\sharp_{\widetilde{g}}} = V$. *From*

$$\widetilde{g}((\omega^S)^{\sharp_{\widetilde{g}}}, X) = X(h) = g_0((dh)^{\sharp_{g_0}}, X),$$

we get $V = e^{-2h}\nabla h$. *From (3.17), we have*

$$(\nabla^{g_0})^*_X V = e^{-2h}\nabla^{g_0}_X \nabla h, \tag{3.18}$$

for any vector field X *in* M. *Since* $(\nabla^{g_0})_{\vec{i}}\vec{i} = (\nabla^{g_0})_{\vec{i}}\vec{j} = (\nabla^{g_0})_{\vec{j}}\vec{i} = (\nabla^{g_0})_{\vec{j}}\vec{j} = \vec{0}$, *we have*

$$(\nabla^{g_0})_{\vec{i}}\nabla h = -\frac{1}{x+y}\nabla h \ \ \textit{and} \ \ (\nabla^{g_0})_{\vec{j}}\nabla h = -\frac{1}{x+y}\nabla h. \tag{3.19}$$

So, using (3.19), for any vector field $X = X^1\vec{i} + X^2\vec{j}$ *in* M, *we have*

$$\begin{aligned}\nabla^{g_0}_X \nabla h &= -\frac{1}{x+y}[X^1 + X^2]\nabla h \\ &= 2g_0(\nabla h, X)\nabla h \\ &= 2dh(X)\nabla h \\ &= \omega^*(X)\nabla h.\end{aligned}$$

From (3.18), we get

$$(\nabla^{g_0})^*_X V = \omega^*(X)V.$$

Thus, V *is a torse-forming vector on the 3S-structure* $(M, \widetilde{g}, \omega^*, \omega, (\nabla^{g_0})^*)$ *with respect to* ω^*.

Proposition 3.1. *For a 3S-structure* $(g, \omega, \omega^*, \nabla)$ *on* M, *for any 1-form* η, *we set* $\rho = \eta - \omega$ *and* $\Omega = \zeta - 2V$, *where* V *and* ζ *are vector fields associated with* ω^S *and* η *with respect to* g *respectively. The relationship between the curvature* R *of* $(g, \omega, \omega^*, \nabla)$ *and the curvature* $\widetilde{R}$ *of the statistical structure* $(g, \widetilde{\nabla})$ *associated to* $(g, \omega, \omega^*, \nabla)$ *is given by*

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z &- (d\omega)(X,Y)Z + (d\eta)(X,Y)Z \\ &+ g(Y,Z)\nabla_X\Omega - g(X,Z)\nabla_Y\Omega + \{g(Y,Z)\rho(X) - g(X,Z)\rho(Y)\}\Omega \\ &+ \big\{X\eta(Z) - \eta(X)\eta(Z) + \omega(X)\eta(Z) - \eta(\nabla_X Z) - \eta(\Omega)g(X,Z)\big\}Y \\ &- \big\{Y\eta(Z) - \eta(Y)\eta(Z) + \omega(Y)\eta(Z) - \eta(\nabla_Y Z) - \eta(\Omega)g(Y,Z)\big\}X,\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 3.2. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. We assume that the sectional curvatures of ∇ and $\widetilde{\nabla}$ defined as (3.15) are well-defined. Then, these sectional curvatures are the same if and only if the mean vector field V is torse-forming with respect to ω.*

Proof. Let X, Y be two linear independent vector fields on M. Let $\widetilde{R}$ and R the curvature of $\widetilde{\nabla}$ and ∇ respectively. From proposition 3.1, taking $\eta = 0$ we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - (d\omega)(X,Y)Z \\ &\quad + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\}.\end{aligned}$$

Also we have

$$\widetilde{R}(X,Y,Y,X) = R(X,Y,Y,X) + 2g(X, \omega(X)V - \nabla_X V), \tag{3.20}$$

where $\widetilde{R}(X,Y,Y,X) = g(\widetilde{R}(X,Y)Y,X)$ and $R(X,Y,Y,X) = g(R(X,Y)Y,X)$. Moreover, from (3.20), $\widetilde{R}(X,Y,Y,X) = R(X,Y,Y,X)$ if and only if the vector field V is torse-forming with respect ω. □

We define the tensor S by

$$S = \nabla - \nabla^g. \tag{3.21}$$

Lemma 3.1. *Let ∇ be an affine connection on M. Then, we have*

$$g(X, S(Y,Z)) - g(Y, S(X,Z)) = g(T^{\nabla^*}(X,Y), Z), \tag{3.22}$$

$$S(X,Y) - S(Y,X) = T^{\nabla}(X,Y), \tag{3.23}$$

for all vectors fields X, Y, Z on M.

Proof. Let X, Y, Z be three vectors fields on M. From the duality condition of ∇^g

$$Xg(Y,Z) = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z),$$

and eq. (3.21), we get

$$g(\nabla^*_X Y, Z) = g(\nabla^g_X Y, Z) - g(Y, S(X,Z)). \tag{3.24}$$

Therefore, from (3.24) we obtain

$$g(X, S(Y,Z)) - g(Y, S(X,Z)) = g(T^{\nabla^*}(X,Y), Z).$$

Moreover,

$$S(X,Y) - S(Y,X) = \nabla_X Y - \nabla^g_X Y - \nabla_Y X + \nabla^g_Y X$$
$$= T^{\nabla}(X,Y).$$

□

Proposition 3.3. *Let ∇ be an affine connection on M. Then, $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M if and only if*

$$S(X,Y) - S(Y,X) = S_\omega(X,Y), \tag{3.25}$$

$$g(X, S(Y,Z)) - g(Y, S(X,Z)) = g(S_{\omega^*}(X,Y), Z). \tag{3.26}$$

Proof. This follows easily from lemma 3.1. □

If $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M, from Proposition 3.3 we give the Levi-Civita connection ∇^g in terms of g, mean 1-form ω^S, mean vector field V, ∇ and ∇^*. For this purpose, we set

$$K^{\omega,\omega^*}(X,Y) = \omega^S(Y)X - g(X,Y)V,$$

Proposition 3.4. *If $(g,\omega,\omega^*,\nabla)$ is a 3S-structure on M, then the Levi-Civita connection ∇^g is given by:*

$$\nabla^g_X Y = \frac{1}{2}\big(\nabla_X Y + \nabla^*_X Y\big) + K^{\omega,\omega^*}(X,Y), \tag{3.27}$$

for all vector fields X,Y on M.

Proof. Let X, Y, Z be vector fields on M, ξ and ξ^* be vector fields g-associated to ω and ω^* respectively. From the duality condition of ∇ and ∇^g with respect to g we obtain

$$g(Z, S(X,Y)) = g(Y, \nabla^g_X Z - \nabla^*_X Z). \tag{3.28}$$

From (3.25) and (3.28) we get

$$g(Z, S(Y,X)) = g(Y, \nabla^g_X Z - \nabla^*_X Z) + g(Y, -\omega(X)Z + g(X,Z)\xi). \tag{3.29}$$

As (3.26) is equivalent to

$$g(Z, S(Y,X)) = g(Y, S(Z,X)) + g(Y, \omega^*(Z)X - g(Z,X)\xi^*), \tag{3.30}$$

then from (3.29) and (3.30) we have

$$g(Y,S(Z,X))+g(Y,\omega^*(Z)X-g(Z,X)\xi^*)$$
$$=g(Y,\nabla^g_X Z-\nabla^*_X Z-\omega(X)Z+g(X,Z)\xi),$$

that is

$$S(Z,X)=\nabla^g_X Z-\nabla^*_X Z-\omega(X)Z-\omega^*(Z)X+g(X,Z)(\xi+\xi^*). \tag{3.31}$$

According to (3.25) and (3.31) we have

$$S(X,Z)+S_\omega(Z,X)=\nabla^g_X Z-\nabla^*_X Z-\omega(X)Z-\omega^*(Z)X+g(X,Z)(\xi+\xi^*),$$

that is

$$S(X,Z)=\nabla^g_X Z-\nabla^*_X Z-(\omega+\omega^*)(Z)X+g(X,Z)(\xi+\xi^*). \tag{3.32}$$

Using (3.21) and (3.32), we get

$$2\nabla^g_X Z=\nabla_X Z+\nabla^*_X Z+(\omega+\omega^*)(Z)X-g(X,Z)(\xi+\xi^*).$$

□

Corollary 3.1. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M. Then, the dual connection ∇^* of ∇ is given by*

$$S=\nabla^g-\nabla^*-2K^{\omega,\omega^*}=\frac{1}{2}(\nabla-\nabla^*)-K^{\omega,\omega^*}. \tag{3.33}$$

For a given totally symmetric (0, 3)-tensor field, we can define mutually dual semi-symmetric affine connections.

Proposition 3.5. *Assume that (g,M) is a Riemannian manifold, C is a totally symmetric (0, 3)-tensor field on M, ω and ω^* two 1-forms on M such that $2\omega^S=\omega+\omega^*$ and V the vector field g-associated to ω^S. We define the mapping ∇ and ∇^* by*

$$g(\nabla_X Y,Z)=g(\nabla^g_X Y,Z)-\frac{1}{2}C(X,Y,Z)+g(Z,\omega(X)Y+2g(X,Y)V), \tag{3.34}$$
$$g(\nabla^*_X Y,Z)=g(\nabla^g_X Y,Z)+\frac{1}{2}C(X,Y,Z)-g(Z,\omega(X)Y+2\omega^S(Y)X). \tag{3.35}$$

Then ∇ and ∇^* are mutually dual connections. Moreover $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M.

Proposition 3.6. *Let ∇ and ∇^* be mutually dual connections on M with respect to g. Let C be a (0, 3)-tensor field on M, $2\omega^S=\omega+\omega^*$ where ω,ω^* are two 1-forms on M and V the vector field g-associated with ω^S. We suppose that ∇ and C verify (3.34) and $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M. Then C is totally symmetric.*

Proof. We set $S = \nabla - \nabla^g$ and $\widetilde{S} = \widetilde{\nabla} - \nabla^g$ such that ∇ and $\widetilde{\nabla}$ verify (3.15). Let $X, Y, Z \in \mathfrak{X}(M)$. Since ∇ and C verify (3.34), so we get

$$C(X,Y,Z) = -2g(S(X,Y) - \omega(X)Y - 2g(X,Y), Z). \tag{3.36}$$

From (3.36), we have

$$C(X,Y,Z) = -2g(\widetilde{S}(X,Y), Z).$$

$(g, \widetilde{\nabla})$ is statistical structure implies that $C(X,Y,Z) = (\widetilde{\nabla}_X g)(Y,Z)$, thus, C is totally symmetric. □

3.1. **Some results on the alpha-connections of a 3S-structure.** For $\alpha \in \mathbb{R}$, we define a family of connections $\nabla^{(\alpha)}$ by,

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*. \tag{3.37}$$

$\nabla^{(\alpha)}$ is called an α-connection of dualistic structure (∇, ∇^*). The dual of $\nabla^{(\alpha)}$ with respect to g is given by $\nabla^{(-\alpha)}$.

If $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, we set

$$\omega_\alpha = \frac{1+\alpha}{2}\omega + \frac{1-\alpha}{2}\omega^* \quad \text{and} \quad \omega^*_\alpha = \omega_{-\alpha} = \frac{1-\alpha}{2}\omega + \frac{1+\alpha}{2}\omega^*. \tag{3.38}$$

In particular,

$$\omega_0 = \omega^g, \omega_1 = \omega \ \text{ and } \ \omega_{-1} = \omega^*. \tag{3.39}$$

We define the tensors K by

$$K = \nabla^* - \nabla. \tag{3.40}$$

Proposition 3.7. *If $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, then $(g, \omega_\alpha, \omega^*_\alpha, \nabla^{(\alpha)})$ is a 3S-structure.*

Proof. Using (3.37) and (3.38) we get

$$T^{\nabla^{(\alpha)}} = S_{\omega_\alpha} \ \text{ and } \ T^{\nabla^{(-\alpha)}} = S_{\omega^*_\alpha}.$$

□

When $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M we have the following equality,

$$\nabla^{(\alpha)} = \nabla^g - \frac{\alpha}{2}K - K^{\omega,\omega^*}. \tag{3.41}$$

In particular

$$\nabla = \nabla^g - \frac{1}{2}K - K^{\omega,\omega^*}, \tag{3.42}$$

$$\nabla^* = \nabla^g + \frac{1}{2}K - K^{\omega,\omega^*}. \tag{3.43}$$

From (3.40) we get

$$K(X,Y) - K(Y,X) = S_{\omega^*}(X,Y) - S_\omega(X,Y). \tag{3.44}$$

It was shown in [3] that for a pair of conjugate connections, their curvature tensors satisfy

$$g(R(X,Y)Z,T) + g(Z,R^*(X,Y)T) = 0, \tag{3.45}$$

and more generally

$$g(R^{(\alpha)}(X,Y)Z,T) + g(Z,R^{(-\alpha)}(X,Y)T) = 0 \tag{3.46}$$

where $R^{(-\alpha)} = R^{*(\alpha)}$.

Denote $K(X,Y,Z) = (\nabla_X g)(Y,Z)$, where

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z).$$

The 3-tensor $K(.,.,.)$ is called the cubic form and satisfies $K(.,Y,Z) = K(.,Z,Y)$ by its definition.

Recall the difference tensor $K(X,Y)$ introduced in (3.40), which can be verified to be related to $K(X,Y,Z)$ via

$$g(K(X,Y),Z) = K(X,Y,Z). \tag{3.47}$$

Proposition 3.8. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M, the curvature tensor $R^{(\alpha)}$ for the α-connection $\nabla^{(\alpha)}$ satisfies*

$$\begin{aligned} R^{(\alpha)}(X,Y)Z = {} & \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z \\ & + \frac{1-\alpha^2}{4}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big). \end{aligned} \tag{3.48}$$

Proof. We assume that $(g,\omega,\omega^*,\nabla)$ is a 3S-structure on M. By definition of the α-connection and of curvature tensor

$$
\begin{aligned}
R^{(\alpha)}(X,Y)Z = &\left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z \\
&+ \left(\frac{1-\alpha^2}{4}\right)\Big(\nabla_X\nabla^*_Y Z + \nabla^*_X\nabla_Y Z - \nabla_Y\nabla^*_X Z \\
&- \nabla^*_Y\nabla_X Z - \nabla_{[X,Y]}Z - \nabla^*_{[X,Y]}Z\Big).
\end{aligned} \tag{3.49}
$$

From (3.42) and (3.43) we have

$$
\begin{aligned}
\nabla_X\nabla^*_Y Z = &\nabla^g_X\nabla^g_Y Z - \frac{1}{2}K(X,\nabla^g_Y Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) \\
&+ \frac{1}{2}\nabla^g_X K(Y,Z) - \frac{1}{4}K(X,K(Y,Z)) - \frac{1}{2}K^{\omega,\omega^*}(X,K(Y,Z)) \\
&- \nabla^g_X K^{\omega,\omega^*}(Y,Z) + \frac{1}{2}K(X,K^{\omega,\omega^*}(Y,Z)) + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)),
\end{aligned}
$$

$$
\begin{aligned}
-\nabla_Y\nabla^*_X Z = &-\nabla^g_Y\nabla^g_X Z + \frac{1}{2}K(Y,\nabla^g_X Z) + K^{\omega,\omega^*}(Y,\nabla^g_X Z) \\
&- \frac{1}{2}\nabla^g_Y K(X,Z) + \frac{1}{4}K(Y,K(X,Z)) + \frac{1}{2}K^{\omega,\omega^*}(Y,K(X,Z)) \\
&+ \nabla^g_Y K^{\omega,\omega^*}(X,Z) - \frac{1}{2}K(Y,K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z)),
\end{aligned}
$$

$$
\begin{aligned}
\nabla^*_X\nabla_Y Z = &\nabla^g_X\nabla^g_Y Z + \frac{1}{2}K(X,\nabla^g_Y Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) \\
&- \frac{1}{2}\nabla^g_X K(Y,Z) - \frac{1}{4}K(X,K(Y,Z)) + \frac{1}{2}K^{\omega,\omega^*}(X,K(Y,Z)) \\
&- \nabla^g_X K^{\omega,\omega^*}(Y,Z) - \frac{1}{2}K(X,K^{\omega,\omega^*}(Y,Z)) + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)),
\end{aligned}
$$

$$
\begin{aligned}
-\nabla^*_Y\nabla_X Z = &-\nabla^g_Y\nabla^g_X Z - \frac{1}{2}K(Y,\nabla^g_X Z) + K^{\omega,\omega^*}(Y,\nabla^g_X Z) \\
&+ \frac{1}{2}\nabla^g_Y K(X,Z) + \frac{1}{4}K(Y,K(X,Z)) - \frac{1}{2}K^{\omega,\omega^*}(Y,K(X,Z)) \\
&+ \nabla^g_Y K^{\omega,\omega^*}(X,Z) + \frac{1}{2}K(Y,K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))
\end{aligned}
$$

and

$$
-\nabla_{[X,Y]}Z - \nabla^*_{[X,Y]}Z = 2K^{\omega,\omega^*}([X,Y],Z) - 2\nabla^g_{[X,Y]}Z.
$$

Therefore, the last parentheses in (3.49) become

$$\begin{aligned}
&2R^g(X,Y)Z + \frac{1}{2}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big) \\
&+ 2\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big) - 2K^{\omega,\omega^*}(X,\nabla^g_Y Z) \\
&+ 2K^{\omega,\omega^*}(Y,\nabla^g_X Z) - 2\nabla^g_X K^{\omega,\omega^*}(Y,Z) + 2\nabla^g_Y K^{\omega,\omega^*}(X,Z) + 2K^{\omega,\omega^*}([X,Y],Z),
\end{aligned}$$

where R^g is the Riemann curvature tensor, i.e, the curvature of the Levi–Civita connection ∇^g. This last expression can simplify to

$$\begin{aligned}
&2R^g(X,Y)Z + \frac{1}{2}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big) \\
&+ 2\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big) \\
&+ 2\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big).
\end{aligned}$$

Therefore, (3.49) becomes

$$\begin{aligned}
R^{(\alpha)}(X,Y)Z = {} & \left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z \\
&+ \frac{1-\alpha^2}{2}\Big(R^g(X,Y)Z + \frac{1}{4}\big(K(Y,K(X,Z)) - K(X,K(Y,Z))\big)\Big) \\
&+ \frac{1-\alpha^2}{2}\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big) \\
&+ \frac{1-\alpha^2}{2}\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big). \qquad (3.50)
\end{aligned}$$

Taking $\alpha = 0$ and using (3.50) we get

$$\begin{aligned}
R^{(0)}(X,Y)Z = {} & \frac{1}{4}R(X,Y)Z + \frac{1}{4}R^*(X,Y)Z + \frac{1}{8}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big) \\
&+ \frac{1}{2}R^g(X,Y)Z + \frac{1}{2}\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big) \\
&+ \frac{1}{2}\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big). \qquad (3.51)
\end{aligned}$$

Taking $\alpha = 0$ and using (3.41) we get

$$\begin{aligned}
R^{(0)}(X,Y)Z = {} & R^g(X,Y)Z + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z)) \\
&+ K^{\omega,\omega^*}(Y,\nabla^g_X Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) + \nabla^g_Y K^{\omega,\omega^*}(X,Z) \\
&- \nabla^g_X K^{\omega,\omega^*}(Y,Z) + K^{\omega,\omega^*}([X,Y],Z),
\end{aligned}$$

that is

$$R^{(0)}(X,Y)Z = R^g(X,Y)Z + K^{\omega,\omega^*}(X, K_g^{\omega}(Y,Z)) - K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z)) \\ + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z). \tag{3.52}$$

Using (3.51) and (3.52) we obtain

$$R^g(X,Y)Z = \frac{1}{2}R(X,Y)Z + \frac{1}{2}R^*(X,Y)Z + \frac{1}{4}\big(K(Y,K(X,Z)) - K(X,K(Y,Z))\big) \\ + K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,Z)) \\ + (\nabla^g_X K^{\omega,\omega^*})(Y,Z) - (\nabla^g_Y K^{\omega,\omega^*})(X,Z). \tag{3.53}$$

Using (3.53) in (3.50) leads to :

$$R^{(\alpha)}(X,Y)Z = \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z \\ + \frac{1-\alpha^2}{4}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big).$$

□

From (3.48)

$$R^{(\alpha)}(X,Y)Z - R^{(-\alpha)}(X,Y)Z = \alpha(R(X,Y)Z - R^*(X,Y)Z). \tag{3.54}$$

Remark 3.2. *Surprisinely, the transformation $R \longmapsto R^{(\alpha)}$ from (3.48) is the same form for 3S-structure and statistical structure* [21].

Remark 3.3. *In the theory of Semi-Symmetric statistical manifold, from (3.52), we get $R^{(0)} \neq R^g$.*

3.2. **Statistical curvature of semi-Symmetric statistical manifolds.** In this section, we firstly give symmetry properties of curvatures R, R^* and give these properties for the statistical curvature R^S of semi-symmetric statistical manifolds.

Lemma 3.2. *For a 3S-structure $(g, \omega, \omega^*, \nabla)$ on M, the following formulas hold for $X, Y, Z, T \in \mathfrak{X}(M)$:*

$$R(X,Y)Z = -R(Y,X)Z, \tag{3.55}$$

$$g(R(X,Y)Z,T) + g(R(Y,X)Z,T) = 0, \tag{3.56}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = (d\omega(X,Y))Z$$
$$+ (d\omega(Y,Z))X + (d\omega(Z,X))Y, \tag{3.57}$$

$$g(R(X,Y)Z,T) + g(R(X,Y)T,Z) = 2g\Big((\nabla^g_X S)(Y,T) - (\nabla^g_Y S)(X,T), Z\Big)$$
$$+ 2g\Big((\nabla^g_X K^{\omega,\omega^*})(Y,T) - (\nabla^g_Y K^{\omega,\omega^*})(X,T), Z\Big) + 4g\Big(K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,T))$$
$$- K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,T)), Z\Big) + 2g\Big(K^{\omega,\omega^*}(Y, S(X,T)) - K^{\omega,\omega^*}(X, S(Y,T)), Z\Big)$$
$$+ 2g\Big(S(Y, K^{\omega,\omega^*}(X,T)) - S(X, K^{\omega,\omega^*}(Y,T)), Z\Big), \tag{3.58}$$

and

$$R^*(X,Y)Z = -R^*(Y,X)Z, \tag{3.59}$$

$$g(R^*(X,Y)Z,T) + g(R^*(Y,X)Z,T) = 0, \tag{3.60}$$

$$R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = (d\omega^*(X,Y))Z$$
$$+ (d\omega^*(Y,Z))X + (d\omega^*(Z,X))Y, \tag{3.61}$$

$$g(R^*(X,Y)Z,T) + g(R^*(X,Y)T,Z) = 2g\Big((\nabla^g_Y S)(X,T) - (\nabla^g_X S)(Y,T), Z\Big)$$
$$+ 2g\Big((\nabla^g_Y K^{\omega,\omega^*})(X,T) - (\nabla^g_X K^{\omega,\omega^*})(Y,T), Z\Big) + 4g\Big(K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,T))$$
$$- K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,T)), Z\Big) + 2g\Big(K^{\omega,\omega^*}(X, S(Y,T)) - K^{\omega,\omega^*}(Y, S(X,T)), Z\Big)$$
$$+ 2g\Big(S(X, K^{\omega,\omega^*}(Y,T)) - S(Y, K^{\omega,\omega^*}(X,T)), Z\Big). \tag{3.62}$$

Proof. For the proof of (3.55), (3.56), (3.57), (3.59), (3.60) and (3.61) refer to [12]. We now, prove (3.58) and (3.62) which depend on the tensor K^{ω,ω^*}.

Using (3.33) we get

$$\nabla^*_X \nabla^*_Y T = \nabla_X \nabla_Y T - 2\nabla_X S(Y,T) - 2\nabla_X K^{\omega,\omega^*}(Y,T) - 2S(X, \nabla_Y T)$$
$$+ 4S(X, S(Y,T)) + 4S(X, K^{\omega,\omega^*}(Y,T)) - 2K^{\omega,\omega^*}(X, \nabla_Y T)$$
$$+ 4K^{\omega,\omega^*}(X, S(Y,T)) + 4K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,T)).$$

Using (3.21), the last equation becomes

$$\begin{aligned}\nabla^*_X\nabla^*_Y T = \nabla_X\nabla_Y T &- 2\nabla^g_X S(Y,T) - 2\nabla^g_X K^{\omega,\omega^*}(Y,T) - 2S(X,\nabla^g_Y T)\\ &- 2K^{\omega,\omega^*}(X,\nabla^g_Y T) + 2S(X,K^{\omega,\omega^*}(Y,T)) + 2K^{\omega,\omega^*}(X,S(Y,T))\\ &+ 4K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T)). \end{aligned} \tag{3.63}$$

Moreover,

$$\begin{aligned}-\nabla^*_Y\nabla^*_X T = -\nabla_Y\nabla_X T &+ 2\nabla^g_Y S(X,T) + 2\nabla^g_Y K^{\omega,\omega^*}(X,T) + 2S(Y,\nabla^g_X T)\\ &+ 2K^{\omega,\omega^*}(Y,\nabla^g_X T) - 2S(Y,K^{\omega,\omega^*}(X,T)) - 2K^{\omega,\omega^*}(Y,S(X,T))\\ &- 4K^{\omega,\omega^*}(Y,K(X,T)). \end{aligned} \tag{3.64}$$

Since $[X,Y] = \nabla^g_X Y - \nabla^g_Y X$, we get

$$\begin{aligned}-\nabla^*_{[X,Y]}T = -\nabla_{[X,Y]}T &+ 2S(\nabla^g_X Y,T) - 2S(Y,\nabla^g_X T)\\ &+ 2K^{\omega,\omega^*}(\nabla^g_X Y,T) - 2K^{\omega,\omega^*}(Y,\nabla^g_X T). \end{aligned} \tag{3.65}$$

Summing up (3.63), (3.64) and (3.65), we obtain

$$\begin{aligned}R^*(X,Y)T = R(X,Y)T &- 2(\nabla^g_X S)(Y,T) + 2(\nabla^g_Y S)(X,T) - 2(\nabla^g_X K^{\omega,\omega^*})(Y,T)\\ &+ (\nabla^g_Y K^{\omega,\omega^*})(X,T) - 4K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,T)) + 4K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T))\\ &- 2K^{\omega,\omega^*}(Y,S(X,T)) + 2K^{\omega,\omega^*}(X,S(Y,T)) - 2S(Y,K^{\omega,\omega^*}(X,T))\\ &+ 2S(X,K^{\omega,\omega^*}(Y,T)). \end{aligned} \tag{3.66}$$

Using (3.45) and (3.66) we get (3.58). Similarly, the proof can be done for Eq. (3.62). □

For a 3S-structure $(g,\omega,\omega^*,\nabla)$ on M, we denote the curvature tensor of ∇ by R^∇ or R for short, and R^{∇^*} by R^* in the similar fashion. We define

$$R^S(X,Y)Z = \frac{1}{2}\{R(X,Y)Z + R^*(X,Y)Z\}, \tag{3.67}$$

for all $X,Y,Z \in \mathfrak{X}(M)$, and call R^S the semi-symmetric statistical curvature tensor field of $(g,\omega,\omega^*,\nabla)$. We define the (0, 4) tensor $\overline{R}$, $\overline{R}^*$ and $\overline{R}^S$ by

$$\overline{R}(X,Y,Z,T) = g(R(X,Y)Z,T), \quad \overline{R}^*(X,Y,Z,T) = g(R^*(X,Y)Z,T),$$
$$\overline{R}^S(X,Y,Z,T) = \frac{1}{2}\{\overline{R}(X,Y,Z,T) + \overline{R}^*(X,Y,Z,T)\}.$$

From Lemma 3.2, we can state the following.

Proposition 3.9. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, we get*

$$
\begin{aligned}
R^S(X,Y)Z = R^g(X,Y)Z &+ S(X,S(Y,Z)) - S(Y,S(X,Z)) \\
&+ S(X,K^{\omega,\omega^*}(Y,Z)) - S(Y,K^{\omega,\omega^*}(X,Z)) \\
&+ K^{\omega,\omega^*}(X,S(Y,Z)) - K^{\omega,\omega^*}(Y,S(X,Z)) \\
&- (\nabla^g_X K^{\omega,\omega^*})(Y,Z) + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) \\
&+ 2\{K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,(X,Z))\}.
\end{aligned}
$$

Proof. Using (3.21) in $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

$$
\begin{aligned}
R(X,Y)Z = R^g(X,Y)Z &+ (\nabla^g_X S)(Y,Z) - (\nabla^g_Y S)(X,Z) \\
&+ S(X,S(Y,Z)) - S(Y,S(X,Z)). \qquad (3.68)
\end{aligned}
$$

Similarly, using (3.33) $R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$, we get

$$
\begin{aligned}
R^*(X,Y)Z = R^g(X,Y)Z &- (\nabla^g_X S)(Y,Z) + (\nabla^g_Y S)(X,Z) \\
&+ S(X,S(Y,Z)) - S(Y,S(X,Z)) \\
&+ 2\{K^{\omega,\omega^*}(X,S(Y,Z)) - K^{\omega,\omega^*}(Y,S(X,Z))\} \\
&+ 2\{(\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\} \\
&+ 4\{K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\}. \qquad (3.69)
\end{aligned}
$$

And finally, using (3.68) and (3.69) in (3.67), we reach that

$$
\begin{aligned}
R^S(X,Y)Z = R^g(X,Y)Z &+ S(X,S(Y,Z)) - S(Y,S(X,Z)) \\
&+ S(X,K^{\omega,\omega^*}(Y,Z)) - S(Y,K^{\omega,\omega^*}(X,Z)) \\
&+ K^{\omega,\omega^*}(X,S(Y,Z)) - K^{\omega,\omega^*}(Y,S(X,Z)) \\
&- (\nabla^g_X K^{\omega,\omega^*})(Y,Z) + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) \\
&+ 2\{K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\}.
\end{aligned}
$$

□

Remark 3.4. *If a 3S-structure $(g, \omega, \omega^*, \nabla)$ is a statistical structure, that is, $\omega = \omega^* = 0$, then, the semi-symmetric statistical curvature tensor field R^S is given by*

$$
R^S(X,Y)Z = R^g(X,Y)Z + S(X,S(Y,Z)) - S(Y,S(X,Z)),
$$

and coincides with the statistical curvature tensor. Thus, the semi-symmetric statistical curvature tensor field R^S is a generalization of the statistical curvature tensor field.

The following series of lemmas and theorem give the symmetrical properties of R^S for a 3S-structure $(g, \omega, \omega^*, \nabla)$.

Lemma 3.3. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, the semi-symmetric statistical curvature field R^S satisfies the following property.*

$$\begin{aligned}&R^S(X,Y)Z + R^S(Y,Z)X + R^S(Z,X)Y \\ &= (d\omega^S(X,Y))Z + (d\omega^S(Y,Z))X + (d\omega^S(Z,X))Y. \end{aligned} \tag{3.70}$$

Proof. It follows from Lemma 3.2. □

Hence, from Lemma 3.3 we can state the following lemma :

Lemma 3.4. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, we have*

$$\begin{aligned}&\overline{R}^S(Z,X,Y,T) + \overline{R}^S(T,Y,Z,X) + \overline{R}^S(X,Z,T,Y) \\ &+ \overline{R}^S(Y,T,X,Z) = 2g(Z,T)(d\omega^S)(X,Y) + 2g(X,T)(d\omega^S)(Y,Z) \\ &+ 2g(X,Y)(d\omega^S)(Z,T) + 2g(Y,Z)(d\omega^S)(T,X).\end{aligned}$$

Proof. Using the Bianchi identity of R and R^* given by the lemma 3.2 we get

$$\begin{aligned}&g(R(X,Y)Z,T) + g(R(Y,Z)X,T) + g(R(Z,X)Y,T) \\ &= g((d\omega)(X,Y)Z + (d\omega)(Y,Z)X + (d\omega)(Z,X)Y, T),\end{aligned}$$

$$\begin{aligned}&g(R^*(X,Y)T,Z) + g(R^*(Y,T)X,Z) + g(R^*(T,X)Y,Z) \\ &= g((d\omega^*)(X,Y)T + (d\omega^*)(Y,T)X + (d\omega^*)(T,X)Y, Z),\end{aligned}$$

$$\begin{aligned}&g(R(X,Z)T,Y) + g(R(Z,T)X,Y) + g(R(T,X)Z,Y) \\ &= g((d\omega)(X,Z)T + (d\omega)(Z,T)X + (d\omega)(T,X)Z, Y),\end{aligned}$$

and

$$\begin{aligned}&g(R^*(Y,Z)T,X) + g(R^*(Z,T)Y,X) + g(R^*(T,Y)Z,X) \\ &= g((d\omega^*)(Y,Z)T + (d\omega^*)(Z,T)Y + (d\omega^*)(T,Y)Z, X).\end{aligned}$$

Using (3.45) and summing up these equations, we get

$$\begin{aligned}
&\overline{R}(Z,X,Y,T)+\overline{R}^*(T,Y,Z,X)+\overline{R}(X,Z,T,Y)\\
&+\overline{R}^*(Y,T,X,Z)=2g(Z,T)(d\omega^S)(X,Y)+2g(X,T)(d\omega^S)(Y,Z)\\
&+2g(X,Y)(d\omega^S)(Z,T)+2g(Y,Z)(d\omega^S)(T,X). \qquad (3.71)
\end{aligned}$$

Moreover, using similar reasoning as above, we have the following dual version of (3.71)

$$\begin{aligned}
&\overline{R}^*(Z,X,Y,T)+\overline{R}(T,Y,Z,X)+\overline{R}^*(X,Z,T,Y)\\
&+\overline{R}(Y,T,X,Z)=2g(Z,T)(d\omega^S)(X,Y)+2g(X,T)(d\omega^S)(Y,Z)\\
&+2g(X,Y)(d\omega^S)(Z,T)+2g(Y,Z)(d\omega^S)(T,X). \qquad (3.72)
\end{aligned}$$

Summing up (3.71) and (3.72) we get

$$\begin{aligned}
&2\overline{R}^S(Z,X,Y,T)+2\overline{R}^S(T,Y,Z,X)+2\overline{R}^S(X,Z,T,Y)\\
&+2\overline{R}^S(Y,T,X,Z)=4g(Z,T)(d\omega^S)(X,Y)+4g(X,T)(d\omega^S)(Y,Z)\\
&+4g(X,Y)(d\omega^S)(Z,T)+4g(Y,Z)(d\omega^S)(T,X).
\end{aligned}$$

We deduce the result.

□

Proposition 3.10. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M. Then, we have*

$$\begin{aligned}
&\overline{R}^S(X,Z,T,Y)-\overline{R}^S(T,Y,X,Z)=g(Z,T)(d\omega^S)(X,Y)\\
&+g(X,T)(d\omega^S)(Y,Z)+g(X,Y)(d\omega^S)(Z,T)+g(Y,Z)(d\omega^S)(T,X). \qquad (3.73)
\end{aligned}$$

Proof. Using (3.55), (3.59) and (3.77) we have

$$\begin{aligned}
\overline{R}^S(Z,X,Y,T)+\overline{R}^S(X,Z,T,Y)&=-\overline{R}^S(X,Z,Y,T)+\overline{R}^S(X,Z,T,Y)\\
&=\overline{R}^S(X,Z,T,Y)+\overline{R}^S(X,Z,T,Y)\\
&=2\overline{R}^S(X,Z,T,Y).
\end{aligned}$$

Therefore,

$$\overline{R}^S(Z,X,Y,T)+\overline{R}^S(X,Z,T,Y)=2\overline{R}^S(X,Z,T,Y), \qquad (3.74)$$

$$\overline{R}^S(T,Y,Z,X)+\overline{R}^S(Y,T,X,Z)=-2\overline{R}^S(T,Y,X,Z). \qquad (3.75)$$

Using the lemma 3.4, (3.74) and (3.75) we have the result. □

Proposition 3.11. *For a 3S-structure $(g, \omega, \omega^*, \nabla)$ on M, we have following equations*

$$g(R^S(X,Y)Z,T) + g(R^S(Y,X)Z,T) = 0, \tag{3.76}$$

$$g(R^S(X,Y)Z,T) + g(R^S(X,Y)T,Z) = 0. \tag{3.77}$$

Moreover, if ω^S is closed, we get

$$R^S(X,Y)Z + R^S(Y,Z)X + R^S(Z,X)Y = 0, \tag{3.78}$$

$$g(R^S(X,Y)Z,T) - g(R^S(Z,T)X,Y) = 0. \tag{3.79}$$

Proof. Summing up Eqs. (3.58) and (3.62) we get (3.77). □

Remark 3.5. *Using the preceding properties of the statistical curvature R^S, we easily check that the statistical sectional curvature given as (3.83), is well defined.*

Proposition 3.12. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M, V its mean vector field and $\widetilde{\nabla}$ its associated statistical connection with respect to ω. If V is torse-forming with respect to ω, then the statistical sectional curvatures of ∇ and $\widetilde{\nabla}$ are the same if and only if*

$$\omega(X)\omega^S(X) + 4\omega^S(X)\omega^S(X) - 2X\omega^S(X) + 2\omega^S(\nabla^*_X X) = 0,$$

for any unitary vector field X.

Proof. From (3.15) and (3.16) we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - (d\omega)(X,Y)Z \\ &\quad + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\},\end{aligned}$$

and

$$\begin{aligned}(\widetilde{R})^*(X,Y)Z &= R^*(X,Y)Z + (d\omega)(X,Y)Z + \{-\omega(X)\omega^S(Z) - 4\omega^S(X)\omega^*(Z) + 2X\omega^S(Z) \\ &\quad - 2\omega^S(\nabla^*_X Z)\}Y + \{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^*(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\}X.\end{aligned}$$

After summing up these equations we get

$$\begin{aligned}2\widetilde{R}^S(X,Y)Z &= 2R^S(X,Y)Z + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\} \\ &\quad + \Big\{ - \omega(X)\omega^S(Z) - 4\omega^S(X)\omega^S(Z) + 2X\omega^S(Z) - 2\omega^S(\nabla^S_X Z)\Big\}Y \\ &\quad + \Big\{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^S(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\Big\}X.\end{aligned} \tag{3.80}$$

From (3.80), since the mean vector field V is torse-forming with respect to ω then we have

$$2\widetilde{R}^S(X,Y)Z = 2R^S(X,Y)Z + \Big\{ -\omega(X)\omega^S(Z) - 4\omega^S(X)\omega^S(Z) + 2X\omega^S(Z) - 2\omega^S(\nabla^S_X Z)\Big\}Y$$
$$+ \Big\{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^S(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\Big\}X. \quad (3.81)$$

Taking $\{X,Y\}$ an orthonormal basis on M. Then (3.81) becomes

$$2g(\widetilde{R}^S(X,Y)Y,X) = 2g(R^S(X,Y)Y,X)+\omega(Y)\omega^S(Y)+4\omega^S(Y)\omega^S(Y)-2Y\omega^S(Y)+2\omega^S(\nabla^*_Y Y).$$

This ends the proof. □

A multilinear function $F : Tp(M)^4 \longrightarrow \mathbb{R}$ is curvature-like provided F has the symmetries stated in Proposition 3.11. Thus, $F(X,Y,Y,X) = 0$ for all $X,Y \in T_p(M)$ spanning a nondegenerate plane implies F vanishes on M [19].

Theorem 3.3. *A 3S-structure $(g,\omega,\omega^*,\nabla)$ on M is of constant semi-symmetric statistical sectional curvature $k \in \mathbb{R}$ if and only if*

$$R^S(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} + \frac{1}{2}\{d\omega^S(Y,Z)X - d\omega^S(X,Z)Y\}$$
$$+ \frac{1}{2}\{g(Y,Z)d\omega^S(X,.)^\sharp - g(X,Z)d\omega^S(Y,.)^\sharp\}, \quad (3.82)$$

for $X,Y,Z \in \Gamma(TM)$.

Proof. Let X,Y,Z and T be vector fields on M. We set

$$\Omega(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} + \frac{1}{2}\{d\omega^S(Y,Z)X - d\omega^S(X,Z)Y\}$$
$$+ \frac{1}{2}\{g(Y,Z)d\omega^S(X,.)^\sharp - g(X,Z)d\omega^S(Y,.)^\sharp\}.$$

It is easy to see that Ω verifies (3.70), (3.73), (3.76) and (3.77). We now set

$$F(X,Y,Z,T) = g(R^S(X,Y)Z,T) - g(\Omega(X,Y)Z,T).$$

Since R^S and Ω verify (3.70), (3.73), (3.76) and (3.77) then F is also curvature-like. That is, F verifies the symmetries properties given by proposition 3.11. Moreover, by hypothesis, we have $F(X,Y,Y,X) = 0$. Thus, F vanishes on M, that is $R^S(X,Y)Z = \Omega(X,Y)Z$.

Conversely, if R^S verify (3.82), it is easy to see that the semi-symmetric statistical sectional curvature tensor field given by

$$K(\pi) = \frac{g(R^S(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \quad (3.83)$$

is equal to k, where π is the 2-dimensional plane spanned by X and Y.

□

The Ricci curvature tensor with respect to ∇, denoted by Ric^{∇} is defined by

$$\mathrm{Ric}^{\nabla}(X,Y)=\mathrm{tr}\{Z\longmapsto R^{\nabla}(X,Z)Y\}.$$

We denote Ric^{∇} by Ric, Ric^{∇^*} by Ric^* and Ric^{∇^g} by Ric^g respectively, for short. For a 3S-structure $(g,\omega,\omega^*,\nabla)$, we can also define the Ricci curvature tensor relative to the semi-symmetric statistical curvature tensor field R^S as follows.

Definition 3.2. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M and R^S the semi-symmetric statistical curvature tensor field of ∇.*

1. *The 3S-Ricci curvature tensor, denoted by Ric^S, is defined as*

$$Ric^S(X,Y)=tr_g\{Z\longmapsto R^S(X,Z)Y\}.$$

2. *The 3S-Ricci curvature tensor field Ric^S of $(M,g,\omega,\omega^*,\nabla)$ is said to be 3S-Einstein manifold if there exists $\lambda\in\mathbb{R}$ such that*

$$Ric^S=\lambda g.$$

3. *The 3S-scalar curvature, denoted by ρ^S, is defined as*

$$\rho^S=tr_gRic^S=\sum_{i=1}^{n}Ric^S(e_i,e_i),$$

where $\{e_1,e_2,...,e_n\}$ is an orthonormal basis of T_xM with respect to g, for $x\in M$.

Remark 3.6. *For a 3S-structure $(g,\omega,\omega^*,\nabla)$ on M, it is easy to see that $\rho^S=\rho=\rho^*$, where*

$$\rho=tr_gRic,\qquad\qquad \rho^*=tr_gRic^*.$$

From proposition 3.10, we obtain the following corollary.

Corollary 3.2. *Let (M,g) be a Riemannian manifold of dimension n, let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M. Then, the Ricci tensor Ric^S of $(g,\omega,\omega^*,\nabla)$ verifies*

$$Ric^S(X,Y)-Ric^S(Y,X)=(2-n)(d\omega^S)(X,Y).$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space at any point p of the 3S-structure manifold. Taking $Z = Y = e_i$ in (3.73), summing over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
Ric^S(X,T) - Ric^S(T,X) &= \sum_{i=1}^{n} \Big(\overline{R}^S(X,e_i,T,e_i) - \overline{R}^S(T,e_i,X,e_i) \Big) \\
&= \sum_{i=1}^{n} \Big(g(e_i,T)(d\omega^S)(X,e_i) + g(X,T)(d\omega^S)(e_i,e_i) \\
&+ g(X,e_i)(d\omega^S)(e_i,T) + g(e_i,e_i)(d\omega^S)(T,X) \Big) \\
&= \sum_{i=1}^{n} \Big((d\omega^S)(X, g(e_i,T)e_i) + (d\omega^S)(g(X,e_i)e_i, T) \\
&+ g(e_i,e_i)(d\omega^S)(T,X) \Big) \\
&= (d\omega^S)(X,T) + (d\omega^S)(g(X,T) + n(d\omega^S)(T,X) \\
&= (2-n)(d\omega^S)(X,T).
\end{aligned}$$

□

From Corollary 3.2, we obtain the following Corollaries :

Corollary 3.3. *Let (M, g) be a Riemannian manifold of dimension 2. The semi-symmetric statistical Ricci tensor of a 3S-structure on M is always symmetric.*

Corollary 3.4. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, the semi-symmetric statistical Ricci tensor Ric^S of $(g, \omega, \omega^*, \nabla)$ is symmetric if and only if ω^S is closed. Thus, for a statistical manifold (M, g, ∇), the statistical Ricci tensor Ric^S is always symmetric.*

Proposition 3.13. *Let (M, g) be a Riemannian manifold of dimension n, let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M such that the 3S-sectional curvature of R^S is constant $k \in \mathbb{R}$. Then, we have*

$$Ric^S(X,Y) = k(1-n)g(X,Y) + (1 - \frac{n}{2})d\omega^S(X,Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Let $X, Y \in \mathfrak{X}(M)$, we have

$$
\begin{aligned}
\mathrm{Ric}^S(X,Y) &= \sum_{i=1}^{n} g(R^S(X,e_i)Y, e_i) \\
&= \sum_{i=1}^{n} g\Big(k\{g(e_i,Y)X - g(X,Y)e_i\} + \frac{1}{2}\{d\omega^S(e_i,Y)X - d\omega^S(X,Y)e_i\} \\
&+ \frac{1}{2}\{g(e_i,Y)d\omega^S(X,.)^\sharp - g(X,Y)d\omega^S(e_i,.)^\sharp\}, e_i \Big) \\
&= k\sum_{i=1}^{n}\{g(e_i,Y)g(X,e_i) - g(X,Y)g(e_i,e_i)\} + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(e_i,Y)g(X,e_i) \\
&- \frac{1}{2}\sum_{i=1}^{n} d\omega^S(X,Y)g(e_i,e_i) + \frac{1}{2}\sum_{i=1}^{n} g(e_i,Y)d\omega^S(X,e_i).
\end{aligned}
$$

That is:

$$
\begin{aligned}
\mathrm{Ric}^S(X,Y) &= k\sum_{i=1}^{n}\{g(X, g(e_i,Y)e_i) - g(X,Y)\} + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(g(X,e_i)e_i, Y) \\
&- \frac{n}{2}d\omega^S(X,Y) + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(X, g(e_i,Y)e_i) \\
&= kg(X,Y) - nkg(X,Y) + \frac{1}{2}d\omega^S(X,Y) - \frac{n}{2}d\omega^S(X,Y) + \frac{1}{2}d\omega^S(X,Y) \\
&= k(1-n)g(X,Y) + (1-\frac{n}{2})d\omega^S(X,Y).
\end{aligned}
$$

□

Corollary 3.5. *Let $(M, g, \omega, \omega^*, \nabla)$ be a 3S-manifold such that its 3S-sectional curvature is constant. Then, M is 3S-Einstein if ω^S is closed.*

Definition 3.3. *Let $(M, g, \omega, \omega^*, \nabla)$ be a 3S-manifold and Ric^S its 3S-Ricci tensor. The symmetrized $\widetilde{R}ic^S$ of Ric^S is given by*

$$\widetilde{R}ic^S(X,Y) = \frac{1}{2}\{Ric^S(X,Y) + Ric^S(Y,X)\},$$

for all vector fields X, Y on M.

Definition 3.4. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M, Ric^S the 3S-Ricci curvature tensor field. $(M, g, \omega, \omega^*, \nabla)$ is said to be symmetrically Einstein manifold if the symmetrized $\widetilde{R}ic^S$ of Ric^S is of the form*

$$\widetilde{R}ic^S = \lambda g,$$

where $\lambda = \frac{\rho^S}{n}$ is a constant.

Corollary 3.6. *If a 3S-manifold $(M, g, \omega, \omega^*, \nabla)$ of dimension n is of constant 3S-sectional curvature, then $(M, g, \omega, \omega^*, \nabla)$ is symmetrically Einstein manifold.*

Proof. Let $X, Y \in \mathfrak{X}(M)$, from the proposition 3.11 we get

$$\widetilde{R}\mathrm{ic}^S(X,Y) = k(1-n)g(X,Y).$$

We also said that $(M, g, \omega, \omega^*, \nabla)$ is symmetrically Einstein manifold. □

4. Semi-symmetric statistical submanifolds

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold, $(\overline{M}, \overline{g}, \overline{\omega}, \overline{\omega}^*, \overline{\nabla})$ be a 3S-manifold, M be a submanifold of $\overline{M}$, g the induced metric $\overline{g}$ and two 1-forms ω, ω^* such that

$$\omega(X) = \overline{\omega}(X), \omega^*(X) = \overline{\omega}^*(X),$$

for all vector field X on M.

Let $TM^\perp$ be the normal bundle of M in $\overline{M}$ with respect to $\overline{g}$. We define the second fundamental form of M for $\overline{\nabla}$ and $\overline{\nabla}^*$ by

$$h(X,Y) = (\overline{\nabla}_X Y)^\perp, \tag{4.84}$$

$$h^*(X,Y) = (\overline{\nabla}^*_X Y)^\perp \tag{4.85}$$

respectively, where $()^\perp$ denotes the orthogonal projection on the normal bundle $TM^\perp$.
Let consider the connections ∇ and ∇^* given by

$$\nabla_X Y = (\overline{\nabla}_X Y)^\top, \tag{4.86}$$

$$\nabla^*_X Y = (\overline{\nabla}^*_X Y)^\top. \tag{4.87}$$

It is well known in the literature that ∇ and ∇^* are dual connections with respect to g and h, h^* are bilinear and symmetric.

Proposition 4.1. *When $(\overline{M}, \overline{g}, \overline{\omega}, \overline{\omega}^*, \overline{\nabla})$ is a 3S-manifold, the induced structure $(M, g, \omega, \omega^*, \nabla)$ on a submanifold M is a 3S-manifold.*

Proof. Let $X, Y \in \mathfrak{X}(M)$. From (4.84) and (4.86) we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{4.88}$$

Since $\overline{\nabla}$ has semi-symmetric connection, we can write

$$\begin{aligned}\overline{\omega}(X)Y - \overline{\omega}(Y)X &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]\\ &= \nabla_X Y + h(X,Y) - \nabla_Y X + h(Y,X) - [X,Y]\\ &= T^{\nabla}(X,Y) + h(X,Y) - h(Y,X). \qquad (4.89)\end{aligned}$$

Grouping the normal and tangential components of (4.89), we have

$$T^{\nabla}(X,Y) = \omega(X)Y - \omega(Y)X. \qquad (4.90)$$

This equation show that ∇ has semi-symmetric. The proof for ∇^* follows immediately by substituting ∇^* for ∇ in the preceding argument, and we obtain

$$T^{\nabla^*}(X,Y) = \omega^*(X)Y - \omega^*(Y)X. \qquad (4.91)$$

□

From (4.90) and (4.91), $(M, g, \omega, \omega^*, \nabla)$ is a 3S-manifold.

5. Semi-symmetric dualistic structures on warped product spaces

In this section, we give a method to construct 3S-structures on warped product manifolds, starting from 3S-strutures on the fiber and base manifolds.

Let $(M; g)$ and $(N; h)$ be two Riemannian manifolds of dimension m and n respectively and $f \in \mathcal{C}^\infty(M)$ a positive function on M. The warped product of $(M; g)$ and $(N; h)$, with warping function f, is the $(m+n)$-dimensional manifold $M \times N$ endowed with the metric G_f given by:

$$G_f = \pi^* g + (f \circ \pi)^2 \sigma^* h,$$

where π^* and σ^* are the pull-backs of the projections π and σ of $M \times N$ on M and N respectively.

This warped product is sometime denoted by $M \times_f N$, but for simplicity we keep $M \times N$ in the sequel.

The tangent space $T_{(p;q)}(M \times N)$ at a point $(p; q) \in M \times N$ is isomorph to the direct sum $T_pM \oplus T_qN$. Let $\mathcal{L}_H(M)$ (resp. $\mathcal{L}_V(N)$) denote the set of the horizontal lifts (resp. the vertical lifts) to $T(M \times N)$ of all the tangent vectors on M. (resp. on N). Hence from [20], we have the following:

$$\Gamma(T(M \times N)) \simeq \mathcal{L}_H(M) \oplus \mathcal{L}_V(N),$$

and thus a vector field A on $M \times N$ can be written as

$$A = X + U; \text{ with } X \in \mathcal{L}_H(M) \text{ and } U \in \mathcal{L}_V(N).$$

For any vector field $X \in \mathcal{L}_H(M)$ we denote $\pi_*(X)$ by $\overline{X}$, and for any vector field $U \in \mathcal{L}_V(N)$ we denote $\sigma_*(U)$ by $\widetilde{U}$. Furthermore we denote the horizontal lift on $M \times N$ of a vector field $X \in \Gamma(TM)$ by $(X)^H$, and the vertical lift on $M \times N$ of a vector field $U \in \Gamma(TN)$ by $(U)^V$.

Obviously

$$\pi_*(\mathcal{L}_H(M)) = \Gamma(TM) \text{ and } \sigma_*(\mathcal{L}_V(N)) = \Gamma(TN).$$

Let $(G_f; \widetilde{D}; \widetilde{D}^*)$ be a dualistic structure on $M \times N$. For $X; Y, Z \in \mathcal{L}_H(M)$ and $U; V, W \in \mathcal{L}_V(N)$ we define the following four connections [20] :

$${}^M\widetilde{\nabla}_{\overline{X}}\overline{Y} = \pi_*(\widetilde{D}_X Y) \text{ and } {}^M\widetilde{\nabla}'_{\overline{X}}\overline{Y} = \pi_*(\widetilde{D}^*_X Y),$$

and

$${}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V} = \sigma_*(\widetilde{D}_U V) \text{ and } {}^N\widetilde{\nabla}'_{\widetilde{U}}\widetilde{V} = \sigma_*(\widetilde{D}^*_U V).$$

We also recall that[19] :

$$\overline{X}G_f(\overline{Y}, \overline{Z}) \circ \pi = XG_f(Y, Z) \text{ and } \widetilde{U}G_f(\widetilde{V}, \widetilde{W}) \circ \sigma = UG_f(V, W).$$

Hence we have the following result from [20]:

Proposition 5.1. *The triplet* $(g;^M \widetilde{\nabla};^M \widetilde{\nabla}')$ *is a dualistic structure on* M *and the triplet* $(h;^N \widetilde{\nabla};^N \widetilde{\nabla}')$ *is a dualistic structure on* N*; that is*

$${}^M\widetilde{\nabla}' =^M \widetilde{\nabla}^* w.r.t.g \text{ and } {}^N\widetilde{\nabla}' =^N \widetilde{\nabla}^* w.r.t.h.$$

Conversely, in [20], it has been given a method to construct statistical structure on the warped product, starting from statistical structures on the fiber and the base manifolds as it follows:

Let $(g,^M \widetilde{\nabla},^M \widetilde{\nabla}^*)$ and $(h,^N \widetilde{\nabla},^N \widetilde{\nabla}^*)$ be dualistic structures on M and N respectively. For all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$ we set:

i) $\widetilde{D}_X Y = ({}^M\widetilde{\nabla}_{\overline{X}}\overline{Y})^H$

ii) $\widetilde{D}_X U = \widetilde{D}_U X = \frac{X(f)}{f} U$

iii) $\widetilde{D}_U V = -\frac{<U,V>}{f} grad(f) + ({}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V})^V$

and

a) $\widetilde{D}^*_X Y = ({}^M\widetilde{\nabla}^*_{\overline{X}}\overline{Y})^H$

b) $\widetilde{D}^*_X U = \widetilde{D}^*_U X = \frac{X(f)}{f} U$

c) $\widetilde{D}^*_U W = -\frac{<U,W>}{f} grad(f) + ({}^N\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{W})^V$,

where we simplify the notation by writing f for $f \circ \pi$ and $grad(f)$ for $grad(f \circ \pi)$, and we denote by $<,>$ the inner product w.r.t. G_f. Obviously $\widetilde{D}$ and $\widetilde{D}^*$ define affine connections on $T(M \times N)$ and the following proposition holds:

Proposition 5.2. *The triplet* $(G_f, \widetilde{D}, \widetilde{D}^*)$ *is a dualistic structure on* $M \times N$.

We call $(G_f, \widetilde{D}, \widetilde{D}^*)$ the dualistic structure on $M \times N$ induced from $(g, {}^M\widetilde{\nabla}, {}^M\widetilde{\nabla}^*)$ on M and $(h, {}^N\widetilde{\nabla}, {}^N\widetilde{\nabla}^*)$ on N.

Now, using [5, 7], and the fact that $[X, U] = 0$, $[\overline{X}, \overline{Y}]^H = [X, Y]$, $[\widetilde{U}, \widetilde{W}]^V = [U, W]$ for all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$, one obtains the following result.

Corollary 5.1. *If* $(g, {}^M\widetilde{\nabla}, {}^M\widetilde{\nabla}^*)$ *and* $(h, {}^N\widetilde{\nabla}, {}^N\widetilde{\nabla}^*)$ *are statistical structures on* M *and* N *respectively, then* $(G_f, \widetilde{D}, \widetilde{D}^*)$ *is statistical structure on* $M \times N$.

Inspired by the reasoning from [5, 7, 20] we introduce here similar but different construction for 3S-structures on the warped product. We first notice that a 3S-structure on $M \times N$ projects to 3S-structures on M and N.

Let $(G_f; \eta, \eta*, D)$ be a 3S-structure on $M \times N$, ${}^M\nabla$ and ${}^N\nabla$ the connections such that

$$ {}^M\nabla_{\overline{X}}\overline{Y} = \pi_*(D_X Y) \text{ and } {}^N\nabla_{\widetilde{U}}\widetilde{W} = \sigma_*(D^*{}_U W), $$

for all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$, where D^* is the dual connection of D with respect to G_f. Let ${}^M\nabla^*$ and ${}^N\nabla^*$ be defined by

$$ {}^M\nabla^*_{\overline{X}}\overline{Y} = \pi_*(D^*_X Y) \text{ and } {}^N\nabla^*_{\widetilde{U}}\widetilde{W} = \sigma_*(D^*_U W), $$

The last two connections are dual connections of ${}^M\nabla$ and ${}^N\nabla$ respectively. We set

$$ \omega(\overline{X}) = \eta(X), \omega^*(\overline{X}) = \eta^*(X), \widetilde{\omega}(\widetilde{U}) = \eta(U) \text{ and } \widetilde{\omega}^*(\widetilde{U}) = \eta^*(U), $$

for all $X \in \mathcal{L}_H(M)$ and $U \in \mathcal{L}_V(N)$. Then the following holds:

Proposition 5.3. $(M, g, \omega, \omega^*, {}^M\nabla)$ *and* $(N, h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$ *are 3S-manifolds.*

Proof. Direct computations give $T^{{}^M\nabla}(\overline{X}, \overline{Y}) = \eta(X)\overline{Y} - \eta(Y)\overline{X}$ and $T^{{}^M\nabla^*}(\overline{X}, \overline{Y}) = \eta^*(X)\overline{Y} - \eta^*(Y)\overline{X}$. Thus, $(M, g, \omega, \omega^*, {}^M\nabla)$ is a 3S-manifold. Similarly for $(N, h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$. □

We now give a converse of the preceding construction. Assume that $(g, \omega, \omega^*, {}^M\nabla)$ and $(h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$ are 3S-structures on M and N respectively and let ${}^M\widetilde{\nabla}$ and ${}^N\widetilde{\nabla}$ be the affine connections on M and N respectively defined by

$$
\begin{aligned}
{}^M\widetilde{\nabla}_{\overline{X}}\overline{Y} &= {}^M\nabla_{\overline{X}}\overline{Y} - \omega(\overline{X})\overline{Y} - 2g(\overline{X}, \overline{Y})\,{}^MV,\\
{}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{W} &= {}^N\nabla_{\widetilde{U}}\widetilde{W} - \widetilde{\omega}(\widetilde{U})\widetilde{W} - 2h(\widetilde{U}, \widetilde{W})\,{}^NV,
\end{aligned}
$$

where ${}^MV$ and ${}^NV$ are the vector field g-associated and h-associated with $\omega^S = \frac{1}{2}(\omega + \omega^*)$ and $\widetilde{\omega}^S = \frac{1}{2}(\widetilde{\omega} + \widetilde{\omega}^*)$ respectively. Let η and η^* be the 1-forms defined by:

$$\eta = \omega \oplus \widetilde{\omega} \ \text{ and } \ \eta^* = \omega^* \oplus \widetilde{\omega}^*$$

namely, $\eta(X+U) = \omega(\overline{X}) + \widetilde{\omega}(\widetilde{U})$ and $\eta^*(X+U) = \omega^*(\overline{X}) + \widetilde{\omega}^*(\widetilde{U})$ for all $X \in \mathcal{L}_H(M)$ and $U \in \mathcal{L}_V(N)$. We set V^S to be the vector field G_f-associated with $\Omega^S = \frac{1}{2}(\eta + \eta^*)$.

It is easy to see from Theorem 3.2 that $(M, g, {}^M\widetilde{\nabla})$ and $(N, h, {}^N\widetilde{\nabla})$ are statistical manifolds. Then, from corollary 5.1, define a connection $\widetilde{D}$ on $M \times N$ by the following formula:

i) $\widetilde{D}_XY = ({}^M\widetilde{\nabla}_{\overline{X}}\overline{Y})^H$
ii) $\widetilde{D}_XU = \widetilde{D}_UX = \frac{X(f)}{f}U$
iii) $\widetilde{D}_UV = -\frac{<U,V>}{f}grad(f) + ({}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V})^V$

and

a) $\widetilde{D}^*_XY = ({}^M\widetilde{\nabla}^*_{\overline{X}}\overline{Y})^H$
b) $\widetilde{D}^*_XU = \widetilde{D}^*_UX = \frac{X(f)}{f}U$
c) $\widetilde{D}^*_UW = -\frac{<U,W>}{f}grad(f) + ({}^N\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{W})^V$,

Then, $(G_f, \widetilde{D}, \widetilde{D}^*)$ is a statistical structure on $M \times N$.

From this, we deduce a 3S-structure as it follows:

Proposition 5.4. *Let D be the connection defined by*

$$D = \widetilde{D} + \eta \otimes I + 2G_f(.,.)V^S. \tag{5.92}$$

Then, (G_f, η, η^*, D) is 3S-structure on $M \times N$.

Proof. Let T^D and T^{D^*} be the torsion tensors of D and D^* respectively, where D^* is the dual connection of D with respect to G_f. From 3.2 we have

$$D^* = \widetilde{D}^* - \eta \otimes I - 2I \otimes \Omega^S. \tag{5.93}$$

Let $A = X + U$ and $B = Y + W$ such that $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$. Since $G_f(X,W) = 0 = G_f(U,Y)$ and $[X,W] = 0$, $[U,Y] = 0$, $[\overline{X}, \overline{Y}]^H = [X,Y]$, $[\widetilde{U}, \widetilde{W}]^V = [U,W]$ [4] and $\widetilde{D}$ and $\widetilde{D}^*$ are torsion-free connections on $M \times N$, we get:

$$T^D(A,B) = \eta(A)B - \eta(B)A, \quad T^{D^*}(A,B) = \eta^*(A)B - \eta^*(B)A.$$

Thus, (G_f, η, η^*, D) is a 3S-structure on $M \times N$. □

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