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ON THE ZAGREB INDICES OF THE ZERO-DIVISOR GRAPH OF $\mathbb{Z}_n$

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Abstract. In this paper, we compute the first and second Zagreb indices, first and second multiplicative Zagreb indices, first and second Zagreb coindices, first and second multiplicative Zagreb coindices of the Beck's zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$, where $n = p^k, pq, pqr$ and p, q, r are distinct primes. These indices are important in chemical graph theory and network analysis, providing information about the structural characteristics of graph.

Keywords: Zero-divisor graph, Commutative ring, Zagreb indices, Vertex degree.

MSC (2020): 05C25, 05C07, 05C30, 05A18.

1. Introduction

The zero-divisor graph is a graphical presentation of an algebraic structure and has emerged as a powerful tool for investigating its structural properties of the structure. In 1988, while studying coloring of a ring, Beck [3] took all the elements of a ring as the vertices of the graph, connecting two vertices by an edge if and only if the product of corresponding elements in the ring is zero. After modification of Beck's definition, Anderson and Livingston [1] introduced the zero-divisor graph by removing all the non zero-divisors from the vertex set, denoted by $\Gamma(R)$. Subsequently, extensive research has been conducted on zero-divisor graphs within both commutative and non-commutative context, encompassing thorough analyses of their spectral and non-spectral graph characteristics.
Graph invariants are structural properties of the graph that gives quantitative measures its characteristic. The Zagreb indices are a set of topological indices in graph theory, used to characterize molecular graphs, particularly in the field of chemical graph theory. These indices can record information of degree-based information of the molecular graphs and have been been successfully employed in modeling a host of physico-chemical properties of organic compounds; including boiling points, stability, strain energy.

The first Zagreb index M_1 was first introduced by Gutman et al. [8] in 1972 is the sum of the squares of the degrees of all vertices in the graph

$$M_1(G) = \sum_{v \in V(G)} d(v)^2.$$

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It can be also expressed as a sum over edges of G,

$$M_1(G) = \sum_{(u,v)\in E(G)} [d(u) + d(v)].$$

The second Zagreb index $M_2(G)$ is the sum of the product of the degrees of pairs of adjacent vertices in the graph.

$$M_2(G) = \sum_{(u,v)\in E(G)} d(u)d(v).$$

In [4, 9, 11], the principal characteristics of $M_1(G)$ and $M_2(G)$ are explained in detail. The first and second Zagreb indices are strongly related with the total π-electrons energy of conjugated hydrocarbons. Zagreb indices provides information about the topological features of a graph which are used in quantitative structure activity relationships (QSAR). They offers various chemical applications to predict several properties and activities of molecules. Different molecular structures can have different Zagreb indices, which can be helpful in comparing and classifying compounds based on their structural properties.
Eliasi et al. [6] and Todeschini et al. [13, 14] have introduced modified version of conventional Zagreb indices using multiplication operations, are

$$\textstyle\prod_1(G) = \prod_{v\in V(G)} d(v)^2,$$
$$\textstyle\prod_2(G) = \prod_{(u,v)\in E(G)} d(u)d(v).$$

Došlić [5] introduced these calculations involving summations over edges of complements of graph G and referred as Zagreb coindices. The first Zagreb coindex of graph G is

$$\overline{M_1}(G) = \sum_{(u,v)\notin E(G)} [d(u) + d(v)].$$

The second Zagreb coindex is defined in [5] as

$$\overline{M_2}(G) = \sum_{(u,v)\notin E(G)} [d(u)d(v)].$$

In 2013, Xu et al. [15] introduced the concept of multiplicative Zagreb coindices.

$$\overline{\textstyle\prod}_1(G) = \prod_{(u,v)\notin E(G)} [d(u) + d(v)],$$
$$\overline{\textstyle\prod}_2(G) = \prod_{(u,v)\notin E(G)} d(u)d(v).$$

In this article, we use Beck's definition of the zero-divisor graph to compute the first Zagreb index, second Zagreb index, first multiplicative Zagreb index, second multiplicative Zagreb index, first Zagreb coindex, second Zagreb coindex, first multiplicative Zagreb coindex and second multiplicative Zagreb coindex of the zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$ for $n = p^k,\ n = pq,\ n = pqr$, where $p,\ q,\ r$ are distinct primes.

2. Preliminaries

We use the partition of vertex set of zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$ given in [10, Remark 2.2].

Remark 2.1. (Partition of zero-divisor graph) *Let $\mathbb{Z}_{p^k}$ be the ring of integers modulo p^k. Then we can write $\Gamma_0(\mathbb{Z}_{p^k}) = \bigcup_{i=1}^{k+1} A_i$, where A_i are disjoint subsets of $\Gamma_0(\mathbb{Z}_{p^k})$, for*

$1 \leq i \leq k+1$, where $A_1,\ A_2, \cdots, A_{k+1}$ for $1 \leq i \leq k+1$, which are defined as follows

$$\begin{aligned} A_1 &= (p^{k-1})\backslash\{0\}, \\ A_2 &= (p^{k-2})\backslash\{\ A_1 \cup \{0\}\}, \\ A_3 &= (p^{k-3})\backslash\{\ A_1 \cup A_2 \cup \{0\}\}, \\ &\vdots \\ A_{k-1} &= (p)\backslash\{\{\textstyle\bigcup_{i=1}^{k-2} A_i\} \cup \{0\}\}, \\ A_k &= \{unit\ vertices\}, \\ A_{k+1} &= \{0\} \end{aligned}$$

Also, we can write $A_i = \{k_i p^{k-i} : k_i = 1, 2, \ldots, (p^i - 1);\ p \nmid k_i\}$, for $1 \leq i \leq k-1$. Thus for any $1 \leq i \leq k$, $|A_i| = p^i - p^{(i-1)} = \phi(p^i)$ and $|A_{k+1}| = 1$.

Lemma 2.1. [10] *Let A_i be subsets of $V(\Gamma_0(\mathbb{Z}_{p^k}))$ which are defined in the above Remark 2.1, for $1 \leq i \leq k$; and let s, t be two integers with $1 \leq s \leq t \leq k$, then $\sum_{i=s}^{t} |A_i| = p^t - p^{s-1}$.*

The following lemma gives the adjacency among the vertices in the partitioning sets $A_i,\ A_j$.

Lemma 2.2. [10] *Let p be a prime, $\Gamma_0(\mathbb{Z}_{p^k})$ be zero-divisor graph of $\mathbb{Z}_{p^k}$ and $x \in A_i$, $y \in A_j$, for $i, j \in \{1, 2, \cdots, k+1\}$. Then* $d(x,y) = \begin{cases} 1, & if\ \ x \neq y, \quad i+j \leq k; \\ 2, & if\ \ x \neq y, \quad i+j > k. \end{cases}$

First we state the results given in Mahale et al. [10], which we use frequently.

Lemma 2.3. [10] *Let A_i be subsets of $V(\Gamma_0(\mathbb{Z}_{p^k}))$, for $1 \leq i \leq k+1$, as in given Remark 2.1. Then the following assertions hold.*

1. *A vertex in A_{k+1} is adjacent to every vertex in $\bigcup_{i=1}^{k} A_i$.*
2. *Each vertex of A_i is adjacent to every vertex in $\bigcup_{j=1}^{k-i} A_j$.*
3. *If k is odd, then any two vertices in $\bigcup_{i=1}^{\frac{k-1}{2}} A_i$ are adjacent; and any two vertices in $\bigcup_{i=\frac{k+1}{2}}^{k-1} A_i$ are non adjacent.*
4. *If k is even, then any two vertices in $\bigcup_{i=1}^{\frac{k}{2}} A_i$ are adjacent; and any two vertices in $\bigcup_{i=\frac{k+2}{2}}^{k-1} A_i$ are non adjacent.*

Next result gives the partition of $\Gamma_0(\mathbb{Z}_{pq})$.

Lemma 2.4. [10] *Let p and q be distinct primes. Then $\Gamma_0(\mathbb{Z}_{pq}) = \dot{\bigcup}_{i=1}^{4} A_i$, where*

$$\begin{aligned} A_1 &= \{pk_1\colon k_1 = 1, 2, \cdots, (q-1),\ p \nmid k_1\}, \\ A_2 &= \{qk_2\colon k_2 = 1, 2, \cdots, (p-1),\ q \nmid k_2\}, \\ A_3 &= \{unit\ verices\}, \\ A_4 &= \{0\} \ \text{ are such that,} \end{aligned}$$

1. *Any vertex of A_4 is adjacent to every vertex of $\bigcup_{i=1}^{3} A_i$.*
2. *A_1, A_2 and A_3 are independent sets.*
3. *Each element of A_1 is adjacent to every element in A_2.*

The following result of Mahale et al. [10] gives the partition of vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$, for the sake of completeness we provide the proof.

Lemma 2.5. [10] *Let p, q and r be distinct primes. Then* $\Gamma_0(\mathbb{Z}_{pqr}) = A_1 \dot\cup A_2 \dot\cup \cdots \dot\cup A_8$.

Proof. Here p, q and r are distinct primes. The vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$ can be partitioned as

$$\begin{aligned}
A_1 &= \{pk_1 \colon k_1 = 1, 2, \cdots, (qr-1) \text{ with } q \nmid k_1 \text{ and } r \nmid k_1\},\\
A_2 &= \{qk_2 \colon k_2 = 1, 2, 3, \cdots, (pr-1),\ p \nmid k_2, r \nmid k_2\},\\
A_3 &= \{rk_3 \colon k_3 = 1, 2, 3, \cdots, (pq-1),\ p \nmid k_3, q \nmid k_3\},\\
A_4 &= \{qrk_4 \colon k_4 = 1, 2, 3, \cdots, (p-1)\},\\
A_5 &= \{prk_5 \colon k_5 = 1, 2, 3, \cdots, (q-1)\},\\
A_6 &= \{pqk_6 \colon k_6 = 1, 2, 3, \cdots, (r-1)\},\\
A_7 &= \{unit\ vertices\}\\
A_8 &= \{0\}
\end{aligned}$$

Then the sets $A_1, A_2, \cdots, A_8$ are as desired. □

Lemma 2.6. [2] *Let G be a simple graph on n vertices and m edges. Then*

(1) $\overline{M_1}(G) = 2m(n-1) - M_1(G)$.
(2) $\overline{M_2}(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$.

Lemma 2.7. *The number of edges of zero-divisor graph $\Gamma_0(\mathbb{Z}_{p^k})$ is*

$$E(\Gamma_0(\mathbb{Z}_{p^k})) = \begin{cases} \frac{1}{2}[kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}}], & \text{if } k \text{ is odd;} \\ \frac{1}{2}[kp^k - kp^{k-1} - p^{\frac{k}{2}} + 2p^k - p^{k-1}], & \text{if } k \text{ is even.} \end{cases}$$

Proof. In the view of Lemma 2.3, the number of vertices of zero-divisor graph $\Gamma_0(\mathbb{Z}_{p^k})$ is p^k and we have the degree of each vertex. Thus by Handshaking lemma[7], we get number of edges of $\Gamma_0(\mathbb{Z}_{p^k})$ as $\sum_{i=1, v\in A_i}^{k+1} d(v)|A_i| = 2|E|$.
Now we have two cases for k.
Case I) If k is odd. From (3) of Lemma 2.3, we get

$$\begin{aligned}
\sum_{\substack{i=1\\ v\in A_i}}^{k+1} d(v)|A_i| &= \sum_{i=1}^{\frac{k-1}{2}} d(v)|A_i| + \sum_{i=\frac{k+1}{2}}^{k-1} d(v)|A_i| + d(v)|A_k| + d(v)|A_{(k+1)}| \\
&= \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i} - 1)(p^i - p^{i-1}) + \sum_{i=\frac{k+1}{2}}^{k-1} p^{(k-i)}(p^i - p^{i-1}) + 1.(p^k - p^{(k-1)}) \\
&\quad + (p^k - 1)
\end{aligned}$$

Therefore, $|E(\Gamma_0(\mathbb{Z}_{p^k}))| = \frac{1}{2}[kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}}]$.
Case II) If k is even, then $|E(\Gamma_0(\mathbb{Z}_{p^k}))| = \frac{1}{2}[kp^k - kp^{k-1} - p^{\frac{k}{2}} + 2p^k - p^{k-1}]$. □

3. Zagreb indices

In this section, we compute the Zagreb indices for the zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$, for $n = p^k$, $n = pq$, $n = pqr$ where p, q and r are distinct primes.

Theorem 3.1. *Let p be a prime. Then the first Zagreb index M_1 of $\Gamma_0(\mathbb{Z}_{p^k})$ is*

$$M_1(\Gamma_0(\mathbb{Z}_{p^k})) = \begin{cases} p^{2k} + p^{(2k-1)} + (k-3)p^k - kp^{(k-1)} + p^{(\frac{k-1}{2})}, & \text{if } k \text{ is odd;} \\ p^{2k} + p^{2k-1} + (k-2)p^k - (k+1)p^{k-1} + p^{\frac{k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. Using the fact given in the Remark 2.1, the vertex set $V(\Gamma_0(\mathbb{Z}_{p^k}))$ is partitioned into $k+1$ sets $A_1, A_2, \dots, A_{k+1}$ with $|A_i| = p^i - p^{i-1}$, for any $1 \le i \le k$ and $|A_{k+1}| = 1$. There are two cases for k.

Case I) If k is odd. In view of Lemma 2.3, we have following subcases.

Subcase 1) Consider the set A_i, for $1 \le i \le \frac{k-1}{2}$, the degree of each A_i is $(p^{k-i} - 1)$. Let $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$. Then

$$\sum_{v \in V_1} d(v)^2 = \sum_{i=1}^{\frac{k-1}{2}} \sum_{v \in A_i} d(v)^2 = \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i} - 1)^2 |A_i| = \sum_{i=1}^{\frac{k-1}{2}} (p^{2(k-i)} - 2p^{k-i} + 1)(p^i - p^{i-1}).$$

Now consider

$$\sum_{i=1}^{\frac{k-1}{2}} (p^{2(k-i)})(p^i - p^{i-1}) = p^{2k}(1 - p^{-1}) \left(\sum_{i=1}^{\frac{k-1}{2}} p^{-i} \right) = p^{2k}(1 - p^{-1}) \left(\frac{p^{-1} - p^{-(\frac{k+1}{2})}}{(1 - p^{-1})} \right) = p^{2k-1} - p^{\frac{3k-1}{2}}.$$

Also, $2\sum_{i=1}^{\frac{k-1}{2}} p^{k-i}(p^i - p^{i-1}) = 2\sum_{i=1}^{\frac{k-1}{2}} (p^k - p^{k-1}) = (k-1)(p^k - p^{k-1})$

and $\sum_{i=1}^{\frac{k-1}{2}} (p^i - p^{i-1}) = p^{\frac{k-1}{2}} - 1$ (from Lemma 2.1).

Thus,

$$\sum_{v \in V_1} d(v)^2 = p^{2k-1} - p^{\frac{3k-1}{2}} + (k-1)(p^k - p^{k-1}) + (p^{\frac{k-1}{2}} - 1). \tag{3.1}$$

Subcase 2) Let $V_2 = \bigcup_{i=\frac{k+1}{2}}^{k-1} A_i$. Then degree of each vertex in A_i is $p^{\frac{k-1}{2}}, \dots, p^3, p^2, p$ respectively.

$$\begin{aligned} \sum_{v \in V_2} d(v)^2 = \sum_{i=\frac{k+1}{2}}^{k-1} \sum_{v \in A_i} d(v)^2 &= \sum_{\frac{k+1}{2}}^{k-1} (p^{(k-i)})^2 |A_i| \\ &= \sum_{\frac{k+1}{2}}^{k-1} p^{(2k-2i)} (p^i - p^{(i-1)}) \\ &= p^{2k}(1 - p^{-1}) \sum_{\frac{k+1}{2}}^{k-1} p^{-i} \\ &= p^{\frac{3k-1}{2}} - p^k. \end{aligned} \tag{3.2}$$

Subcase 3) If $v \in A_K$ then $d(v) = 1$ and $|A_k| = p^k - p^{k-1}$. Thus,

$$\sum d(v)^2 = \sum d(v)^2 |A_k| = p^k - p^{k-1}. \tag{3.3}$$

Subcase 4) If $v \in A_{k+1}$ then $d(v) = p^k - 1$ and $|A_{k+1}| = 1$. Thus,

$$\sum d(v)^2 = \sum d(v)^2 |A_{k+1}| = (p^k - 1)^2. \tag{3.4}$$

Hence, using Equations (3.1) to (3.4) we get
$$\sum_{i=1}^{k+1} d(v)^2 = p^{2k} + p^{(2k-1)} + (k-3)p^k - kp^{(k-1)} + p^{(\frac{k-1}{2})}.$$

Case II) If k is even. The techniques of Case (I) can be applied to obtain Case (II).
$$\sum_{i=1}^{k+1} d(v)^2 = p^{2k} + p^{2k-1} + (k-2)p^k - (k+1)p^{k-1} + p^{\frac{k}{2}}.$$ □

Theorem 3.2. *Let p be a prime. Then the second Zagreb index M_2 of $\Gamma_0(\mathbb{Z}_{p^k})$ is*
$$\begin{cases} \frac{1}{8p^2(p+1)}\Big((3k^2+2k+3)p^{(2k+3)} - (3k^2-14k+7)p^{(2k+2)} - (3k^2-6k+7)p^{(2k+1)} \\ +(3k^2-6k+3)p^{2k} - (k^2-9)p^{(k+3)} - (k^2-12k+23)p^{(k+2)} + (k^2-8k+11)p^{(k+1)} \\ -(k^2-4k+3)p^k - (4k+16)p^{\frac{3k+5}{2}} + (4k+4)p^{\frac{3k+1}{2}} + (4k-16)p^{\frac{k+3}{2}} - (4k+8)p^{\frac{3(k+1)}{2}} \\ -(4k-4)p^{\frac{k-1}{2}} - 4p^{\frac{3k-1}{2}} - 4p^{\frac{k+5}{2}} - 8p^{\frac{k+1}{2}} + 8p^2 + 8p\Big), & \text{if } k \text{ is odd;} \\ \frac{1}{8p^2(p+1)}\Big((3k^2+4k)p^{(2k+3)} - (3k^2-4k-4)p^{(2k+2)} - (3k^2+4k-4)p^{(2k+1)} + (3k^2-4k) \\ p^{2k} - (k^2-2k-20)p^{(k+3)} - (k^2-2k+28)p^{(k+2)} + (k^2-2k)p^{(k+1)} - (k^2-2k)p^k \\ -(4k+20)p^{\frac{3k+4}{2}} + (4k-4)p^{\frac{3k+2}{2}} + (4k-4)p^{\frac{k+4}{2}} - (4k-16)p^{\frac{3(k+2)}{2}} + 4p^{\frac{k+6}{2}} + 4kp^{\frac{3k}{2}} \\ -4kp^{\frac{k}{2}} + 8p^{\frac{k+2}{2}} - 8p^3 - 8p^2 + 8p\Big), & \text{if } k \text{ is even.} \end{cases}$$

Proof. In the view of Lemma 2.3. There are two cases for k.

Case I) If k is odd.

Subcase 1) As given in (3) of Lemma 2.3, the vertex set $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$ forms a complete subgraph of $\Gamma_0(\mathbb{Z}_{p^k})$ and degree of each vertex in the respective set is $(p^{(k-i)}-1)$. Thus in this case we get,

$$\begin{aligned} d_1 &= \sum_{i=1}^{\frac{k-1}{2}} d(v_i) \sum_{v_i,v_j\in V_1} d(v_j) = \sum_{i=1}^{\frac{k-1}{2}} d(v)|A_i|\left(\tfrac{1}{2}d(v)(|A_i|-1)\right) \\ &= \tfrac{1}{2}\sum_{i=1}^{\frac{k-1}{2}} (p^{(k-i)}-1)(p^i-p^{(i-1)})((p^{(k-i)}-1)(p^i-p^{(i-1)}-1)) \\ &= \tfrac{1}{2}\sum_{i=1}^{\frac{k-1}{2}} (p^k-p^i)(1-p^{(-1)})\left[(p^k-p^i)(1-p^{(-1)}) + 1 - p^{(k-i)}\right] \\ &= \tfrac{1}{2}(1-p^{(-1)})^2\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i)^2 + \tfrac{1}{2}(1-p^{(-1)})\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i) - \tfrac{1}{2}(1-p^{(-1)})\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i)p^{(k-i)} \\ &= (p-1)^2p^{(2k-2)}(\tfrac{k-1}{4}) + (p-1)p^{(k-1)}(1-p^{\frac{(k-1)}{2}}) + \frac{(p-1)(p^{(k-1)}-1)}{2(p+1)} \\ &\quad + \tfrac{1}{2}[(p-1)p^{(k-1)}(\tfrac{k-1}{2}) + (1-p^{\frac{k-1}{2}}) + p^{(2k-1)} - p^{(\frac{3k-1}{2})} - p^{(k-1)}(p-1)(\tfrac{k-1}{2})]. \end{aligned} \tag{3.5}$$

Subcase 2) In this case, we take the adjacency in the complete graph $\bigcup_{i=1}^{\frac{k-1}{2}} A_i$, from vertex set $V_1 = \bigcup_{i=1}^{\frac{k-3}{2}} A_i$ to the vertex set $V_2 = \bigcup_{j=i+1}^{\frac{k-1}{2}} A_j$. Hence,

$$
\begin{aligned}
d_2 &= \sum_{i=1}^{\frac{k-1}{2}} \sum_{v_i\in V_1, v_j\in V_2} d(v_i)d(v_j) = \sum_{i=1}^{\frac{k-3}{2}} d(v)|A_i| \left(\sum_{j=i+1}^{\frac{k-1}{2}} d(v)|A_j| \right) \\
&= \sum_{i=1}^{\frac{k-3}{2}} (p^{k-i}-1)(p^i-p^{i-1}) \left(\sum_{j=i+1}^{\frac{k-1}{2}} (p^{k-j}-1)(p^j-p^{j-1}) \right) \\
&= (1-p^{-1})^2 \sum_{i=1}^{\frac{k-3}{2}} (p^k-p^i) \left(\sum_{j=i+1}^{\frac{k-1}{2}} (p^k-p^j) \right) \\
&= (1-p^{-1})^2 \sum_{i=1}^{\frac{k-3}{2}} (p^k-p^i) \left(\frac{p^k[(k-1)-2i]}{2} - \frac{p^{\frac{k+1}{2}}-p^{i+1}}{(p-1)} \right) \\
&= (1-p^{-1})^2 \left(\frac{p^{2k}(k-1)(k-3)}{4} - \frac{p^k(k-3)(k-1)}{8} - \frac{p^{\frac{3k+1}{2}}(k-3)}{2(p-1)} \right. \\
&\quad + \frac{p^{k+1}p(p^{\frac{(k-3)}{2}}-1)}{(p-1)^2} - \frac{p^k(k-1)p(p^{\frac{k-3}{2}})}{2(p-1)} + \frac{2p-(k-1)p^{\frac{k-1}{2}}+(k-3)p^{\frac{k+1}{2}}}{2(p-1)^2} \\
&\quad \left. + \frac{p^{\frac{k+1}{2}}p(p^{\frac{k-3}{2}}-1)}{(p-1)^2} - \frac{(p^k-p^3)}{(p-1)(p^2-1)} \right).
\end{aligned} \tag{3.6}
$$

Subcase 3) By using (2) of Lemma 2.3, we get the adjacency of the sets from vertex set $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$ to the vertex set $V_4 = \bigcup_{j=\frac{k+1}{2}}^{k-i} A_j$ and degree of each vertex of the set V_4 is $p^{(k-j)}$ respectively.

$$
\begin{aligned}
d_3 &= \sum_{i=1}^{\frac{k-1}{2}} \sum_{v\in A_i} d(v)|A_i| \left(\sum_{v\in A_j} \sum_{j=\frac{k+1}{2}}^{k-i} d(v)|A_j| \right) \\
&= \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i}-1)(p^i-p^{i-1}) \left(\sum_{j=\frac{k+1}{2}}^{k-i} (p^{k-j})(p^j-p^{j-1}) \right) \\
&= \sum_{i=1}^{\frac{k-1}{2}} (1-p^{-1})(p^k-p^i) \left(\sum_{j=\frac{k+1}{2}}^{k-i} (p^k-p^{k-1}) \right) \\
&= (1-p^{-1}) \sum_{i=1}^{\frac{k-1}{2}} (p^k-p^i) \left((p^k-p^{k-1})(\frac{k+1-2i}{2}) \right) \\
&= (1-p^{-1})(p^k-p^{k-1}) \sum_{i=1}^{\frac{k-1}{2}} (p^k-p^i) \left(\frac{k+1-2i}{2} \right) \\
&= \left(\frac{(p-1)^2p^k}{2p^2} \right) \left((k+1)p^k(\frac{k-1}{2}) - \frac{p^k(k^2-1)}{4} - (k+1)\frac{p(1-p^{\frac{k-1}{2}})}{1-p} \right. \\
&\quad \left. +(\frac{2p-(k+1)p^{\frac{k+1}{2}}+(k-1)p^{\frac{k+3}{2}}}{(p-1)^2}) \right) \\
&= \left(\frac{(p-1)^2p^{k-1}}{2} \right) \left(\frac{p^{k-1}(k^2-1)}{4} + \frac{(k+1)(1-p^{\frac{k-1}{2}})}{p-1} + \frac{2-(k+1)p^{\frac{k-1}{2}}+(k-1)p^{\frac{k+1}{2}}}{(p-1)^2} \right) \\
&= (\frac{p^{k-2}}{8}) \left[(k^2-1)(p^{k+2}-2p^{k+1}+p^k)+4(k+1)p^2-4(k-1)p-8p^{\frac{k+3}{2}} \right]
\end{aligned} \tag{3.7}
$$

Subcase 4) By using (1) of Lemma 2.3, a vertex in A_{k+1} is adjacent to every vertex in $\bigcup_{i=1}^{k} A_i$.

$$\begin{aligned}
d_4 &= \sum_{i=1}^{k}\sum_{v\in A_i} d(v_{k+1})|A_{k+1}|d(v)|A_i| \\
&= \left(\sum_{i=1}^{\frac{k-1}{2}} d(v_{k+1})d(v)|A_i|\right) + \left(\sum_{i=\frac{k+1}{2}}^{k-1} d(v_{k+1})d(v)|A_i|\right) + \Big((p^k-1)d(v)|A_k|\Big) \\
&= \left(\sum_{i=1}^{\frac{k-1}{2}}(p^k-1)(p^{k-i}-1)(p^i-p^{i-1})\right) + \left(\sum_{i=\frac{k+1}{2}}^{k-1}(p^k-1)(p^{k-i})(p^i-p^{i-1})\right) \\
&\quad +(p^k-1)(p^k-p^{k-1}) \\
&= (p^k-1)(1-p^{-1})\left(p^k(\frac{k-1}{2}) - (\frac{p-p^{\frac{k+1}{2}}}{1-p})\right) + (p^k-1)(p^k-p^{k-1})(\frac{k-1}{2}) \\
&\quad +(p^k-1)p^k(1-p^{-1}).
\end{aligned} \tag{3.8}$$

Hence, using Equations (3.5) to (3.8) we get

$$\begin{aligned}
M_2(\Gamma_0(R)) &= \frac{1}{8p^2(p+1)}\Big((3k^2+2k+3)p^{2k+3} - (3k^2-14k+7)p^{2k+2} - (3k^2-6k+7) \\
&\quad p^{2k+1} + (3k^2-6k+3)p^{2k} - (k^2-9)p^{k+3} - (k^2-12k+23)p^{k+2} + (k^2-8k \\
&\quad +11)p^{k+1} - (k^2-4k+3)p^k - (4k+16)p^{\frac{3k+5}{2}} + (4k+4)p^{\frac{3k+1}{2}} + (4k-16)p^{\frac{k+3}{2}} \\
&\quad -(4k+8)p^{\frac{3(k+1)}{2}} - (4k-4)p^{\frac{k-1}{2}} - 4p^{\frac{3k-1}{2}} - 4p^{\frac{k+5}{2}} - 8p^{\frac{k+1}{2}} + 8p^2 + 8p\Big).
\end{aligned}$$

Case II) If k is even: The techniques of Case (I) can be applied to obtain Case (II).

$$\begin{aligned}
M_2(\Gamma_0(R)) &= \frac{1}{8p^2(p+1)}\Big((3k^2+4k)p^{2k+3} - (3k^2-4k-4)p^{2k+2} - (3k^2+4k-4)p^{2k+1} \\
&\quad +(3k^2-4k)p^{2k} - (k^2-2k-20)p^{k+3} - (k^2-2k+28)p^{k+2} + (k^2-2k)p^{k+1} \\
&\quad -(k^2-2k)p^k - (4k+20)p^{\frac{3k+4}{2}} + (4k-4)p^{\frac{3k+2}{2}} + (4k-4)p^{\frac{k+4}{2}} \\
&\quad -(4k-16)p^{\frac{3(k+2)}{2}} + 4p^{\frac{k+6}{2}} + 4kp^{\frac{3k}{2}} - 4kp^{\frac{k}{2}} + 8p^{\frac{k+2}{2}} - 8p^3 - 8p^2 + 8p\Big).
\end{aligned}$$

□

Theorem 3.3. *Let p be a prime. Then the first Zagreb coindex $\overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k}))$ is given by*

$$\begin{cases} kp^{2k} - (k+1)p^{2k-1} - 2(k-1)p^k + 2kp^{k-1} - p^{\frac{3k-1}{2}}, & \text{if } k \text{ is odd;} \\ (k+1)p^{2k} - (k+2)p^{2k-1} - 2kp^k + 2(k+1)p^{k-1} - p^{\frac{3k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. We have the vertices in $\Gamma_0(\mathbb{Z}_{p^k})$ is p^k and the number of edges are $\frac{1}{2}(kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}})$, if k is odd and $\frac{1}{2}[kp^k - kp^{k-1} - p^{\frac{k}{2}} + 2p^k - p^{k-1}]$, if k is even. By using (1) of Lemma 2.6, the first Zagreb coindex is

If k is odd:

$$\begin{aligned}
\overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k})) &= 2m(n-1) - M_1(\Gamma_0(\mathbb{Z}_{p^k})) \\
&= 2(\tfrac{1}{2}[kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}}])(p^k-1) - [p^{(2k)} + p^{(2k-1)} + (k-3)p^k \\
&\quad -kp^{(k-1)} + p^{(\frac{k-1}{2})}] \\
\overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k})) &= kp^{2k} - (k+1)p^{2k-1} - 2(k-1)p^k + 2kp^{k-1} - p^{\frac{3k-1}{2}}.
\end{aligned}$$

If k is even, then $\overline{M}_1(\Gamma_0(\mathbb{Z}_{p^k})) = (k+1)p^{2k} - (k+2)p^{2k-1} - 2kp^k + 2(k+1)p^{k-1} - p^{\frac{3k}{2}}$. □

Theorem 3.4. *Let p be a prime. Then the second Zagreb coindex $\overline{M_2}(\Gamma_0(\mathbb{Z}_{p^k}))$ is given by*

$$\begin{cases} \frac{1}{8p^2(p+1)}[4(kp^{k+1}+p^{k+1}-kp^k-p^{\frac{k+1}{2}})^2(p+1)-(3k^2+2k+7)p^{2k+3}+(3k^2-14k-1) \\ p^{2k+2}+(3k^2-6k+3)p^{2k+1}-(3k^2-6k+3)p^{2k}+(k^2-4k+3)p^{k+3}+(k^2-12k+35) \\ p^{k+2}-(k^2-12k+11)p^{k+1}+(k^2-4k+3)p^k+4(k+4)p^{\frac{3k+5}{2}}-4(k+1)p^{\frac{3k+1}{2}}-4(k-3) \\ p^{\frac{k+3}{2}}+4(k+2)p^{\frac{3(k+1)}{2}}+4(k-1)p^{\frac{k-1}{2}}+4p^{\frac{3k-1}{2}}+8p^{\frac{k+1}{2}}-8p^2-8p], \text{ if } k \text{ is odd;} \\ \frac{1}{8p^2(p+1)}[4((k+2)p^{k+1}-(k+1)p^k-p^{\frac{k+1}{2}})^2(p+1)-(k^2+2k)p^{k+1}+(3k^2-4k-12) \\ p^{2k+2}+(3k^2+4k-8)p^{2k+1}-(3k^2+4k)p^{2k}+(k^2-2k+8)p^{k+3}+(k^2-2k+40)p^{k+2} \\ -(3k^2-4k-4)p^{2k+3}+(k^2-2k)p^k+4(k+5)p^{\frac{3k+4}{2}}-4(k-1)p^{\frac{3k+2}{2}}-4kp^{\frac{k+4}{2}}+12p^2 \\ +4(k-4)p^{\frac{3(k+2)}{2}}-8p^{\frac{k+6}{2}}-4kp^{\frac{3k}{2}}+4kp^{\frac{k}{2}}-8p^{\frac{k+2}{2}}+12p^3-8p], \text{ if } k \text{ is even.} \end{cases}$$

Proof. Using (2) of Lemma 2.6 we get result. □

Theorem 3.5. *Let p and q be distinct primes with $p<q$. Then Zagreb indices of $\Gamma_0(\mathbb{Z}_{pq})$ are given by*

(1) $M_1(\Gamma_0(\mathbb{Z}_{pq}))=(pq-1)(pq+p+q)-(p^2+q^2)+2.$
(2) $M_2(\Gamma_0(\mathbb{Z}_{pq}))=4(pq)^2-(p+q)(3pq-2)-pq-1.$
(3) $\prod_1(\Gamma_0(\mathbb{Z}_{pq}))=p^{2(q-1)}q^{2(p-1)}(pq-1)^2.$
(4) $\prod_2(\Gamma_0(\mathbb{Z}_{pq}))=p^{p(q-1)}q^{q(p-1)}(pq-1)^{(pq-1)}.$
(5) $\overline{M_1}(\Gamma_0(\mathbb{Z}_{pq}))=3(pq)^2-3(p^2q+pq^2+pq-p-q)+p^2+q^2-2.$
(6) $\overline{M_2}(\Gamma_0(\mathbb{Z}_{pq}))=\frac{1}{2}[7(pq)^2-11(p^2q+pq^2)+11pq+5(p^2+q^2)-3(p+q)].$
(7) $\overline{\prod_1}(\Gamma_0(\mathbb{Z}_{pq}))=(p+1)^{(p-1)(q-1)^2}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(q+1)^{(p-1)^2(q-1)}\times(2q)^{\frac{(p-1)(p-2)}{2}}$ $\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}.$
(8) $\overline{\prod_2}(\Gamma_0(\mathbb{Z}_{pq}))=p^{(q-1)(pq-p-1)}q^{(p-1)(pq-q-1)}.$

Proof. The vertex set $V(\Gamma_0(\mathbb{Z}_{pq}))$ is partitioned in given 2.4 as $A_1,\ A_2,\ A_3,\ A_4$. The degree of each vertex in the set $A_1,\dots,A_4$ is $p,\ q,\ 1,\ pq-1$ and the cardinality of the these sets are $(q-1),\ (p-1),\ (p-1)(q-1),\ 1$ respectively.
The first Zagreb index $M_1(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned} M_1(\Gamma_0(\mathbb{Z}_{pq})) &= \sum_{u\in A_1}(p)^2+\sum_{u\in A_2}(q)^2+\sum_{u\in A_3}(1)^2+\sum_{u\in A_4}(pq-1)^2 \\ &= (q-1)p^2+(p-1)q^2+(p-1)(q-1)+(pq-1)^2 \\ &= (pq-1)(pq+p+q)-(p^2+q^2)+2. \end{aligned}$$

The adjacency of the vertices within the respective sets is provided in the Lemma 2.4. The second Zagreb index $M_2(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned} M_2(\Gamma_0(\mathbb{Z}_{pq})) &= \sum_{u\in A_1,v\in A_2}pq+\sum_{u\in A_1,v\in A_4}p(pq-1)+\sum_{u\in A_2,v\in A_4}q(pq-1)+\sum_{u\in A_3,v\in A_4}(pq-1)1 \\ &= pq(q-1)(p-1)+p(pq-1)(q-1)+q(pq-1)(p-1)+(pq-1)(p-1)(q-1) \\ &= 4(pq)^2-(p+q)(3pq-2)-pq-1. \end{aligned}$$

The first multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned} \prod\nolimits_1(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{v\in A_1}(p)^2\times\prod_{v\in A_2}(q)^2\times\prod_{v\in A_3}(1)^2\times\prod_{v\in A_4}(pq-1)^2 \\ &= (p^2)^{(q-1)}(q^2)^{(p-1)}(1^2)^{(p-1)(q-1)}[(pq-1)^2]^1 \\ &= p^{2(q-1)}q^{2(p-1)}(pq-1)^2. \end{aligned}$$

The second multiplicative Zagreb indices of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned}\prod\nolimits_2(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{u\in A_1,v\in A_2}(p)(q)\times\prod_{u\in A_1,v\in A_4}(p)(pq-1)\times\prod_{u\in A_2,v\in A_4}(q)(pq-1)\\ &\times\prod_{u\in A_3,v\in A_4}(1)(pq-1)\\ &=(pq)^{(q-1)(p-1)}\times[p(pq-1)]^{(q-1)}\times[q(pq-1)]^{(p-1)}\times(pq-1)^{(p-1)(q-1)}\\ &=p^{p(q-1)}q^{q(p-1)}(pq-1)^{(pq-1)}.\end{aligned}$$

The number of edges of $\Gamma_0(\mathbb{Z}_{pq})$ are $2pq-p-q$. Using (1) of Lemma 2.6, the first Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned}\overline{M_1}(\Gamma_0(\mathbb{Z}_{pq})) &= 2m(n-1)-M_1(\Gamma_0(\mathbb{Z}_{pq}))\\ &=2(2pq-p-q)(pq-1)-[(pq-1)(pq+p+q)-(p^2+q^2)+2]\\ &=3(pq)^2-3(p^2q+pq^2+pq-p-q)+p^2+q^2-2.\end{aligned}$$

Using (2) of Lemma 2.6, the second Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\overline{M_2}(\Gamma_0(\mathbb{Z}_{pq})) = \tfrac{1}{2}[7(pq)^2-11(p^2q+pq^2)+11pq+5(p^2+q^2)-3(p+q)].$$

The second multiplicative Zagreb coindex of $(\Gamma_0(\mathbb{Z}_{pq}))$is

$$\begin{aligned}\overline{\prod}_1(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{u\in A_1,v\in A_3}(p+1)\times\prod_{u,v\in A_1}(p+p)\times\prod_{u,v\in A_2}(q+q)\times\prod_{u\in A_2,v\in A_3}(q+1)\\ &\times\prod_{u,v\in A_3}(1+1)\\ &=(p+1)^{(q-1)(p-1)(q-1)}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(2q)^{\frac{(p-1)(p-2)}{2}}\\ &\times(q+1)^{(p-1)(p-1)(q-1)}\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\ &=(p+1)^{(p-1)(q-1)^2}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(q+1)^{(p-1)^2(q-1)}\\ &\times(2q)^{\frac{(p-1)(p-2)}{2}}\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}.\end{aligned}$$

The second multiplicative Zagreb coindex of $(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned}\overline{\prod}_2(\Gamma_0(\mathbb{Z}_{pq})) &= p^{(q-1)(p-1)(q-1)}\times(p^2)^{\frac{(q-1)(q-2)}{2}}\times q^{(p-1)(p-1)(q-1)}\times(q^2)^{\frac{(p-1)(p-2)}{2}}\\ &\times 1^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\ &=p^{(q-1)(p-1)(q-1)}\times p^{(q-1)(q-2)}\times q^{(p-1)(p-1)(q-1)}\times q^{(p-1)(p-2)}\\ &=p^{(q-1)(pq-p-1)}q^{(p-1)(pq-q-1)}.\end{aligned}$$

□

Theorem 3.6. *Let p, q and r be distinct primes with $p < q < r$. Then Zagreb indices of $\Gamma_0(R)$ where R is $\mathbb{Z}_{pqr}$ are given by*

(1) $M_1(\Gamma_0(R)) = (pqr)^2 + pqr(p+q+r+pq+qr+pr-1) + (p^2+q^2+r^2) - pr(1+p+r+pr) - pq(1+p+q+pq) - qr(1+q+r+qr) + (p+q+r)$.

(2) $M_2(\Gamma_0(R)) = 13(p^2q^2r^2) - 9(p^2q^2r + p^2r^2q + q^2r^2p) + 6(p^2qr + q^2pr + r^2pq) - 11pqr + 4(pq+pr+qr) - 2(p+q+r) + 1$.

(3) $\prod_1(\Gamma_0(R)) = p^{(2qr-2)}q^{(2pr-2)}r^{(2pq-2)}(pqr-1)^2$.

(4) $\prod_2(\Gamma_0(R)) = p^{[3pqr-2(pq+pr)+p]}\times q^{[3pqr-2(pq+qr)+q]}\times r^{[3pqr-(pr+qr)+r]}\times(pqr-1)^{(pqr-1)}$.

(5) $\overline{M_1}(\Gamma_0(R)) = 7(pqr)^2 - 5(p^2q^2r + p^2r^2q + q^2r^2p) + (p^2qr + q^2pr + r^2pq) - 9pqr + 5(pq + pr + qr) - 3(p+q+r) + (p^2q^2 + p^2r^2 + q^2r^2) - (p^2+q^2+r^2) + (p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) + 2$.

(6) $\overline{M_2}(\Gamma_0(R)) = \frac{1}{2}[5p^2q^2r^2 - 15(p^2q^2r + p^2r^2q + q^2r^2p) + 9(p^2q^2 + p^2r^2 + q^2r^2) + 19(p^2qr + q^2pr + r^2pq) - 17pqr - 5(pq+pr+qr) - (p+q+r) + 7(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) + (p^2+q^2+r^2)]$.

$$(7)\ \overline{\prod}_1(\Gamma_0(R)) = (p+q)^{(p-1)(q-1)(r-1)^2} \times (p+r)^{(p-1)(q-1)^2(r-1)} \times (q+r)^{(p-1)^2(q-1)(r-1)} \times p^{\frac{1}{2}[(qr)^2-5qr+2q+2r]} \times q^{\frac{1}{2}[(pr)^2-5pr+2p+2r]} \times r^{\frac{1}{2}[(pq)^2-5pq+2p+2q]} \times (1+p)^{(p-1)[(q-1)^2+(r-1)^2+(q-1)^2(r-1)^2]} \times (1+q)^{(q-1)[(p-1)^2+(r-1)^2+(p-1)^2(r-1)^2]} \times (1+r)^{(r-1)[(p-1)^2+(q-1)^2+(p-1)^2(q-1)^2]} \times 2^{(p-1)^2(q-1)^2(r-1)^2} \times 2^{(p-1)^2(q-1)^2} \times 2^{(p-1)^2(r-1)^2+(q-1)^2(r-1)^2-(p-1)(q-1)(r-1)} \times 2^{(1-p)(q-p+1)+(1-r)(p-r+1)+(1-q)(r-q+1)}.$$

$$(8)\ \overline{\prod}_2(\Gamma_0(R)) = p^{[q^2r^2p+2(pq+pr)-(p+qr)-4pqr+1]} \times q^{[p^2r^2q+2(pq+qr)-(q+pr)-4pqr+1]} \times r^{[p^2q^2r+2(pr+qr)-(r+pq)-4pqr+1]}$$

Proof. The vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$ can be partitioned into $\bigcup_{i=1}^{8} A_i$ as given in the proof of Lemma 2.5. The degree of each vertex in the sets $A_1,\ A_2,\ \ldots,\ A_8$ is $p,\ q,\ r,\ qr,\ pr,\ pq,\ 1,\ (pqr-1)$ respectively.

The total number of vertices in sets $A_1,\ A_2,\ A_3,\ \cdots,\ A_8$ is $(q-1)(r-1),\ (p-1)(r-1),\ (p-1)(q-1),\ (p-1),\ (q-1),\ (r-1),\ (p-1)(q-1)(r-1),\ 1$ respectively. Then we have the result.

The first Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned} M_1(\Gamma_0(R)) &= \sum_{u\in A_1}(p)^2 + \sum_{u\in A_2}(q)^2 + \sum_{u\in A_3}(r)^2 + \sum_{u\in A_4}(qr)^2 + \sum_{u\in A_5}(pr)^2 + \sum_{u\in A_6}(pq)^2 \\ &\quad + \sum_{u\in A_7}(1)^2 + \sum_{u\in A_8}(pqr-1)^2 \\ &= (q-1)(r-1)p^2 + (p-1)(r-1)q^2 + (p-1)(q-1)r^2 + (p-1)(qr)^2 \\ &\quad +(q-1)(pr)^2 + (r-1)(pq)^2 + (p-1)(q-1)(r-1) + (pqr-1)^2 \\ &= (pqr)^2 + pqr(p+q+r+pq+qr+pr-1) + (p^2+q^2+r^2) - pr(1+p+r \\ &\quad +pr) - pq(1+p+q+pq) - qr(1+q+r+qr) + (p+q+r). \end{aligned}$$

By the adjacency between the vertices as given in the Lemma 2.5, the second Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned} M_2(\Gamma_0(R)) &= \sum_{u\in A_1,v\in A_4} p(qr) + \sum_{u\in A_1,v\in A_8} p(pqr-1) + \sum_{u\in A_2,v\in A_5} q(pq) + \sum_{u\in A_2,v\in A_8} q(pqr-1) \\ &\quad + \sum_{u\in A_3,v\in A_6} r(pq) + \sum_{u\in A_3,v\in A_8} r(pqr-1) + \sum_{u\in A_4,v\in A_8} qr(pqr-1) \\ &\quad + \sum_{u\in A_4,v\in A_5} qr(pr) + \sum_{u\in A_4,v\in A_6} qr(pq) + \sum_{u\in A_5,v\in A_6} pr(pq) + \sum_{u\in A_5,v\in A_8} pr(pqr-1) \\ &\quad + \sum_{u\in A_6,v\in A_8} pq(pqr-1) + \sum_{u\in A_7,v\in A_8} 1(pqr-1) \\ &= (pqr)(q-1)(r-1)(p-1) + p(pqr-1)(q-1)(r-1) + pqr(p-1) \\ &\qquad (q-1)(r-1) + q(pqr-1)(p-1)(r-1) + pqr(p-1)(q-1)(r-1) \\ &\quad +r(pqr-1)(p-1)(q-1) + qr(pqr-1)(p-1) + (qrpr)(p-1)(q-1) \\ &\quad +(qrpq)(p-1)(r-1) + (prpq)(q-1)(r-1) + pr(pqr-1)(q-1) \\ &\quad +pq(pqr-1)(r-1) + (pqr-1)(p-1)(q-1)(r-1)1q1 \\ &= 13(p^2q^2r^2) - 9(p^2q^2r + p^2r^2q + q^2r^2p) + 6(p^2qr + q^2pr + r^2pq) - 11pqr \\ &\quad +4(pq+pr+qr) - 2(p+q+r) + 1. \end{aligned}$$

The first multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\prod_1(\Gamma_0(R)) &= \prod_{v\in A_1}(p)^2\times\prod_{v\in A_2}(q)^2\times\prod_{v\in A_3}(r)^2\times\prod_{v\in A_4}(qr)^2\times\prod_{v\in A_5}(pr)^2\times\prod_{v\in A_6}(pq)^2\\ &\quad\times\prod_{v\in A_7}(1)^2\times\prod_{v\in A_8}(pqr-1)^2\\ &= (p)^{2(q-1)(r-1)}\times(q)^{2(p-1)(r-1)}\times(r)^{2(p-1)(q-1)}\times(qr)^{2)(p-1)}\times(pr)^{2(q-1)}\\ &\quad\times(pq)^{2(r-1)}\times 1\times(pqr-1)^2\\ &= p^{(2qr-2)}q^{(2pr-2)}r^{(2pq-2)}(pqr-1)^2.\end{aligned}$$

The second multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\prod_2(\Gamma_0(R)) &= \prod_{u\in A_1,v\in A_4}p(qr)\times\prod_{u\in A_1,v\in A_8}p(pqr-1)\times\prod_{u\in A_2,v\in A_5}q(pr)\times\prod_{u\in A_2 v\in A_8}q(pqr-1)\\ &\quad\times\prod_{u\in A_3,v\in A_6}r(pq)\times\prod_{u\in A_3,v\in A_8}r(pqr-1)\times\prod_{u\in A_4,v\in A_5}qr(pr)\times\prod_{u\in A_4,v\in A_6}qr(pq)\\ &\quad\times\prod_{u\in A_4,v\in A_8}qr(pqr-1)\times\prod_{u\in A_5 v\in A_6}pr(pq)\times\prod_{u\in A_5,v\in A_8}pr(pqr-1)\\ &\quad\times\prod_{u\in A_6,v\in A_8}pq(pqr-1)\times\prod_{u\in A_7,v\in A_8}1(pqr-1)\\ &= (pqr)^{(q-1)(r-1)(p-1)}\times[p(pqr-1)]^{(q-1)(r-1)}\times(pqr)^{(p-1)(r-1)(q-1)}\\ &\quad\times[q(pqr-1)]^{(p-1)(r-1)}\times(pqr)^{(p-1)(q-1)(r-1)}\times[r(pqr-1)]^{(p-1)(q-1)}\\ &\quad\times(pqr^2)^{(p-1)(q-1)}\times(pq^2r)^{(p-1)(r-1)}\times[qr(pqr-1)]^{(p-1)}\times(p^2qr)^{(q-1)(r-1)}\\ &\quad\times[pr(pqr-1)]^{(q-1)}\times[pq(pqr-1)]^{(r-1)}\times(pqr-1)^{(p-1)(q-1)(r-1)}\\ &= (pqr)^{3pqr-2(pq+qr+pr)+(p+q+r)}\times p^{2qr-q-r}\times q^{2pr-p-r}\times r^{2pq-p-q}\\ &\quad\times(pqr-1)^{(pqr-1)}\\ &= p^{[3pqr-2(pq+pr)+p]}\times q^{[3pqr-2(pq+qr)+q]}\times r^{[3pqr-(pr+qr)+r]}\times(pqr-1)^{(pqr-1)}.\end{aligned}$$

The first Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\overline{M_1}(\Gamma_0(R)) &= 2m(n-1)-M_1(R)\\ &= 2[4pqr-2(pq+pr+qr)+(p+q+r)-1](pqr-1)-[(pqr)^2+pqr(p+q\\ &\quad+r+pq+qr+pr-1)-pr(1+p+r+pr)-pq(1+p+q+pq)-qr(1\\ &\quad+q+r+qr)+(p+q+r)+(p^2+q^2+r^2)]\\ &= 7(pqr)^2-5(p^2q^2r+p^2r^2q+q^2r^2p)+(p^2qr+q^2pr+r^2pq)-9pqr+5(pq\\ &\quad+pr+qr)-3(p+q+r)+(p^2q^2+p^2r^2+q^2r^2)-(p^2+q^2+r^2)+(p^2q\\ &\quad+p^2r+q^2p+q^2r+r^2p+r^2q)+2.\end{aligned}$$

By using (2) of Lemma 2.6, the second Zagreb coindex $M_2(\Gamma_0(\mathbb{Z}_{pqr}))$ is given by

$$\begin{aligned}\overline{M_2}(\Gamma_0(R)) &= \tfrac{1}{2}[5p^2q^2r^2-15(p^2q^2r+p^2r^2q+q^2r^2p)+9(p^2q^2+p^2r^2+q^2r^2)+19(p^2qr\\ &\quad+q^2pr+r^2pq)-17pqr-5(pq+pr+qr)-(p+q+r)+7(p^2q+p^2r+q^2p\\ &\quad+q^2r+r^2p+r^2q)+(p^2+q^2+r^2)].\end{aligned}$$

The first multiplicative Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\overline{\prod}_1(\Gamma_0(R)) &= \prod_{u\in A_1,v\in A_2}(p+q)\times\prod_{u\in A_1,v\in A_3}(p+r)\times\prod_{u\in A_1,v\in A_5}(p+pr)\times\prod_{u\in A_1,v\in A_6}(p+pq)\\ &\quad\times\prod_{u\in A_1,v\in A_7}(p+1)\times\prod_{u,v\in A_1}(p+p)\times\prod_{u\in A_2,v\in A_3}(q+r)\times\prod_{u\in A_2,v\in A_4}(q+qr)\\ &\quad\times\prod_{u\in A_2,v\in A_6}(q+pq)\times\prod_{u\in A_2,v\in A_7}(q+1)\times\prod_{u,v\in A_2}(q+q)\times\prod_{u\in A_3,v\in A_4}(r+qr)\\ &\quad\times\prod_{u\in A_3,v\in A_5}(r+pr)\times\prod_{u\in A_3,v\in A_7}(r+1)\times\prod_{u,v\in A_3}(r+r)\times\prod_{u\in A_4,v\in A_7}(qr+1)\\ &\quad\times\prod_{u,v\in A_4}(qr+qr)\times\prod_{u\in A_5,v\in A_7}(pr+1)\times\prod_{u,v\in A_5}(pr+pr)\times\prod_{u\in A_6,v\in A_7}(pq+1)\end{aligned}$$

$$\begin{aligned}
&\times \prod_{u,v\in A_6} (pq+pq) \times \prod_{u,v\in A_7} (1+1)\\
&= (p+q)^{(q-1)(r-1)(p-1)(r-1)} \times (p+r)^{(q-1)(r-1)(p-1)(q-1)} \times (p+pr)^{(q-1)(r-1)(q-1)}\\
&\times (p+pq)^{(q-1)(r-1)(r-1)} \times (p+1)^{(q-1)(r-1)(p-1)(q-1)(r-1)} \times (2p)^{\frac{(q-1)(r-1)[(q-1)(r-1)-1]}{2}}\\
&\times (q+r)^{(p-1)(r-1)(p-1)(q-1)} \times (q+qr)^{(p-1)(r-1)(p-1)} \times (q+pq)^{(p-1)(r-1)(r-1)}\\
&\times (q+1)^{(p-1)(r-1)(p-1)(q-1)(r-1)} \times (2q)^{\frac{(p-1)(r-1)[(p-1)(r-1)-1]}{2}} \times (r+qr)^{(p-1)(q-1)(p-1)}\\
&\times (r+pr)^{(p-1)(q-1)(q-1)} \times (r+1)^{(p-1)(q-1)(p-1)(q-1)(r-1)} \times (2r)^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\
&\times (qr+1)^{(p-1)(p-1)(q-1)(r-1)} \times (2qr)^{\frac{(p-1)(p-2)}{2}} \times (pr+1)^{(q-1)(p-1)(q-1)(r-1)}\\
&\times (2pr)^{\frac{(q-1)(q-2)}{2}} \times (pq+1)^{(r-1)(p-1)(q-1)(r-1)} \times (2pr)^{\frac{(r-1)(r-2)}{2}}\\
&\times 2^{\frac{(p-1)(q-1)(r-1)[(p-1)(q-1)(r-1)-1]}{2}}\\
&= (p+q)^{(p-1)(q-1)(r-1)^2} \times (p+r)^{(p-1)(q-1)^2(r-1)} \times (q+r)^{(p-1)^2(q-1)(r-1)}\\
&\times p^{\frac{1}{2}[(qr)^2-5qr+2q+2r]} \times q^{\frac{1}{2}[(pr)^2-5pr+2p+2r]} \times r^{\frac{1}{2}[(pq)^2-5pq+2p+2q]}\\
&\times (1+p)^{(p-1)[(q-1)^2+(r-1)^2+(q-1)^2(r-1)^2]} \times (1+q)^{(q-1)[(p-1)^2+(r-1)^2+(p-1)^2(r-1)^2]}\\
&\times (1+r)^{(r-1)[(p-1)^2+(q-1)^2+(p-1)^2(q-1)^2]} \times 2^{(p-1)^2(q-1)^2(r-1)^2} \times 2^{(p-1)^2(q-1)^2}\\
&\times 2^{(p-1)^2(r-1)^2+(q-1)^2(r-1)^2-(p-1)(q-1)(r-1)} \times 2^{(1-p)(q-p+1)+(1-r)(p-r+1)+(1-q)(r-q+1)}.
\end{aligned}$$

The second multiplicative Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}
\overline{\prod}_2(\Gamma_0(R)) &= (pq)^{(q-1)(r-1)(p-1)(r-1)} \times (pr)^{(q-1)(r-1)(p-1)(q-1)} \times (p^2r)^{(q-1)(r-1)(q-1)}\\
&\times (p^2q)^{(q-1)(r-1)(r-1)} \times (p)^{(q-1)(r-1)(p-1)(q-1)(r-1)} \times (p^2)^{\frac{(q-1)(r-1)[(q-1)(r-1)-1]}{2}}\\
&\times (qr)^{(p-1)(r-1)(p-1)(q-1)} \times (q^2r)^{(p-1)(r-1)(p-1)} \times (pq^2)^{(p-1)(r-1)(r-1)}\\
&\times (q)^{(p-1)(r-1)(p-1)(q-1)(r-1)} \times (q^2)^{\frac{(p-1)(r-1)[(p-1)(r-1)-1]}{2}} \times (qr^2)^{(p-1)(q-1)(p-1)}\\
&\times (pr^2)^{(p-1)(q-1)(q-1)} \times (r)^{(p-1)(q-1)(p-1)(q-1)(r-1)} \times (r^2)^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\
&\times (qr)^{(p-1)(p-1)(q-1)(r-1)} \times [(qr)^2]^{\frac{(p-1)(p-2)}{2}} \times (pr)^{(q-1)(p-1)(q-1)(r-1)}\\
&\times [(pr)^2]^{\frac{(q-1)(q-2)}{2}} \times (pq)^{(r-1)(p-1)(q-1)(r-1)} \times [(pq)^2]^{\frac{(r-1)(r-2)}{2}}\\
&\times (1)^{\frac{(p-1)(q-1)(r-1)[(p-1)(q-1)(r-1)-1]}{2}}\\
&= p^{[q^2r^2p+2(pq+pr)-(p+qr)-4pqr+1]} \times q^{[p^2r^2q+2(pq+qr)-(q+pr)-4pqr+1]}\\
&\times r^{[p^2q^2r+2(pr+qr)-(r+pq)-4pqr+1]}.
\end{aligned}$$

□

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