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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. This article presents the concepts of statistically bounded, statistically bounded convergence, statistically bounded null, statistically regular convergence, statistically regular null double sequences of bi-complex numbers, statistically convergent double sequences in Pringsheim's sense, and statistically null double sequences in Pringsheim's sense. We have established that these spaces are linear, and we have demonstrated their many algebraic, topological, and geometric properties using the Euclidean norm defined on bi-complex numbers. Suitable examples have been discussed.

Keywords: Double sequences, Statistical convergence, Bi-complex numbers, $\mathbb{BC}$-convex, $\mathbb{BC}$-strictly convex, $\mathbb{BC}$- uniformly convex.

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1. Introduction

The notion of convergence double sequence was first proposed by Pringsheim [14]. Bromwich [2] has some of the earliest works on double sequence spaces. The concept of regular convergence of double sequence was later introduced by Hardy [5]. Additionally, double sequences of bi-complex numbers were introduced by Kumar and Tripathy in various directions [6], [7], and [8].

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Definition 1.1. [13] *Norm (Euclidean Norm) on $\mathbb{C}_2$ is defined by*

$$\|\gamma\|_{\mathbb{C}_2} = \sqrt{w^2+x^2+y^2+z^2} = \sqrt{|u_1|^2+|u_2|^2} = \sqrt{\frac{|\mu_1|^2+|\mu_2|^2}{2}}.$$

$\mathbb{C}_2$ becomes a modified Banach algebra with respect to this norm in the sense that

$$\|\gamma\cdot s\|_{\mathbb{C}_2} \leq \sqrt{2}\|\gamma\|_{\mathbb{C}_2}\cdot\|s\|_{\mathbb{C}_2}.$$

Definition 1.2. *Three types of conjugations are defined in the bi-complex numbers (Rochon, Shapiro* [16]*) as follows,*

(1) *i_1- conjugation of bi-complex number γ is $\gamma^* = \overline{u_1} + i_2\overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of u_1, u_2 respectively.*

(2) *i_2- conjugation of bi-complex number γ is $\tilde{\gamma} = u_1 - i_2u_2$, for all $u_1, u_2 \in \mathbb{C}_1$.*

(3) *i_1i_2- conjugation of bi-complex number γ is $\gamma^{'} = \overline{u_1} - i_2\overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of u_1, u_2 respectively.*

The concept of statistical convergence was introduced Fast [3] and reintroduced by Schoenberg [18]. It was also discussed in the work of Zygmund [20]. Subsequently, several researchers including Fridy and Orhan [4], Maddox [11], Salat [17], Mursaleen and Edely [12], Rath and Tripathy [15], Tripathy [19] and others explored this notion in various contexts".

A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty}\frac{1}{n}\sum_{l\leq n}\chi_E(l) \text{ exists },$$

where χ_E is the characteristic function of E.

A single sequence (γ_l) is said to be statistically convergent to L if for each $\varepsilon > 0, \delta(\{l \in \mathbb{N} : \|\gamma_l - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$ and write $\gamma_l \xrightarrow{\text{stat}} L$ or $stat - \lim \gamma_l = L$. A sequence that is statistically convergent to zero is called a statistically null sequence.

The density of a subset E of $\mathbb{N}\times\mathbb{N}$ is defined as

$$\delta_2(E) = \lim_{n,k\to\infty}\frac{1}{nk}\sum_{l\leq n}\sum_{m\leq k}\chi_E(l,m) \text{ exists.}$$

A double sequence (γ_{lm}) is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0, \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$, written as $st - \lim_{l,m\to\infty}\gamma_{lm} = L$.

A double sequence (γ_{lm}) is said to be statistically null if it is statistically convergent to 0 in Pringsheim's sense.

A double sequence (γ_{lm}) is said to be statistically regular convergent if it converges in Pringsheim's sense and the following statistical limits exist

$$stat - \lim_{l\to\infty} \gamma_{lm} = P_m, \text{ exists, for each } m \in \mathbb{N},$$

and

$$stat - \lim_{m\to\infty} \gamma_{lm} = Q_l, \text{ exists, for each } l \in \mathbb{N}.$$

For regularly null sequences we have $P_m = Q_l = L = 0$, for all $l, m \in \mathbb{N}$.

Definition 1.3. *Let (γ_{lm}) and (t_{lm}) be two double sequences, then we say that $\gamma_{lm} = t_{lm}$, for all most all l and m (in short a.a.l and m) if $\delta_2(\{(l,m) : \gamma_{lm} \neq t_{lm}\}) = 0$.*

Definition 1.4. *A double sequence (γ_{lm}) of bi-complex numbers is said to be statistically divergent to ∞ if for any given $G, \delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} > G\}) = 0$. Similarly, statistically divergent to $-\infty$ is defined.*

Definition 1.5. *A double sequence (γ_{lm}) is said to be statistically Cauchy if for every $\varepsilon > 0$, there exists $n = n(\varepsilon)$ and $k = k(\varepsilon)$ such that $\delta_2(\{(l,m) : \|\gamma_{lm} - \gamma_{nk}\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$.*

Definition 1.6. [10] *The set of bi-complex numbers is a commutative ring. Modules over rings are defined in the same way as vector spaces are over fields. A module defined over the bi-complex number ring $\mathbb{BC}$ is known as a $\mathbb{BC}-$ module or simply module.*

Definition 1.7. [6] *A double sequence (γ_{lm}) of bi-complex numbers is called bounded, if there exists a real number $M > 0$ such that*

$$\|\gamma_{lm}\|_{\mathbb{C}_2} \leq M, \text{ for all } l, m \in \mathbb{N}.$$

and the set of all bounded double sequences of bi-complex numbers, defined by;

$${}_2\ell_\infty(\mathbb{C}_2) := \left\{ \gamma = (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : \sup_{l,m\in\mathbb{N}} \|\gamma_{lm}\|_{\mathbb{C}_2} < \infty \right\}.$$

The sequence space ${}_2\ell_\infty(\mathbb{C}_2)$ is a normed linear space with respect to

$$\|A\| = \sup_{l,m} \|\gamma_{lm}\|_{\mathbb{C}_2}.$$

2. Definitions and Preliminaries

In this paper, the notations ${}_2\bar{\ell}_\infty(\mathbb{C}_2), {}_2\bar{c}(\mathbb{C}_2), {}_2\bar{c}_0(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ are used to denote the spaces of bi-complex double sequences that are statistically bounded, statistically convergence in Pringsheim's sense, statistically null in Pringsheim's sense, statistically regularly convergent, statistically regularly null, statistically bounded convergent in Pringsheim's sense, statistically bounded null in Pringsheim's sense, respectively.

$$ {}_2\bar{\ell}_\infty(\mathbb{C}_2) := \{(\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : \exists\, 0 < M \in \mathbb{C}_0 : \delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} \geq M\}) = 0\}; $$

$$ {}_2\bar{c}(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : \text{ there exists } L \in \mathbb{C}_2 \text{ such that } st - \lim_{\substack{l\to\infty \\ m\to\infty}} \gamma_{lm} = L \right\}; $$

$$ {}_2\bar{c}_0(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : st - \lim_{\substack{l\to\infty \\ m\to\infty}} \gamma_{lm} = 0 \right\}; $$

$$ {}_2\bar{c}^R(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\bar{c}(\mathbb{C}_2) : st - \lim_{l\to\infty} \gamma_{lm} = \gamma_m, \text{ exists for each } \right. $$
$$ \left. m \in \mathbb{N} \text{ and } st - \lim_{m\to\infty} \gamma_{lm} = \gamma_l, \text{ exists for each } l \in \mathbb{N} \right\}; $$

$$ {}_2\bar{c}_0^R(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\bar{c}^R(\mathbb{C}_2) : \gamma_m = \gamma_l = L = 0, \text{ for all } l, m \in \mathbb{N} \right\}; $$

$$ {}_2\bar{c}^B(\mathbb{C}_2) := {}_2\bar{c}(\mathbb{C}_2) \cap {}_2\ell_\infty(\mathbb{C}_2) \text{ and } {}_2\bar{c}_0^B(\mathbb{C}_2) = {}_2\bar{c}_0(\mathbb{C}_2) \cap {}_2\ell_\infty(\mathbb{C}_2). $$

Definition 2.1. [7] *Let E be a subset of a linear space X. Then E is said to be convex (or $\mathbb{BC}-$ convex) if $(1-\lambda)(\gamma_{lm}) + \lambda(t_{lm}) \in E$ for all $(\gamma_{lm}), (t_{lm}) \in E$ and scalar $\lambda \in [0,1]$.*

Definition 2.2. [8] *A Banach space X is said to be strictly convex (or $\mathbb{BC}-$ strictly convex) if $(\gamma_{lm}), (t_{lm}) \in S_X$ with $(\gamma_{lm}) \neq (t_{lm})$ implies that $\|\lambda(\gamma_{lm}) + (1-\lambda)(t_{lm})\|_X < 1$, for all $\lambda \in (0,1)$, where S_X is unit sphere.*

Definition 2.3. [9] *A Banach space X is considered uniformly convex (or $\mathbb{BC}-$ uniformly convex) if, for any ε with $0 < \varepsilon \leq 2$, the following inequalities hold true: $\|\gamma_{lm}\|_X \leq 1, \|t_{lm}\|_X \leq 1$ and $\|(\gamma_{lm}) - (t_{lm})\|_X \geq \varepsilon$ imply that there is a $\delta = \delta(\varepsilon) > 0$ such that*

$$ \left\| \frac{(\gamma_{lm}) + (t_{lm})}{2} \right\|_X \leq 1 - \delta. $$

3. Main Result

In this section, the following results are established.

Theorem 3.1. *If a double sequence* (γ_{lm}) *of bi-complex numbers* $\gamma_{lm} = u_{1lm} + i_2 u_{2lm}$ *for all* $l, m \in \mathbb{N}$ *is a statistically bounded double sequence of bi-complex numbers, then the double sequences* (u_{1lm}) *and* (u_{2lm}) *of bi-complex numbers are also statistically bounded double sequences of bi-complex numbers.*

Proof. Let (γ_{lm}) be a statistically bounded double sequence of bi-complex numbers. There exists a positive real number M, such that $\delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} \geq M\}) = 0$, which implies $\delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0$ and $\delta_2(\{(l,m) : \|u_{plm}\|_{\mathbb{C}_2} \geq M\}) \leq \delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0$ for $p = 1, 2$. Hence, (u_{1lm}) and (u_{2lm}) are statistically bounded double sequences of bi-complex numbers.
Conversely, let (u_{1lm}) and (u_{2lm}) are statistically bounded double sequences of bi-complex numbers. Then, without loss of generality, we can find $M > 0$, such that

$$\delta_2(\{(l,m) : \|u_{1lm}\|_{\mathbb{C}_2} \geq M\}) = 0,$$

and

$$\delta_2(\{(l,m) : \|u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0.$$

Consequently, the following inequality yields the result;

$$\begin{aligned} &\delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) \\ &\leq \delta_2(\{(l,m) : \|u_{1lm}\|_{\mathbb{C}_2} \geq M\}) + \delta_2(\{(l,m) : \|u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0. \end{aligned}$$

(By sub-additive property)
Hence, (γ_{lm}) is statistically bounded. □

We formulate the following corollaries based on the previous theorem:

Corollary 3.1. *If a double sequence* (γ_{lm}) *of bi-complex numbers, where* $\gamma_{lm} = x_{1lm} + i_1 x_{2lm} + i_2 x_{3lm} + i_1 i_2 x_{4lm}$, *is statistically bounded double sequence of bi-complex numbers, then the double sequences* (x_{plm}), $p = 1, 2, 3, 4.$ *of bi-complex numbers are also statistically bounded double sequences.*

Corollary 3.2. *If a double sequence* (γ_{lm}) *of bi-complex numbers, where* $\gamma_{lm} = \mu_{1lm} e_1 + \mu_{2lm} e_2$, *is statistically bounded double sequence of bi-complex numbers, then the double sequences* (μ_{1lm}) *and* (μ_{2lm}) *are also statistically bounded double sequences.*

Theorem 3.2. *If a double sequence (γ_{lm}) of bi-complex numbers, where $\gamma_{lm} = u_{1lm} + i_2 u_{2lm}$ for all $l, m \in \mathbb{N}$ is statistically convergent to $\gamma = u_1 + i_2 u_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if and only if (u_{1lm}) and (u_{2lm}) are statistically convergent to u_1 and u_2 respectively.*

Proof. Consider (γ_{lm}) be statistically convergent to γ. Then, by definition, for every $\varepsilon > 0$

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - \gamma\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\implies \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{1lm} - \gamma_1) + i_2(\gamma_{2lm} - \gamma_2)\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\implies \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Now, consider the set

$$A_\varepsilon = \{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Since, $\|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon$ implies $\|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon$,
we have,

$$A_\varepsilon \subseteq \{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}$$

$$\implies \delta_2(A_\varepsilon) \leq \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Similarly, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, the double sequences (γ_{1lm}) and (γ_{2lm}) of bi-complex numbers are statistically convergent to γ_1 and γ_2 respectively. □

Theorem 3.3. *If a bounded double sequence (γ_{lm}), where $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$ is statistically Cauchy, then (γ_{lm}) is a Cauchy double sequence in $\|\cdot\|_{\mathbb{C}_2}$.*

Proof. Let (γ_{lm}) be statistically Cauchy double sequence of bi-complex numbers; then, for each $\varepsilon > 0$, there exists $n_0, k_0 \in \mathbb{N}$, such that

$$\delta_2(\{(l,m) : \|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Substituting $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$, we have

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = \|e_1(\mu_{1lm} - \mu_{1n_0k_0}) + e_2(\mu_{2lm} - \mu_{2n_0k_0})\|_{\mathbb{C}_2}.$$

Using the properties of the Euclidean norm on $\mathbb{C}_2$, then

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = \sqrt{\|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2} + \|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2}}.$$

Since (γ_{lm}) is a statistically Cauchy double sequence of bi-complex numbers, we have;

$$\delta_2(\{(l,m) : \|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2} \geq \varepsilon^1\}) = 0,$$

and

$$\delta_2(\{(l,m) : \|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2} \geq \varepsilon^2\}) = 0,$$

for some $\varepsilon_1, \varepsilon_2 > 0$, such that $\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2$.
This implies that the statistical bounds of $(\|\mu_{1lm} - \mu_{1n_0k_0}\|_{\mathbb{C}_2})$ and $(\|\mu_{2lm} - \mu_{2n_0k_0}\|_{\mathbb{C}_2})$ are zero as $\varepsilon_1, \varepsilon_2 \to 0$.
Hence, for any $\varepsilon > 0$, we have

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = e_1(\|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2}) + e_2(\|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2}) \to 0.$$

Thus, (γ_{lm}) is a Cauchy double sequence of bi-complex numbers in $\|\cdot\|_{\mathbb{C}_2}$. □

Corollary 3.3. *If a double sequence (γ_{lm}) of bi-complex numbers, where $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$ is statistically convergent, then (γ_{lm}) is a Cauchy sequence in $\|\cdot\|_{\mathbb{C}_2}$.*

Theorem 3.4. *Let (γ_{lm}) be a statistically convergent double sequence of bi-complex numbers to L. If $(t_{lm}) \in [(\gamma_{lm})]$, then (t_{lm}) is also statistically convergent to L in $\|\cdot\|_{\mathbb{C}_2}$*

Proof. Since (γ_{lm}) is statistically convergent double sequence of bi-complex numbers to L, by definition, for every $\varepsilon > 0$;

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Given that $(t_{lm}) \in [(\gamma_{lm})]$, we have:

$$\|\gamma_{lm} - t_{lm}\|_{\mathbb{C}_2} = 0, \text{ for all } l, m \in \mathbb{N}.$$

Now,

$$\|t_{lm} - L\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L\|_{\mathbb{C}_2} + \|t_{lm} - \gamma_{lm}\|_{\mathbb{C}_2}.$$

Substituting $\|\gamma_{lm} - t_{lm}\|_{\mathbb{C}_2} = 0$, we get

$$\|t_{lm} - L\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L\|_{\mathbb{C}_2}.$$

Since (γ_{lm}) is statistically convergent to L, for every $\varepsilon > 0$:

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2}\}) = 0.$$

It follows that:

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) \leq \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, the double sequence of bi-complex numbers (t_{lm}) is statistically convergent to L in $\| \cdot \|_{\mathbb{C}_2}$. □

The decomposition theorem for statistically bounded sequences of bi-complex numbers for single sequences was demonstrated by Bera and Tripathy [1].
The decomposition theorem for double sequences of bi-complex numbers is as follows.

Theorem 3.5. *A bounded double sequence (s_{lm}) of bi-complex numbers and a statistically null double sequence (t_{lm}) of bi-complex numbers exist if a double sequence (γ_{lm}) of bi-complex numbers is statistically bounded. This means that $(\gamma_{lm}) = (s_{lm}) + (t_{lm})$.*

Proof. Let (γ_{lm}), where $\gamma_{lm} = \mu_{1lm}e_1 + \mu_{2lm}e_2$, be a statistically bounded double sequence. Then $\delta_2(B) = 0$, where $B = \{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} \geq M\}$.
Define the double sequences (s_{lm}) and (t_{lm}) as follows:

$$s_{lm} = \begin{cases} \gamma_{lm}, \text{ if } k \in B^c; \\ \theta, \text{ otherwise }. \end{cases}$$

$$t_{lm} = \begin{cases} \theta, \text{ if } k \in B^c; \\ \gamma_{lm}, \text{ otherwise }. \end{cases}$$

From the above construction of (s_{lm}) and (t_{lm}), we have

$$(\gamma_{lm}) = (s_{lm}) + (t_{lm}),$$

where $(s_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)$ and $(t_{lm}) \in {}_2\bar{c}_0(\mathbb{C}_2)$. □

We state the following theorem without a proof that can be established by standard techniques.

Theorem 3.6. *Let (γ_{lm}) be a double sequence of bi-complex numbers and $L, L' \in \mathbb{C}_2$. If $st_2 - lim\|\gamma_{lm}\|_{\mathbb{C}_2} = L$. and $st_2 - lim\|\gamma_{lm}\|_{\mathbb{C}_2} = L'$, then $L = L'$.*

Theorem 3.7. *A double sequence (γ_{lm}) of bi-complex numbers is statistically convergent if and only if (γ_{lm}) is statistically Cauchy.*

Proof. Let (γ_{lm}) be statistically convergent to a number $L \in \mathbb{C}_2$. Then for every $\varepsilon > 0$, the set

$$\{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}$$

has double natural density zero. Choose two numbers p and q such that $\|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon$. Now let

$$A = \{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - \gamma_{pq}\|_{\mathbb{C}_2} \geq \varepsilon\},$$

$$B = \{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\},$$

$$C = \{(l,m), l = p \leq n, m = q \leq k : \|\gamma_{pq} - L\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Then $A \subseteq B \cup C$, and therefore $\delta_2(A) \leq \delta_2(B) + \delta_2(C) = 0$. Hence, (γ_{lm}) is statistically Cauchy.

Conversely, assume that (γ_{lm}) is statistically Cauchy but not statistically convergent. This implies that there does not exist a unique $L \in \mathbb{C}_2$ such that $\|\gamma_{lm} - L\|_{\mathbb{C}_2} \to 0$, in the sense of statistical convergence. Instead, there must exist two distinct points $L_1, L_2 \in \mathbb{C}_2$ and some $\varepsilon > 0$, such that the sets

$$B_1 = \{(l,m) : \|\gamma_{lm} - L_1\|_{\mathbb{C}_2} < \varepsilon\} \text{ and } B_2 = \{(l,m) : \|\gamma_{lm} - L_2\|_{\mathbb{C}_2} < \varepsilon\}$$

both have double natural density greater than zero: $\delta_2(B_1) > 0$ and $\delta_2(B_2) > 0$.

Since $L_1 \neq L_2$, the distance between these two points is positive:

$$\|L_1 - L_2\|_{\mathbb{C}_2} = \delta > 0.$$

For $(l,m) \in B_1 \cap B_2$, we have

$$\|\gamma_{lm} - L_1\|_{\mathbb{C}_2} < \varepsilon, \ \|\gamma_{lm} - L_2\|_{\mathbb{C}_2} < \varepsilon.$$

By the triangle inequality

$$\|L_1 - L_2\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L_1\|_{\mathbb{C}_2} + \|\gamma_{lm} - L_2\|_{\mathbb{C}_2}.$$

Substituting the bounds for $\|\gamma_{lm} - L_1\|_{\mathbb{C}_2}$ and $\|\gamma_{lm} - L_2\|_{\mathbb{C}_2}$, we get

$$\|L_1 - L_2\|_{\mathbb{C}_2} < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $L_1 \neq L_2$, their distance $\|L_1 - L_2\|_{\mathbb{C}_2} = \delta > 0$.

Choose $\varepsilon > 0$, such that $2\varepsilon < \delta$.

This creates a contradiction because the inequality $\|L_1 - L_2\|_{\mathbb{C}_2} \leq 2\varepsilon$ can not hold when $2\varepsilon < \delta$.

The assumption that (γ_{lm}) is statistically Cauchy but not statistically convergent leads to a contradiction.

Therefore, if (γ_{lm}) is statistically Cauchy, it must also be statistically convergent to a unique limit $L \in \mathbb{C}_2$. $\square$

Theorem 3.8. *Let (γ_{lm}) and (t_{lm}) be double sequences of bi-complex numbers. If (t_{lm}) is a convergent double sequence such that $\gamma_{lm} \neq t_{lm}$ for all l and m, then (γ_{lm}) is statistically convergent.*

Proof. Suppose that $\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}) = 0$ and $\lim_{l,m\to\infty} \|t_{lm}\|_{\mathbb{C}_2} = L$. Then for every $\varepsilon > 0$,

$$\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\} \subseteq \{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}.$$

Therefore,

$$\begin{aligned} &\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) \\ &\subseteq \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) + \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}). \end{aligned} \tag{3.1}$$

Since, $\lim_{l,m\to\infty} \|t_{lm}\|_{\mathbb{C}_2} = L$, the set $\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}$ contains finite number of integers. Hence,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Using the inequality Eq. (3.1), we get

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. Consequently, $st - \lim_{l,m\to\infty} \|(\gamma_{lm})\|_{\mathbb{C}_2} = L$. $\square$

Corollary 3.4. *Let (γ_{lm}) be a statistically Cauchy sequence. Then there exists a convergent double sequence (t_{lm}) of bi-complex numbers such that $\gamma_{lm} = t_{lm}$, for almost all l and m.*

The following two theorems, Theorems 3.9 and 3.10, are stated without proof, as they can be established using standard techniques.

Theorem 3.9. *Let the double sequences (γ_{lm}) and (t_{lm}) of bi-complex numbers and $L, L' \in \mathbb{C}_2$ and $\alpha \in \mathbb{C}_2 - \mathcal{O}_2$. If $st_2 - \lim \|(\gamma_{lm})\|_{\mathbb{C}_2} = L$ and $st_2 - \lim \|(t_{lm})\|_{\mathbb{C}_2} = L'$. Then*

(1) $st_2 - \lim \|(\gamma_{lm} + t_{lm})\|_{\mathbb{C}_2} = L + L'$
(2) $st_2 - \lim \|\alpha \cdot (\gamma_{lm})\|_{\mathbb{C}_2} = \|\alpha\|_{\mathbb{C}_2} \cdot L$

Theorem 3.10. *A double sequence (γ_{lm}) of bi-complex numbers is statistically convergent to a bi-complex numbers L if and only if there exists a subset $K = \{(n,k) \subset \mathbb{N}\times\mathbb{N} : n,k = 1,2,...\}$ such that $\delta_2(K) = 1$ and*

$$\lim_{n,k} \gamma_{l_n m_k} = L.$$

Theorem 3.11. *If (γ_{lm}), where $\gamma_{lm} = u_{1lm}+i_2u_{2lm}$ is statistically convergent to $\gamma = u_1+i_2u_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if and only if (u_{1lm}) and (u_{2lm}) are statistically convergent to u_1 and u_2 respectively.*

Proof. Consider (γ_{lm}) be statistically convergent to γ. Then for every $\varepsilon > 0$,

$$\begin{aligned} &\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{lm} - \gamma\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0 \\ \Longrightarrow\ &\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{1lm} - \gamma_1) + i_2(\gamma_{2lm} - \gamma_2)\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0 \\ \Longrightarrow\ &\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0. \end{aligned}$$

Now,

$$\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\} \subseteq \{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Thus, we have

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) \leq \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2}+\|\gamma_{2lm}-\gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Similarly, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, the double sequences (γ_{1lm}) and (γ_{2lm}) are statistically convergent to γ_1 and γ_2, respectively.

Conversely, let (u_{1lm}) and (u_{2lm}) be statistically convergent to u_1 and u_2 respectively.

Then, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.\ \&\ \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

We have

$$\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2}\geq\varepsilon\}\cup\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{2lm}-\gamma_2\|_{\mathbb{C}_2}\geq\varepsilon\}$$
$$=\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|(\gamma_{1lm}-\gamma_1)+i_2(\gamma_{2lm}-\gamma_2)\|_{\mathbb{C}_2}\geq\varepsilon\}.$$

Thus,

$$\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|(\gamma_{1lm}-\gamma_1)+i_2(\gamma_{2lm}-\gamma_2)\|_{\mathbb{C}_2}\geq\varepsilon\})$$
$$\leq\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2}\geq\varepsilon\}+\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{2lm}-\gamma_2\|_{\mathbb{C}_2}\geq\varepsilon\})$$
$$\text{(by subadditive property)}$$
$$\implies\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{lm}-\gamma\|_{\mathbb{C}_2}\geq\varepsilon\})=0,\ \text{for every }\varepsilon>0.$$

Hence, (γ_{lm}) is a statistically convergent to γ with respect to the Euclidean norm on $\mathbb{C}_2$. □

We establish the following results based on the apparent proof.

Corollary 3.5. *If the double sequence of bi-complex numbers (γ_{lm}), where $\gamma_{lm}=u_{1lm}+i_2u_{2lm}$ is statistically convergent to $\gamma=u_1+i_2u_2=\mu_1e_1+\mu_2e_2$ with respect to Euclidean norm on $\mathbb{C}_2$ if and only if (μ_{1lm}) and (μ_{2lm}) are statistically convergent to μ_1 and μ_2 respectively.*

Corollary 3.6. *If double sequences (μ_{1lm}) and (μ_{2lm}) are statistically convergent to $L\in\mathbb{C}_2$, then double sequence of bi-complex numbers (γ_{lm}) is statistically convergent to L with respect to Euclidean norm on $\mathbb{C}_2$.*

Theorem 3.12. *Define the function $d_{{}_2\bar{\ell}_\infty(\mathbb{C}_2)}$ by*

$$d_{{}_2\bar{\ell}_\infty(\mathbb{C}_2)}:{}_2\bar{\ell}_\infty(\mathbb{C}_2)\times{}_2\bar{\ell}_\infty(\mathbb{C}_2)\to[0,\infty),(\gamma,t)\to d_{{}_2\bar{\ell}_\infty(\mathbb{C}_2)}(\gamma,t)=\sup_{l,m\in\mathbb{N}}\{\|\gamma_{lm}-t_{lm}\|_{\mathbb{C}_2}\},$$

where $\gamma=(\gamma_{lm}),t=(t_{lm})\in{}_2\bar{\ell}_\infty(\mathbb{C}_2)$. Then $({}_2\bar{\ell}_\infty(\mathbb{C}_2),d_{{}_2\bar{\ell}_\infty(\mathbb{C}_2)})$ is a complete metric space.

Proof. The proof is trivial from Theorem 9 [7]. □

Remark 3.1. *If (γ_{lm}) be statistically convergent to $L\in\mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$, then*

1. *(γ^*_{lm}) is statistically convergent to γ^* with respect to Euclidean norm on $\mathbb{C}_2$ and converse is also true.*
2. *$(\widetilde{\gamma_{lm}})$ is statistically convergent to $\widetilde{\gamma}$ with respect to Euclidean norm on $\mathbb{C}_2$ and converse is also true.*

(3) (γ'_{lm}) *is statistically convergent to* γ' *with respect to Euclidean norm on* $\mathbb{C}_2$ *and converse is also true.*

Remark 3.2. *If* (γ_{lm}) *be statistically convergent with respect to the Euclidean norm on* $\mathbb{C}_2$*, then*

(1) $(|\gamma_{lm}|^2_{i_1}) = (\gamma_{lm} \cdot \widetilde{\gamma_{lm}})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

(2) $(|\gamma_{lm}|^2_{i_2}) = (\gamma_{lm} \cdot \gamma^*_{lm})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

(3) $(|\gamma_{lm}|^2_{i_1 i_2}) = (\gamma_{lm} \cdot \gamma'_{lm})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

Theorem 3.13. *The sets* ${}_2\bar{\ell}_\infty(\mathbb{C}_2), {}_2\bar{c}(\mathbb{C}_2), {}_2\bar{c}_0(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}^R_0(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2),\ {}_2\bar{c}^B_0(\mathbb{C}_2)$ *are* $\mathbb{BC}$*-module.*

Proof. We prove that the set ${}_2\bar{\ell}_\infty(\mathbb{C}_2)$ is a $\mathbb{BC}$-module. The proofs for the other sets follow analogously based on their respective definitions.

Let, $(\gamma_{lm}), (t_{lm}) \in {}_2\bar{\ell}_\infty(\mathbb{C}_2)$. By definition of vector addition,

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0,$$

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Consider the sum $(\gamma_{lm}) + (t_{lm})$, using the triangle inequality for the norm $\|\cdot\|_{\mathbb{C}_2}$, we have;

$$\|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \leq \|(\gamma_{lm})\|_{\mathbb{C}_2} + \|(t_{lm})\|_{\mathbb{C}_2}.$$

Now, analyze the density condition for $(\gamma_{lm}) + (t_{lm})$;

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

By subadditivity of the density measure, this is bounded by

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| + \lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

Since, both terms on the right-hand side are zero by assumption, we conclude.

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Thus, $(\gamma_{lm}) + (t_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, showing closure under addition Let $a \in \mathbb{C}_2$ and $(\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. By definition of scalar multiplication,

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

For the scalar product $a \cdot (\gamma_{lm})$, using the property of the norm

$$\|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} = |a|_{\mathbb{C}_2} \cdot \|(\gamma_{lm})\|_{\mathbb{C}_2},$$

where $|a|_{\mathbb{C}_2}$ is the modulus of a in $\mathbb{C}_2$.

Now analyze the density condition for $a \cdot (\gamma_{lm})$;

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

This is equivalent to;

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq \frac{M}{|a|_{\mathbb{C}_2}}\}|.$$

Since, the right-hand side is zero by the assumption that $(\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, we conclude

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Thus, $a \cdot (\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, showing closure under scalar multiplication. Since ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ satisfies closure under addition and scalar multiplication, it is a $\mathbb{BC}$-module.

Similarly, using analogous arguments, the other sets can be shown to be $\mathbb{BC}$-modules. □

Theorem 3.14. *The classes of the double sequences* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are* $\mathbb{BC}$*-convex.*

Proof. We first prove the $\mathbb{BC}$-convexity for ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$. The other classes can be established similarly. Let $(\gamma_{lm}), (t_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. Then there exist constants $M_1, M_2 > 0$ such that

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1\}| = 0,$$

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M_2\}| = 0.$$

Define $M = max\{M_1, M_2\}$. For $0 \leq \lambda \leq 1$, consider the convex combination

$$(\delta_{lm}) = \lambda(\gamma_{lm}) + (1-\lambda)(t_{lm}).$$

Using the triangle inequality, we have

$$\|(\delta_{lm})\|_{\mathbb{C}_2} \leq \lambda\|(\gamma_{lm})\|_{\mathbb{C}_2} + (1-\lambda)\|(t_{lm})\|_{\mathbb{C}_2}.$$

For $(l,m) \in \mathbb{N}\times\mathbb{N}$ such that $\|(\delta_{lm})\|_{\mathbb{C}_2} \geq M$, at least one of $\|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1$ or $\|(t_{lm})\|_{\mathbb{C}_2} \geq M_2$, must hold.
Thus,

$$|\{(l,m) : \|(\delta_{lm})\|_{\mathbb{C}_2} \geq M\}| \leq |\{(l,m) : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1\}| + |\{(l,m) : \|(t_{lm})\|_{\mathbb{C}_2} \geq M_2\}|.$$

Dividing by lm and taking the limit as $l,m \to \infty$.

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) : \|(\delta_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Hence, $(\delta_{lm}) \in {}_2\bar{\ell}_\infty(\mathbb{C}_2)$. Proving ${}_2\bar{\ell}_\infty(\mathbb{C}_2)$ is $\mathbb{BC}$-convex. Similarly, the other cases can be established. □

Remark 3.3. *The classes of the double sequences* ${}_2\bar{\ell}_\infty(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are not* $\mathbb{BC}$*-strictly convex.*

This follows from the following example for the case ${}_2\bar{\ell}_\infty(\mathbb{C}_2)$. The other classes can be established similarly.

Example 3.1. *Let the double sequences* (γ_{lm}) & (t_{lm}) *of bi-complex numbers defined by*

$$(\gamma_{lm}) = \begin{pmatrix} (\frac{1}{2}-\frac{\sqrt{3}}{2}i_1)e_1+(\frac{1}{2}+\frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_2 & (\frac{1}{3}+\frac{2\sqrt{2}}{3}i_1)e_1+(\frac{1}{3}-\frac{2\sqrt{2}}{3}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$(t_{lm}) = \begin{pmatrix} \theta & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_2 & (\frac{1}{3}+\frac{2\sqrt{2}}{3}i_1)e_1+(\frac{1}{3}-\frac{2\sqrt{2}}{3}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, $\|(\gamma_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = \|(t_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = 1$ *and*

$$\left\|\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)(\gamma_{lm})+\left\{1-\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\right\}(t_{lm})\right\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)}$$

$$=\sup_{l,m\in\mathbb{N}}\left\|\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\gamma_{lm}+\left\{1-\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\right\}t_{lm}\right\|_{\mathbb{C}_2}$$

$$=\sup_{l,m\in\mathbb{N}}\Bigg[\Bigg\|(\frac{1}{4}-\frac{\sqrt{3}}{4}i_1)e_1+(\frac{1}{4}+\frac{\sqrt{3}}{4}i_1)e_2,(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_2,(\frac{1}{6}+\frac{2\sqrt{2}}{6}i_1)e_1+(\frac{1}{6}-\frac{2\sqrt{2}}{6}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,$$

$$+\Bigg\{\theta,(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_2,(\frac{1}{3}+\frac{2\sqrt{2}}{3}i_1)e_1+(\frac{1}{3}-\frac{2\sqrt{2}}{3}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,$$

$$-\Bigg(\theta,(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_2,(\frac{1}{6}+\frac{2\sqrt{2}}{6}i_1)e_1+(\frac{1}{6}-\frac{2\sqrt{2}}{6}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,\Bigg)\Bigg\}\Bigg\|_k\Bigg]$$

$= \sup_{l,m\in\mathbb{N}}\left\{\frac{1}{2},1,\theta\right\} = 1.$ *for* $\lambda = (\frac{1}{2}e_1+\frac{1}{2}e_2)\in\mathbb{C}_2$.
Hence, ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ *is not* $\mathbb{BC}$*-strictly convex.*

Remark 3.4. *The classes of the double sequences* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are not* $\mathbb{BC}$*-uniformly convex.*

This follows from the following Example.

Example 3.2. *Let the double sequences* (γ_{lm}) & (t_{lm}) *of bi-complex numbers defined by*

$$(\gamma_{lm}) = \begin{pmatrix} (\frac{1}{2}-\frac{\sqrt{3}}{2}i_1)e_1+(\frac{1}{2}+\frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$(t_{lm}) = \begin{pmatrix} (-\frac{1}{2}+\frac{\sqrt{3}}{2}i_1)e_1+(-\frac{1}{2}-\frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\|(\gamma_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = \|(t_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = 1$ *and*

$$\begin{aligned} \|(\gamma_{lm})-(t_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} &= \sup_{l,m\in\mathbb{N}}\{\|\gamma_{lm}-t_{lm}\|_{\mathbb{C}_2} : l,m\in\mathbb{N}\} \\ &= \sup_{l,m\in\mathbb{N}}\left\{\left\|(\frac{2}{2}-\frac{2\sqrt{3}}{2}i_1)e_1+(\frac{2}{2}+\frac{2\sqrt{3}}{2}i_1)e_2\right\|_{\mathbb{C}_2}\right\} \\ &= 2. \end{aligned}$$

and $\varepsilon \le \|(\gamma_{lm})-(t_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = 2.$

On the other hand,

$$\begin{aligned} \left\|\frac{(\gamma_{lm})+(t_{lm})}{2}\right\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} &= \sup_{l,m\in\mathbb{N}}\left\|\frac{\gamma_{lm}+t_{lm}}{2}\right\|_{\mathbb{C}_2} \\ &= \sup_{l,m\in\mathbb{N}}\left\{\left\|\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_1+\left(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_2\right\|_{\mathbb{C}_2},\right. \\ &\left\|\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_1+\left(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_2\right\|_{\mathbb{C}_2}, \\ &\left.\left\|\left(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1\right)e_1+\left(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1\right)e_2\right\|_{\mathbb{C}_2}\right\} \\ &= 1. \end{aligned}$$

Thus, there does not exist $\delta(\varepsilon)$ *such that*

$$\left\|\frac{(\gamma_{lm})+(t_{lm})}{2}\right\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} \le 1-\delta.$$

Therefore, we have ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ *is not* $\mathbb{BC}$*-uniformly convex.*

Similarly, other classes can also be proved.

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