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POINTWISE BI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE NEARLY KÄHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold and provide a characterization theorem for the existence of these submanifolds. Following this, we provide a non-trivial example of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and then derive some conditions for the distributions associated with this class of submanifolds to be involutive. Further, we provide a characterization for a pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold to be a bi-slant lightlike submanifold and investigate the geometry of totally umbilical pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold. Finally, we obtain necessary and sufficient conditions for foliations determined by distributions on pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold to be totally geodesic.

Keywords: r-lightlike submanifold, Metric connection, Slant distribution, Bi-slant lightlike submanifold.

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1. Introduction

Chen [4, 5] introduced the notion of slant submanifolds of Kähler manifolds as a generalization of holomorphic and totally real submanifolds. Following this, Lotta [15, 16] introduced and studied concept of slant submanifolds in contact manifolds. Further, Carbrerizo et al.

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[1] studied slant submanifolds in Sasakian manifolds. Afterwards, several generalizations of slant submanifolds were introduced and studied by Carriazo [2, 3], Sahin [19] and Papaghuic [18]. Etayo [9] generalized the notion of slant submanifolds to quasi- slant submanifolds of Kähler manifolds. On a similar note, Chen and Garay [6] generalized the notion of slant submanifolds to pointwise slant submanifolds of a Kähler manifold.

Due to interesting applications in study of asymptotically flat spacetimes, even horizon of Kerr and Kruskal black holes, electromagnetic fields, focus of geometers shifted towards the study of geometry of manifolds and submanifolds endowed with indefinite metric. The theory of lightlike submanifolds of semi-Riemannian manifolds was introduced by Bejancu and Duggal [8] which differs from it's non-degenerate counterpart due to non-trivial intersection of tangent and normal bundle. Further, Sahin [20, 22] introduced notion of slant and screen slant lightlike submanifolds of Kähler manifolds. Following this, several generalizations of slant and screen slant submanifolds of indefinite Kähler manifolds were introduced and studied by Shukla et al. [23, 24]. Moreover, slant and screen slant lightlike submanifolds in framework of Contact and indefinite nearly Kähler manifolds were studied as in [21, 12, 14]. Gupta et al. [11] studied pointwise slant lightlike submanifolds of indefinite Kähler manifolds. Further, Kumar et al. [13, 17] studied the theory of screen bi-slant and pointwise bi-slant lightlike submanifolds of indefinite Kähler manifolds. However, the concept of pointwise bi-slant lightlike submanifolds is yet to be explored in indefinite nearly Kähler manifolds.

Therefore, in this paper, we introduce the notion of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds. Then, we give a characterization theorem for the existence of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and provide a non-trivial example of this class of lightlike submanifolds. We further derive integrability conditions for the distributions associated with these submanifolds and give some conditions for a pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold to be a bi-slant lightlike submanifold. Finally, we investigate the geometry of totally umbilical pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and obtain necessary and sufficient conditions for foliations determined by distributions on pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold to be totally geodesic.

2. Preliminaries

Definition 2.1. *Let (N, g) be $m-$dimensional submanifold of semi-Riemannian manifold $(\bar{N}, \bar{g})$ of dimension $(m+n)$ equipped with metric $\bar{g}$ of index $q(\neq 0)$, where, $m, n \geq 1$ and $m+n-1 \geq q \geq 1$. We assume that metric $\bar{g}$ on TN is degenerate, then, metric $\bar{g}$ is degenerate on $TN^{\perp}$ which gives rise to a distribution $Rad(TN) : p \in N \to Rad(T_pN)$ given by $Rad(T_pN) = T_pN \ \cap \ T_pN^{\perp}$. We call N as $r-$lightlike submanifold if $Rad(TN)$ is a smooth distribution of rank $r > 0$ $(1 \leq r \leq n\ )$ on N.*

Let $S(TN)$ and $S(TN^{\perp})$ be non-degenerate subbundles of $Rad(TN)$ in TN and $TN^{\perp}$ respectively such that $TN = Rad(TN) \perp S(TN)$ and $TN^{\perp} = Rad(TN) \perp S(TN^{\perp})$. Moreover, for local coordinate neighbourhood U of N and local frame field $\{\xi_i\}$, $\{i \in 1, 2, ..., r\}$ of $\Gamma(Rad(TN))$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TN^{\perp})$ i.e, $S(TN^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for } i, j \in \{1, 2, ..., r\}. \tag{2.1}$$

In view of Theorem (1.3), Chapter 5 (see, [8]), there exists a lightlike transversal vector bundle $ltr(TN)$ complementary to $Rad(TN)$ in $S(TN^{\perp})^{\perp}$ locally spanned by $\{N_i\}$. Next, consider the vector bundle $tr(TN)$ in $T\bar{N}|_N$ defined by

$$tr(TN) = ltr(TN) \perp S(TN^{\perp}),$$

and therefore

$$T\bar{N}|_N = TN \oplus tr(TN) = S(TN) \perp (Rad(TN) \oplus ltr(TN)) \perp S(TN^{\perp}). \tag{2.2}$$

Let $\bar{\nabla}$ be Levi-Civita connection of $\bar{N}$. Then, for $Z_1, Z_2 \in \Gamma(TN)$ and $V \in \Gamma(tr(TN))$, Gauss and Weingarten formulae are given by

$$\overline{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + h(Z_1, Z_2), \ \overline{\nabla}_{Z_1} V = -A_V Z_1 + \nabla^t_{Z_1} V, \tag{2.3}$$

where $\{h(Z_1, Z_2), \nabla^t_{Z_1} V\} \in \Gamma(tr(TN))$, $\{\nabla_{Z_1} Z_2, A_V Z_1\} \in \Gamma(TN)$ and h, A_V represent second fundamental form on $\Gamma(TN)$ and linear shape operator on N respectively. In view of Eq. (2.2), we give the the Gauss and Weingarten formulae as

$$\overline{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + h^l(Z_1, Z_2) + h^s(Z_1, Z_2), \tag{2.4}$$

$$\overline{\nabla}_{Z_1} V = -A_V Z_1 + \nabla^l_{Z_1} LV + \nabla^s_{Z_1} SV + D^l(Z_1, SV) + D^s(Z_1, LV) \tag{2.5}$$

where, $Z_1, Z_2 \in \Gamma(TN)$, $V \in \Gamma(tr(TN))$, h^l and h^s are $\Gamma(ltr(TN))$ and $\Gamma(S(TN^{\perp}))$ valued lightlike second fundamental form and screen second fundamental form of N, ∇^l and ∇^s are lightlike and screen transversal linear connections on N respectively and $D^l : \Gamma(TN) \times \Gamma(S(TN^{\perp})) \to \Gamma(ltr(TN))$, $D^s : \Gamma(TN) \times \Gamma(ltr(TN)) \to \Gamma(S(TN^{\perp}))$ respectively are bi-linear mappings, where L and S are projection morphisms onto $ltr(TN)$ and $S(TM^{\perp})$. In particular, if $N \in \Gamma(ltr(TN))$ and $W \in \Gamma(S(TN^{\perp}))$, then, from Eq.(2.5), we have

$$\overline{\nabla}_{Z_1} N = -A_N Z_1 + \nabla^l_{Z_1} N + D^s(Z_1, N) \tag{2.6}$$

and

$$\overline{\nabla}_{Z_1} W = -A_W Z_1 + D^l(Z_1, W) + \nabla^s_{Z_1} W \tag{2.7}$$

From Eqs. (2.4), (2.6) and (2.7), we obtain

$$g(A_W Z_1, Z_2) = \bar{g}(h^s(Z_1, Z_2), W) + \bar{g}(Z_2, D^l(Z_1, W)), \tag{2.8}$$

$$g(A_W Z_1, N) = \bar{g}(D^s(Z_1, N), W). \tag{2.9}$$

Let Q be the projection of TN onto screen distribution $S(TN)$, then using $TN = Rad(TN) \perp S(TN)$, we get

$$\nabla_{Z_1} QZ_2 = \nabla^*_{Z_1} QZ_2 + h^*(Z_1, QZ_2),\ \nabla_{Z_1}\xi = -A^*_{\xi} Z_1 + \nabla^{*t}_{Z_1}\xi, \tag{2.10}$$

where $\xi \in \Gamma(Rad(TN))$, $\{h^*(Z_1, QZ_2), \nabla^{*t}_{Z_1}\xi\} \in \Gamma(Rad(TN))$ and $\{\nabla^*_{Z_1} QZ_2, A^*_{\xi} Z_1\} \in \Gamma(S(TN))$. Also, $h^* : \Gamma(TN) \times \Gamma(S(TN) \to \Gamma(Rad(TN))$ and $A^* : \Gamma(TN) \times \Gamma(Rad(TN) \to \Gamma(S(TN))$ are bilinear forms called second fundamental form and shape operator of distributions $S(TN)$ and $Rad(TN)$ respectively. Moreover, ∇^* and ∇^{*t} denote the induced Levi-Civita connection on $S(TN)$ and $Rad(TN)$ respectively. Then, from Eqs. (2.5), (2.6) and (2.10), we get

$$\bar{g}(h^l(Z_1, QZ_2), \xi) = g(A^*_{\xi} Z_1, QZ_2). \tag{2.11}$$

As $\bar{\nabla}$ is a metric connection on N, therefore for $Z_1, Z_2, Z_3 \in \Gamma(TN)$, one has

$$(\nabla_{Z_1} g)(Z_2, Z_3) = \bar{g}(h^l(Z_1, Z_2), Z_3) + \bar{g}(h^l(Z_1, Z_3), Z_2). \tag{2.12}$$

which shows that the induced connection ∇ on N is not a metric connection.

Definition 2.2. [10] *An indefinite almost Hermitian manifold $(\bar{N}, \bar{J}, \bar{g}, \bar{\nabla})$ is said to be an indefinite nearly Kähler manifold if*

$$\bar{J}^2 = -I,\ \bar{g}(\bar{J}Z_1, \bar{J}Z_2) = \bar{g}(Z_1, Z_2),\ (\overline{\nabla}_{Z_1}\bar{J})Z_2 + (\overline{\nabla}_{Z_2}\bar{J})Z_1 = 0, \tag{2.13}$$

$\forall Z_1, Z_2 \in \Gamma(T\bar{N})$.

3. Pointwise bi-slant Lightlike Submanifolds

In view of Lemmas (3.1) and (3.2) stated by Sahin [20], we introduce the concept of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds as follows:

Definition 3.1. *A q-lightlike submanifold N of an indefinite nearly Kähler manifold $\bar{N}$ with index $2q$ is said to be a pointwise bi-slant lightlike submanifold if the following conditions hold:*

(i) $\bar{J}(Rad(TN))$ *is a distribution on N such that* $\bar{J}(Rad(TN)) \cap Rad(TN)=\{0\}$.

(ii) *There exists non-degenerate orthogonal distributions D_1 and D_2 on N such that* $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.

(iii) $\bar{J}D_1 \perp D_2$ *and* $\bar{J}D_2 \perp D_1$.

(iv) *For $p \in U \subset N$ and each non-zero tangent vector field $Z \in \Gamma(D_j)_p$ (for $j = 1, 2$), the angle $(\theta_j)_p$ between $\bar{J}Z$ and the vector space $(D_j)_p$ is independent of choice of $Z \in \Gamma(D_j)_p$.*

Note that the angle θ_j is called the slant function on N and the pair $\{\theta_1, \theta_2\}$ is called bi-slant function on N. At each point $p \in U \subset N$, $(\theta_j)_p$ (for $j = 1, 2$) is called the slant angle of the distribution $(D_j)_p$. Moreover, if for $j = 1, 2$, $(D_j)_p \neq \{0\}$ and $(\theta_j)_p \neq 0, \pi/2$, then, the pointwise bi-slant lightlike submanifold is said to be proper.

In view of above definition, the tangent bundle TN of N can be decompsed as:

$$TN = Rad(TN) \perp (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2. \tag{3.14}$$

For $Z \in \Gamma(TN)$, we have,

$$\bar{J}Z = tZ + nZ \tag{3.15}$$

and for $V \in \Gamma(tr(TN))$

$$\bar{J}V = BV + CV, \tag{3.16}$$

where $tZ, BV \in \Gamma(TN)$ and $nZ, CV \in \Gamma(tr(TN))$.

Note: In upcoming sections, we will use **pw.bi-s.l.s.** to denote a pointwise bi-slant lightlike submanifold and an indefinite nearly Kähler manifolds will be denoted by $\bar{N}$, unless otherwise stated.

Consider ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 be the projections of TN on $Rad(TN)$, $\bar{J}(Rad(TN))$, $\bar{J}(ltr(TN))$, D_1 and D_2, respectively. Then, for $Z \in \Gamma(TN)$, we have

$$Z = \phi_1 Z + \phi_2 Z + \phi_3 Z + \phi_4 Z + \phi_5 Z. \tag{3.17}$$

Applying $\bar{J}$ on both sides and using Eq. (3.15), we get

$$\bar{J}Z = \bar{J}\phi_1 Z + \bar{J}\phi_2 Z + \bar{J}\phi_3 Z + t\phi_4 Z + n\phi_4 Z + t\phi_5 Z + n\phi_5 Z, \tag{3.18}$$

where $\bar{J}\phi_1 Z \in \Gamma(\bar{J}(Rad(TN)))$, $\bar{J}\phi_2 Z \in \Gamma(Rad(TN))$ and $\bar{J}\phi_3 Z \in \Gamma(ltr(TN))$.

Lemma 3.1. *For a* ***pw.bi-s.l.s.*** *N of $\bar{N}$, $\{n\phi_4 Z, n\phi_5 Z\} \in \Gamma(S(TN^{\perp}))$, $t\phi_4 Z \in \Gamma(D_1)$ and $t\phi_5 Z \in \Gamma(D_2)$, for $Z \in \Gamma(TN)$.*

Proof. For $\xi \in \Gamma(Rad(TN))$, we have

$$\bar{g}(n\phi_i Z, \xi) = -\bar{g}(\phi_i Z, \bar{J}\xi) = 0 \tag{3.19}$$

for $i = 4, 5$. Therefore, $n\phi_i Z$ has no component in $ltr(TN)$, which implies $\{n\phi_4 Z, n\phi_5 Z\} \in (S(TN^{\perp}))$. On the other hand, Let $N \in \Gamma(ltr(TN))$, then using Eqs.(3.19), (2.13), (3.14) and using the condition $\bar{J}D_2 \perp D_1$, we have
$\bar{g}(t\phi_4 Z, N) = 0 = \bar{g}(t\phi_4 Z, \bar{J}\phi_1 Z) = \bar{g}(t\phi_4 Z, \bar{J}\phi_2 Z) = \bar{g}(t\phi_4 Z, \bar{J}\phi_3 Z) = \bar{g}(t\phi_4 Z, \phi_5 Z)$, which shows that $t\phi_4 Z \in \Gamma(D_1)$. Similarly, using Eqs.(3.19), (2.13), (3.14) and using the condition $\bar{J}D_1 \perp D_2$, it follows that $t\phi_5 Z \in \Gamma(D_2)$.

□

We now provide a classification theorem for the existence of **pw.bi-s.l.s.** N of $\bar{N}$.

Theorem 3.1. *(Existence Theorem) A q-lightlike submanifold N of $\bar{N}$ with index $2q$ is a* ***pw.bi-s.l.s.****, if and only if,*

(i) *$\bar{J}(ltrTN)$ is a distribution on N such that $\bar{J}(ltrTN) \cap Rad(TN) = \{0\}$.*
(ii) *There exists non-degenerate orthogonal distributions D_1 and D_2 on N such that $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.*
(iii) *$\bar{J}D_1 \perp D_2$ and $\bar{J}D_2 \perp D_1$.*
(iv) *for $\{i = 4, 5\}$, there exist functions $\alpha_i \in [0, 1)$ such that $t^2(\phi_i Z) = -\alpha_i(\phi_i Z)$ for all $Z \in \Gamma(S(TN))$, where $\cos^2(\theta_j)_p = \alpha_i$ such that $(\theta_j)_p$ are the respective slant functions of $(D_j)_p$ for $j = i - 3$ and $p \in N$.*

Proof. Let N be a **pw.bi-s.l.s.** of $\bar{N}$. Then, the conditions (ii) and (iii) hold trivially. Let $\bar{J}N' \in \Gamma(Rad(TN))$ for $N' \in \Gamma ltr(TN)$, one has $\bar{J}\bar{J}N' = -N' \in \Gamma(S(TN))$, which is a contradiction. Therefore, we get $\bar{J}N' \notin \Gamma(Rad(TN))$. Again, let $\bar{J}N' \in \Gamma(ltr(TN))$ for $N' \in \Gamma(ltr(TN))$. Choose $\xi \in \Gamma(Rad(TN))$ such that $\bar{g}(N', \xi) = 1$. Then from Eq. (2.13), we derive $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is again a contradiction to our hypothesis. Therefore, $\bar{J}N' \notin \Gamma(ltr(TN))$. Consider $\bar{J}N' \in \Gamma(S(TN^{\perp}))$ for $N' \in \Gamma(ltr(TN))$. Choose $\xi \in \Gamma(Rad(TN))$ such that $\bar{g}(N', \xi) = 1$, then, using Eq. (2.13), we have $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is a contradiction to hypothesis. Therefore, $\bar{J}N' \notin \Gamma(S(TN^{\perp}))$. Thus, we conclude that $\bar{J}(ltr(TN)) \subset S(TN)$ and $\bar{J}(ltr(TN)) \cap Rad(TN)=\{0\}$, which proves condition (i).

As N is **pw.bi-s.l.s.** of $\bar{N}$, the angle between $\bar{J}\phi_i Z$ and D_j at p is constant, therefore for $Z \in \Gamma(S(TN))$ and $p \in N$, we have

$$\cos(\theta_j)_p = \frac{\bar{g}(\bar{J}\phi_i Z, t\phi_i Z)}{|\bar{J}\phi_i Z|\ |t\phi_i Z|} = -\frac{\bar{g}(\phi_i Z, \bar{J}t\phi_i Z)}{|\phi_i Z|\ |t\phi_i Z|} = -\frac{\bar{g}(\phi_i Z, t^2(\phi_i Z))}{|\phi_i Z|\ |t\phi_i Z|}.$$

Since, $\cos(\theta_j)_p=\frac{|t\phi_i Z|}{|\bar{J}(\phi_i Z)|}$, therefore we have

$$\cos^2(\theta_j)_p = -\frac{\bar{g}(\phi_i Z, t^2(\phi_i Z))}{|\phi_i Z|^2}. \tag{3.20}$$

We know that $(\theta_j)_p$ is constant on $(D_j)_p$. Hence, we get

$$\bar{g}(\phi_i Z, t^2\phi_i Z) = -\alpha_i \bar{g}(\phi_i Z, \phi_i Z),$$

which gives $t^2\phi_i Z = -\alpha_i \phi_i Z$ as $g = \bar{g}|_{(D_j)_p \times (D_j)_p}$ is non-degenerate. Hence, (iv) holds.

Conversely, suppose that N be a q-lightlike submanifold of $\bar{N}$ such that the conditions $(i)-(iv)$ are satisfied. Let $\bar{J}\xi \in \Gamma(ltr(TN))$ for $\xi \in \Gamma(Rad(TN))$, one has $\bar{J}\bar{J}\xi = -\xi \in \Gamma(S(TN))$ by condition (i), which is a contradiction. Therefore, we get $\bar{J}\xi \notin \Gamma(ltr(TN))$. Again, let $\bar{J}\xi \in \Gamma(S(TN^{\perp}))$ for $\xi \in \Gamma(Rad(TN))$. Choose $N' \in \Gamma(ltr(TN))$ such that $\bar{g}(N', \xi) = 1$. Then from conditions (i), (ii) and Eq. (2.13), we derive $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is again a contradiction to our hypothesis. Therefore, $\bar{J}\xi \notin \Gamma(S(TN^{\perp}))$. Now, Consider $\bar{J}\xi \in \Gamma(Rad(TN))$ for $\xi \in \Gamma(Rad(TN))$. Choose $N' \in \Gamma(ltr(TN))$ such that $\bar{g}(N', \xi) = 1$, then, using condition (i) and Eq. (2.13), we have $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is a contradiction to hypothesis. Therefore, $\bar{J}\xi \notin \Gamma(Rad(TN))$. Thus, we conclude that $\bar{J}(Rad(TN)) \subset S(TN)$ and $\bar{J}(Rad(TN)) \cap Rad(TN)=\{0\}$. Also, by condition (iv), there exists a function α_i such that $t^2\phi_i Z = -\alpha_i \phi_i Z$ for $Z \in \Gamma(S(TN))$. Then using the Eqs.

(2.13) and (3.20), we obtain

$$\cos^2(\theta_j)_p = \frac{g(t(\phi_i Z), t(\phi_i Z))}{g(\phi_i Z, \phi_i Z)}$$
$$= \alpha_i,$$

which shows that the Wirtinger angle is independent of $\phi_i Z \in (D_j)p$. Hence, the theorem is proved. □

Corollary 3.1. *Assume that N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, for $i = 4, 5$ and $j = i - 3$,*

(i) $g(t\phi_i Z_1, t\phi_i Z_2) = \cos^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$,

(ii) $\overline{g}(n\phi_i Z_1, n\phi_i Z_2) = \sin^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$,

where $Z_1, Z_2 \in \Gamma(TN)$.

Proof. Let $Z_1, Z_2 \in \Gamma(TN)$, then we have

$$g(t\phi_i Z_1, t\phi_i Z_2) = \overline{g}(t\phi_i Z_1, t\phi_i Z_2) = -\overline{g}(\phi_i Z_1, t^2\phi_i Z_2) = -\overline{g}(\phi_i Z_1, -\alpha_i\phi_i Z_2) = \alpha_i g(\phi_i Z_1, \phi_i Z_2),$$

which leads to $g(t\phi_i Z_1, t\phi_i Z_2) = \cos^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$. Similarly, consider

$$g(\phi_i Z_1, \phi_i Z_2) = \overline{g}(\bar{J}\phi_i Z_1, \bar{J}\phi_i Z_2) = g(t\phi_i Z_1, t\phi_i Z_2) + \overline{g}(n\phi_i Z_1, n\phi_i Z_2),$$

which gives $\overline{g}(n\phi_i Z_1, n\phi_i Z_2) = \sin^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$, thus the proof is complete. □

Next, we present a non-trivial example of **pw.bi-s.l.s.** N of an indefinite nearly Kähler manifold $\bar{N}$.

Example 3.1. *Consider N be a 6-dimensional submanifold of $(R^{16}_3, \bar{g})$ with signature $(-,-,-,+,+,+,+,+,+,+,+,+,+,+,+,+)$ given by*

$$x^1 = u^1 = x^4, \quad x^2 = u^2 = -x^3, \quad x^5 = u^3, \quad x^6 = u^3, \quad x^7 = u^4, \quad x^8 = u^4,$$
$$x^9 = u^3u^4, \quad x^{10} = \frac{(u^3)^2}{2} + \frac{(u^4)^2}{2}, \quad x^{11} = u^5, \quad x^{12} = u^5, \quad x^{13} = u^6,$$
$$x^{14} = u^6, \quad x^{15} = u^5u^6, \quad x^{16} = \frac{(u^5)^2}{2} + \frac{(u^6)^2}{2}, \quad u^3 \neq \pm u^4, \quad u^5 \neq \pm u^6.$$

Then TN is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$, where

$$Z_1 = \partial x_1 + \partial x_4, \quad Z_2 = \partial x_2 - \partial x_3,$$

$$Z_3 = \partial x_5 + \partial x_6 + u^4\partial x_9 + u^3\partial x_{10}, \quad Z_4 = \partial x_7 + \partial x_8 + u^3\partial x_9 + u^4\partial x_{10},$$

$$Z_5 = \partial x_{11} + \partial x_{12} + u^6\partial x_{15} + u^5\partial x_{16}, \quad Z_6 = \partial x_{13} + \partial x_{14} + u^5\partial x_{15} + u^6\partial x_{16}.$$

As $Rad(TN) = Span\{Z_1\}$ and $\bar{J}Rad(TN) = Span\{Z_2\}$, where $\bar{J}Z_1 = Z_2$, thus N is a 1-lightlike submanifold with $ltr(TN)$ spanned by $N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_4\}$ and $\bar{J}ltr(TN)$ spanned by $\bar{J}N_1 = \frac{1}{2}\{-\partial x_2 - \partial x_3\}$. On the other hand, by direct calculations, we find that $S(TN^{\perp})$ is spanned by

$$W_1 = -u^3\partial x_5 - u^4\partial x_8 + \partial x_{10}, \quad W_2 = -u^5\partial x_{11} - u^6\partial x_{14} + \partial x_{16}.$$

*Hence, $D_1 = Span\{Z_3, Z_4\}$ and $D_2 = Span\{Z_5, Z_6\}$ are slant distributions with the slant angles $\theta_1 = \frac{(u^4)^2-(u^3)^2}{2+(u^3)^2+(u^4)^2}$ and $\theta_2 = \frac{(u^6)^2-(u^5)^2}{2+(u^5)^2+(u^6)^2}$. Thus, N is a proper **pw.bi-s.l.s.** of R_3^{16}.*

Lemma 3.2. *For a **pw.bi-s.l.s.** N of $\bar{N}$, nD_1 and nD_2 are orthogonal.*

Proof. Since N is **pw.bi-s.l.s.** of $\bar{N}$, therefore using Eq.(3.14) and Eq.(3.16) along with theorem (3.1) for $Z \in \Gamma(TN)$, we have

$$\begin{aligned}
\overline{g}(n\phi_4 Z, n\phi_5 Z) &= \overline{g}(\bar{J}\phi_4 Z - t\phi_4 Z, \bar{J}\phi_5 Z - t\phi_5 Z) \\
&= -\overline{g}(\bar{J}\phi_4 Z, t\phi_5 Z) - \overline{g}(\bar{J}\phi_5 Z, t\phi_4 Z) \\
&= \overline{g}(\phi_4 Z, \bar{J}t\phi_5 Z) + \overline{g}(\phi_5 Z, \bar{J}t\phi_4 Z) \\
&= \overline{g}(\phi_4 Z, t^2\phi_5 Z) + \overline{g}(\phi_5 Z, t^2\phi_4 Z) \\
&= -\cos^2(\theta_2)_p g(\phi_4 Z, \phi_5 Z) - \cos^2(\theta_1)_p g(\phi_5 Z, \phi_4 Z) \\
&= 0,
\end{aligned}$$

which completes the proof. □

In view of Lemma (3.2), there exists a holomorphic subspace $\mu_p \subset S(T_pN^{\perp})$, such that at each $p \in N$, we have

$$S(TN^{\perp}) = nD_1 \perp nD_2 \perp \mu \tag{3.21}$$

and

$$T\bar{N} = S(TN) \perp \{Rad(TN) \oplus ltr(TN)\} \perp nD_1 \perp nD_2 \perp \mu. \tag{3.22}$$

Also, for $V \in \Gamma(tr(TN))$, we have $V = PV + QV$, where $PV \in \Gamma(ltr(TN))$ and $QV \in \Gamma(S(TN^{\perp}))$. Note that as per Eq.(3.21) for $V \in \Gamma(S(TN^{\perp}))$, we have

$$QV = Q_1V + Q_2V + Q_3V,$$

where Q_1, Q_2 and Q_3 denote the projections of $S(TN^{\perp})$ onto nD_1, nD_2 and μ, respectively. Now, applying $\bar{J}$ on both sides, we have

$$\begin{aligned}\bar{J}V &= \bar{J}PV + \bar{J}QV\\ &= \bar{J}PV + BQ_1V + CQ_1V + BQ_2V + CQ_2V + \bar{J}Q_3V,\end{aligned}$$

where, $\bar{J}PV \in \Gamma\bar{J}ltr(TN))$ and using Lemma (3.1), we have $BQ_1V \in \Gamma(D_1)$, $CQ_1V \in \Gamma(S(TN^{\perp}))$, $BQ_2V \in \Gamma(D_2)$, $CQ_2V \in \Gamma(S(TN^{\perp}))$ and $\bar{J}Q_3V \in \Gamma(\mu)$. Then, using Eqs. (2.4), (2.7), (2.13) with (3.15) and (3.16) and equating the components of $Rad(TN)$, $\bar{J}Rad(TN)$, $\bar{J}(ltr(TN))$, D_1, D_2, $ltr(TN)$ and $S(TN)^{\perp}$, we get

$$\begin{aligned}&\phi_1(\nabla_{Z_1}\bar{J}\phi_1Z_2) + \phi_1(\nabla_{Z_1}\bar{J}\phi_2Z_2) + \phi_1(\nabla_{Z_1}t\phi_4Z_2) + \phi_1(\nabla_{Z_1}t\phi_5Z_2)\\ &+\phi_1(\nabla_{Z_2}\bar{J}\phi_1Z_1) + \phi_1(\nabla_{Z_2}\bar{J}\phi_2Z_1) + \phi_1(\nabla_{Z_2}t\phi_4Z_1) + \phi_1(\nabla_{Z_2}t\phi_5Z_1)\\ &= \phi_1(A_{n\phi_3Z_2}Z_1) + \phi_1(A_{n\phi_4Z_2}Z_1) + \phi_1(A_{n\phi_5Z_2}Z_1) + \phi_1(A_{n\phi_3Z_1}Z_2)\\ &\quad + \phi_1(A_{n\phi_4Z_1}Z_2) + \phi_1(A_{n\phi_5Z_1}Z_2) + \bar{J}\phi_2\nabla_{Z_1}Z_2 + \bar{J}\phi_2\nabla_{Z_2}Z_1,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\phi_2(\nabla_{Z_1}\bar{J}\phi_1Z_2) + \phi_2(\nabla_{Z_1}\bar{J}\phi_2Z_2) + \phi_2(\nabla_{Z_1}t\phi_4Z_2) + \phi_2(\nabla_{Z_1}t\phi_5Z_2)\\ &+\phi_2(\nabla_{Z_2}\bar{J}\phi_1Z_1) + \phi_2(\nabla_{Z_2}\bar{J}\phi_2Z_1) + \phi_2(\nabla_{Z_2}t\phi_4Z_1) + \phi_2(\nabla_{Z_2}t\phi_5Z_1)\\ &= \phi_2(A_{n\phi_3Z_2}Z_1) + \phi_2(A_{n\phi_4Z_2}Z_1) + \phi_2(A_{n\phi_5Z_2}Z_1) + \phi_2(A_{n\phi_3Z_1}Z_2)\\ &\quad + \phi_2(A_{n\phi_4Z_1}Z_2) + \phi_2(A_{n\phi_5Z_1}Z_2) + \bar{J}\phi_1\nabla_{Z_1}Z_2 + \bar{J}\phi_1\nabla_{Z_2}Z_1,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\phi_3(\nabla_{Z_1}\bar{J}\phi_1Z_2) + \phi_3(\nabla_{Z_1}\bar{J}\phi_2Z_2) + \phi_3(\nabla_{Z_1}t\phi_4Z_2) + \phi_3(\nabla_{Z_1}t\phi_5Z_2)\\ &+\phi_3(\nabla_{Z_2}\bar{J}\phi_1Z_1) + \phi_3(\nabla_{Z_2}\bar{J}\phi_2Z_1) + \phi_3(\nabla_{Z_2}t\phi_4Z_1) + \phi_3(\nabla_{Z_2}t\phi_5Z_1)\\ &= \phi_3(A_{n\phi_3Z_2}Z_1) + \phi_3(A_{n\phi_4Z_2}Z_1) + \phi_3(A_{n\phi_5Z_2}Z_1) + \phi_3(A_{n\phi_3Z_1}Z_2)\\ &\quad + \phi_3(A_{n\phi_4Z_1}Z_2) + \phi_3(A_{n\phi_5Z_1}Z_2) + 2Bh^l(Z_1, Z_2),\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\phi_4(\nabla_{Z_1}\bar{J}\phi_1Z_2) + \phi_4(\nabla_{Z_1}\bar{J}\phi_2Z_2) + \phi_4(\nabla_{Z_1}t\phi_4Z_2) + \phi_4(\nabla_{Z_1}t\phi_5Z_2)\\ &+\phi_4(\nabla_{Z_2}\bar{J}\phi_1Z_1) + \phi_4(\nabla_{Z_2}\bar{J}\phi_2Z_1) + \phi_4(\nabla_{Z_2}t\phi_4Z_1) + \phi_4(\nabla_{Z_2}t\phi_5Z_1)\\ &= \phi_4(A_{n\phi_3Z_2}Z_1) + \phi_4(A_{n\phi_4Z_2}Z_1) + \phi_4(A_{n\phi_5Z_2}Z_1) + \phi_4(A_{n\phi_3Z_1}Z_2)\\ &\quad + \phi_4(A_{n\phi_4Z_1}Z_2) + \phi_4(A_{n\phi_5Z_1}Z_2) + 2BQ_1h^s(Z_1, Z_2)\\ &\quad + t\phi_4\nabla_{Z_1}Z_2 + t\phi_4\nabla_{Z_2}Z_1,\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\phi_5(\nabla_{Z_1}\bar{J}\phi_1 Z_2)+\phi_5(\nabla_{Z_1}\bar{J}\phi_2 Z_2)+\phi_5(\nabla_{Z_1}t\phi_4 Z_2)+\phi_5(\nabla_{Z_1}t\phi_5 Z_2)\\&+\phi_5(\nabla_{Z_2}\bar{J}\phi_1 Z_1)+\phi_5(\nabla_{Z_2}\bar{J}\phi_2 Z_1)+\phi_5(\nabla_{Z_2}t\phi_4 Z_1)+\phi_5(\nabla_{Z_2}t\phi_5 Z_1)\\&=\phi_5(A_{n\phi_3 Z_2}Z_1)+\phi_5(A_{n\phi_4 Z_2}Z_1)+\phi_5(A_{n\phi_5 Z_2}Z_1)+\phi_5(A_{n\phi_3 Z_1}Z_2)\\&+\phi_5(A_{n\phi_4 Z_1}Z_2)+\phi_5(A_{n\phi_5 Z_1}Z_2)+2BQ_2h^s(Z_1,Z_2)\\&+t\phi_5\nabla_{Z_1}Z_2+t\phi_5\nabla_{Z_2}Z_1,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&h^l(Z_2,\bar{J}\phi_1 Z_1)+h^l(Z_2,\bar{J}\phi_2 Z_1)+h^l(Z_2,t\phi_4 Z_1)+h^l(Z_2,t\phi_5 Z_1)\\&+h^l(Z_1,\bar{J}\phi_1 Z_2)+h^l(Z_1,\bar{J}\phi_2 Z_2)+h^l(Z_1,t\phi_4 Z_2)+h^l(Z_1,t\phi_5 Z_2)\\&=n\phi_3\nabla_{Z_1}Z_2+n\phi_3\nabla_{Z_2}Z_1-\nabla^l_{Z_1}n\phi_3 Z_2-D^l(Z_1,n\phi_4 Z_2)\\&-D^l(Z_1,n\phi_5 Z_2)-\nabla^l_{Z_2}n\phi_3 Z_1-D^l(n\phi_4 Z_1,Z_2)-D^l(Z_2,n\phi_5 Z_1),\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}&h^s(Z_2,\bar{J}\phi_1 Z_1)+h^s(Z_2,\bar{J}\phi_2 Z_1)+h^s(Z_2,t\phi_4 Z_1)+h^s(Z_2,t\phi_5 Z_1)\\&+h^s(Z_1,\bar{J}\phi_1 Z_2)+h^s(Z_1,\bar{J}\phi_2 Z_2)+h^s(Z_1,t\phi_4 Z_2)+h^s(Z_1,t\phi_5 Z_2)\\&=n\phi_4\nabla_{Z_1}Z_2+n\phi_5\nabla_{Z_1}Z_2+n\phi_4\nabla_{Z_2}Z_1+n\phi_5\nabla_{Z_2}Z_1\\&-D^s(Z_1,n\phi_3 Z_2)-\nabla^s_{Z_1}n\phi_4 Z_2-\nabla^s_{Z_1}n\phi_5 Z_2\\&-D^s(Z_2,n\phi_3 Z_1)-\nabla^s_{Z_2}n\phi_4 Z_1-\nabla^s_{Z_2}n\phi_5 Z_1\\&+2CQ_1h^s(Z_1,Z_2)+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2).\end{aligned} \tag{3.29}$$

Next, we investigate conditions for the distributions associated with **pw.bi-s.l.s.** N of $\bar{N}$ to be involutive.

Theorem 3.2. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $Rad(TN)$ is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right)+\phi_1\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right)=2\bar{J}\phi_2\nabla_{Z_2}Z_1,$

(ii) $\phi_4\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right)+\phi_4\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right)=2BQ_1h^s\left(Z_1,Z_2\right)+2t\phi_4\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right)+\phi_5\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right)=2BQ_2h^s\left(Z_1,Z_2\right)+2t\phi_5\nabla_{Z_2}Z_1,$

(iv) $h^l\left(Z_2,\bar{J}\phi_1 Z_1\right)+h^l\left(Z_1,\bar{J}\phi_1 Z_2\right)=2n\phi_3\nabla_{Z_2}Z_1,$

(v) $h^s\left(Z_2,\bar{J}\phi_1 Z_1\right)+h^s\left(Z_1,\bar{J}\phi_1 Z_2\right)=2n\phi_4\nabla_{Z_2}Z_1+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2),$

for any $Z_1,Z_2\in\Gamma(Rad(TN))$.

Proof. Consider $Z_1, Z_2 \in \Gamma(Rad(TN))$, then using Eqs.(3.23), (3.26), (3.27), (3.28) and (3.29), we have

$$\phi_1(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_1(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = \bar{J}\phi_2[Z_1, Z_2] + 2\bar{J}\phi_2\nabla_{Z_2}Z_1, \tag{3.30}$$

$$\phi_4(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_4(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = 2BQ_1h^s(Z_1, Z_2) + t\phi_4[Z_1, Z_2] + 2t\phi_4\nabla_{Z_2}Z_1, \tag{3.31}$$

$$\phi_5(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_5(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = 2BQ_2h^s(Z_1, Z_2) + t\phi_5[Z_1, Z_2] + 2t\phi_5\nabla_{Z_2}Z_1, \tag{3.32}$$

$$h^l(Z_2, \bar{J}\phi_1 Z_1) + h^l(Z_1, \bar{J}\phi_1 Z_2) = n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 \tag{3.33}$$

and

$$\begin{aligned} 2CQ_1h^s(Z_1, Z_2) + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2) + n\phi_4[Z_1, Z_2] + 2n\phi_4\nabla_{Z_2}Z_1 \\ + n\phi_5[Z_1, Z_2] + 2n\phi_5\nabla_{Z_2}Z_1 = h^s(Z_2, \bar{J}\phi_1 Z_1) + h^s(Z_1, \bar{J}\phi_1 Z_2). \end{aligned} \tag{3.34}$$

Then the result follows from Eqs. (3.30), (3.31), (3.32), (3.33) and (3.34). □

Theorem 3.3. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $\bar{J}Rad(TN)$ is involutive, if and only if*

(i) $\phi_2\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_2\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2\bar{J}\phi_1\nabla_{Z_2}Z_1,$

(ii) $\phi_4\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_4\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_1h^s\left(Z_1, Z_2\right) + 2t\phi_4\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_5\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_2h^s\left(Z_1, Z_2\right) + 2t\phi_5\nabla_{Z_2}Z_1,$

(iv) $h^l\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^l\left(Z_1, \bar{J}\phi_2 Z_2\right) = 2n\phi_3\nabla_{Z_2}Z_1,$

(v) $h^s\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^s\left(Z_1, \bar{J}\phi_2 Z_2\right) = 2n\phi_4\nabla_{Z_2}Z_1 + 2n\phi_5\nabla_{Z_2}Z_1 + 2CQ_1h^s(Z_1, Z_2) + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2),$

where $Z_1, Z_2 \in \Gamma(\bar{J}Rad(TN))$.

Proof. Let Z_1, $Z_2 \in \Gamma(\bar{J}(Rad(TN))$, then using Eqs.(3.24), (3.26), (3.27), (3.28) and (3.29), we have

$$\phi_2\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_2\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = \bar{J}\phi_1[Z_1, Z_2] + 2\bar{J}\phi_1\nabla_{Z_2}Z_1, \tag{3.35}$$

$$\phi_4\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_4\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_1h^s\left(Z_1, Z_2\right) + t\phi_4[Z_1, Z_2] + 2t\phi_4\nabla_{Z_2}Z_1, \tag{3.36}$$

$$\phi_5\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_5\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_2h^s\left(Z_1, Z_2\right) + t\phi_5[Z_1, Z_2] + 2t\phi_5\nabla_{Z_2}Z_1, \tag{3.37}$$

$$h^l\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^l\left(Z_1, \bar{J}\phi_2 Z_2\right) = n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 \tag{3.38}$$

and

$$n\phi_4[Z_1,Z_2]+2n\phi_4\nabla_{Z_2}Z_1+n\phi_5[Z_1,Z_2]+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$$
$$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2)=h^s\left(Z_2,\bar{J}\phi_2Z_1\right)+h^s\left(Z_1,\bar{J}\phi_2Z_2\right). \tag{3.39}$$

The result follows from Eqs. (3.35), (3.36), (3.37), (3.38) and (3.39). □

Theorem 3.4. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $\bar{J}(ltr(TN))$ is involutive, if and only if*

(i) $\phi_1(A_{n\phi_3Z_2}Z_1)+\phi_1(A_{n\phi_3Z_1}Z_2)+2\bar{J}\phi_2\nabla_{Z_2}Z_1=0,$

(ii) $\phi_2(A_{n\phi_3Z_2}Z_1)+\phi_2(A_{n\phi_3Z_1}Z_2)+2\bar{J}\phi_1\nabla_{Z_2}Z_1=0,$

(iii) $\phi_4(A_{n\phi_3Z_2}Z_1)+\phi_4(A_{n\phi_3Z_1}Z_2)+2BQ_1h^s\left(Z_1,Z_2\right)+2t\phi_4\nabla_{Z_2}Z_1=0,$

(iv) $\phi_5(A_{n\phi_3Z_2}Z_1)+\phi_5(A_{n\phi_3Z_1}Z_2)+2BQ_2h^s\left(Z_1,Z_2\right)+2t\phi_5\nabla_{Z_2}Z_1=0,$

(v) $D^s(Z_1,n\phi_3Z_2)+D^s(Z_2,n\phi_3Z_1)=2n\phi_4\nabla_{Z_2}Z_1+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$
$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2),$

for any $Z_1,Z_2\in\Gamma(\bar{J}(ltr(TN))$.

Proof. Consider $Z_1,Z_2\in\Gamma(\bar{J}(ltr(TN))$, then from Eqs.(3.23), (3.24), (3.26), (3.27) and (3.29), we have

$$\phi_1(A_{n\phi_3Z_2}Z_1)+\phi_1(A_{n\phi_3Z_1}Z_2)+\bar{J}\phi_2[Z_1,Z_2]+2\bar{J}\phi_2\nabla_{Z_2}Z_1=0, \tag{3.40}$$

$$\phi_2(A_{n\phi_3Z_2}Z_1)+\phi_2(A_{n\phi_3Z_1}Z_2)+\bar{J}\phi_1[Z_1,Z_2]+2\bar{J}\phi_1\nabla_{Z_2}Z_1=0, \tag{3.41}$$

$$\phi_4(A_{n\phi_3Z_2}Z_1)+\phi_4(A_{n\phi_3Z_1}Z_2)+2BQ_1h^s\left(Z_1,Z_2\right)+t\phi_4[Z_1,Z_2]+2t\phi_4\nabla_{Z_2}Z_1=0, \tag{3.42}$$

$$\phi_5(A_{n\phi_3Z_2}Z_1)+\phi_5(A_{n\phi_3Z_1}Z_2)+2BQ_2h^s\left(Z_1,Z_2\right)+t\phi_5[Z_1,Z_2]+2t\phi_5\nabla_{Z_2}Z_1=0 \tag{3.43}$$

and

$$n\phi_4[Z_1,Z_2]+2n\phi_4\nabla_{Z_2}Z_1+n\phi_5[Z_1,Z_2]+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$$
$$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2)=D^s(Z_1,n\phi_3Z_2)+D^s(Z_2,n\phi_3Z_1). \tag{3.44}$$

Using Eqs. (3.40), (3.41), (3.42), (3.43) and (3.44), the assertion follows directly. □

Theorem 3.5. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution D_1 is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1}t\phi_4Z_2\right)+\phi_1\left(\nabla_{Z_2}t\phi_4Z_1\right)=\phi_1(A_{n\phi_4Z_2}Z_1)+\phi_1(A_{n\phi_4Z_1}Z_2)+2\bar{J}\phi_2\nabla_{Z_2}Z_1,$

(ii) $\phi_2\left(\nabla_{Z_1}t\phi_4Z_2\right)+\phi_2\left(\nabla_{Z_2}t\phi_4Z_1\right)=\phi_2(A_{n\phi_4Z_2}Z_1)+\phi_2(A_{n\phi_4Z_1}Z_2)+2\bar{J}\phi_1\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_5\left(\nabla_{Z_2} t\phi_4 Z_1\right)=\phi_5(A_{n\phi_4 Z_2} Z_1)+\phi_5(A_{n\phi_4 Z_1} Z_2)+2BQ_2 h^s\left(Z_1, Z_2\right)+2t\phi_5 \nabla_{Z_2} Z_1$,

(iv) $h^l\left(Z_2, t\phi_4 Z_1\right)+h^l\left(Z_1, t\phi_4 Z_2\right)=2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_4 Z_2)-D^l(n\phi_4 Z_1, Z_2)$,

(v) $n\phi_4 \nabla_{Z_2} Z_1+n\phi_4 \nabla_{Z_1} Z_2+2n\phi_5 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_4 Z_2-\nabla^s_{Z_2} n\phi_4 Z_1+2CQ_1 h^s(Z_1, Z_2)+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right)$,

where $Z_1, Z_2 \in \Gamma(D_1)$.

Proof. For $Z_1, Z_2 \in \Gamma(D_1)$, from Eqs.(3.23), (3.24), (3.27), (3.28) and (3.29), we have

$$\phi_1(A_{n\phi_4 Z_2} Z_1)+\phi_1(A_{n\phi_4 Z_1} Z_2)+2\bar{J}\phi_2 \nabla_{Z_2} Z_1+\bar{J}\phi_2[Z_1, Z_2]=\phi_1\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_1\left(\nabla_{Z_2} t\phi_4 Z_1\right), \tag{3.45}$$

$$\phi_2(A_{n\phi_4 Z_2} Z_1)+\phi_2(A_{n\phi_4 Z_1} Z_2)+2\bar{J}\phi_1 \nabla_{Z_2} Z_1+\bar{J}\phi_1[Z_1, Z_2]=\phi_2\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_2\left(\nabla_{Z_1} t\phi_4 Z_2\right), \tag{3.46}$$

$$\phi_5\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_5\left(\nabla_{Z_2} t\phi_4 Z_1\right)=\phi_5(A_{n\phi_4 Z_2} Z_1)+\phi_5(A_{n\phi_4 Z_1} Z_2)+t\phi_5[Z_1, Z_2]+2BQ_2 h^s\left(Z_1, Z_2\right)+2t\phi_5 \nabla_{Z_2} Z_1, \tag{3.47}$$

$$n\phi_3[Z_1, Z_2]+2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_4 Z_2)-D^l(n\phi_4 Z_1, Z_2)=h^l\left(Z_2, t\phi_4 Z_1\right)+h^l\left(Z_1, t\phi_4 Z_2\right) \tag{3.48}$$

and

$$n\phi_4 \nabla_{Z_2} Z_1+n\phi_4 \nabla_{Z_1} Z_2+2n\phi_5 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_4 Z_2-\nabla^s_{Z_2} n\phi_4 Z_1+2CQ_1 h^s(Z_1, Z_2)+n\phi_5[Z_1, Z_2]+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right). \tag{3.49}$$

Then the assertion follows from Eqs. (3.45), (3.46), (3.47), (3.48) and (3.49). □

Theorem 3.6. *For a* ***pw.bi-s.l.s.*** N *of* $\bar{N}$, *the distribution* D_2 *is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_1\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_1(A_{n\phi_5 Z_2} Z_1)+\phi_1(A_{n\phi_5 Z_1} Z_2)+2\bar{J}\phi_2 \nabla_{Z_2} Z_1$,

(ii) $\phi_2\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_2\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_2(A_{n\phi_5 Z_2} Z_1)+\phi_2(A_{n\phi_5 Z_1} Z_2)+2\bar{J}\phi_1 \nabla_{Z_2} Z_1$,

(iii) $\phi_4\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_4\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_4(A_{n\phi_5 Z_2} Z_1)+\phi_4(A_{n\phi_5 Z_1} Z_2)+2BQ_1 h^s\left(Z_1, Z_2\right)+2t\phi_4 \nabla_{Z_2} Z_1$,

(iv) $h^l\left(Z_2, t\phi_5 Z_1\right)+h^l\left(Z_1, t\phi_5 Z_2\right)=2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_5 Z_2)-D^l(n\phi_5 Z_1, Z_2)$,

(v) $n\phi_5 \nabla_{Z_2} Z_1+n\phi_5 \nabla_{Z_1} Z_2+2n\phi_4 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_5 Z_2-\nabla^s_{Z_2} n\phi_5 Z_1+2CQ_1 h^s(Z_1, Z_2)+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right)$, *for any* $Z_1, Z_2 \in \Gamma(D_2)$.

Proof. Let $Z_1, Z_2 \in \Gamma(D_2)$. Using Eqs.(3.23), (3.24), (3.27), (3.28) and (3.29), we have

$$\phi_1(A_{n\phi_5 Z_2}Z_1) + \phi_1(A_{n\phi_5 Z_1}Z_2) + 2\bar{J}\phi_2\nabla_{Z_2}Z_1 + \bar{J}\phi_2[Z_1, Z_2] = \phi_1\left(\nabla_{Z_1}t\phi_5 Z_2\right) + \phi_1\left(\nabla_{Z_2}t\phi_5 Z_1\right), \quad (3.50)$$

$$\phi_2(A_{n\phi_5 Z_2}Z_1) + \phi_2(A_{n\phi_5 Z_1}Z_2) + 2\bar{J}\phi_1\nabla_{Z_2}Z_1 + \bar{J}\phi_1[Z_1, Z_2] = \phi_2\left(\nabla_{Z_1}t\phi_5 Z_2\right) + \phi_2\left(\nabla_{Z_2}t\phi_5 Z_1\right), \quad (3.51)$$

$$\phi_4\left(\nabla_{Z_1}t\phi_5 Z_2\right) + \phi_4\left(\nabla_{Z_2}t\phi_5 Z_1\right) = \phi_4(A_{n\phi_5 Z_2}Z_1) + \phi_4(A_{n\phi_5 Z_1}Z_2) + t\phi_4[Z_1, Z_2] + 2BQ_1h^s\left(Z_1, Z_2\right) + 2t\phi_4\nabla_{Z_2}Z_1, \quad (3.52)$$

$$n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 - D^l(Z_1, n\phi_5 Z_2) - D^l(n\phi_5 Z_1, Z_2) = h^l\left(Z_2, t\phi_5 Z_1\right) + h^l\left(Z_1, t\phi_5 Z_2\right) \quad (3.53)$$

and

$$n\phi_5\nabla_{Z_2}Z_1 + n\phi_5\nabla_{Z_1}Z_2 + 2n\phi_4\nabla_{Z_2}Z_1 - \nabla^s_{Z_1}n\phi_5 Z_2 - \nabla^s_{Z_2}n\phi_5 Z_1 + 2CQ_1h^s(Z_1, Z_2) + n\phi_4[Z_1, Z_2] + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2) = h^s\left(Z_2, t\phi_5 Z_1\right) + h^s\left(Z_1, t\phi_5 Z_2\right). \quad (3.54)$$

The result follows from Eqs. (3.50), (3.51), (3.52), (3.53) and (3.54). □

We now give a necessary and sufficient condition for the induced connection on a **pw.bi-s.l.s.** N to be a metric connection.

Theorem 3.7. *Assume that N is a **pw.bi-s.l.s.** of $\bar{N}$, then ∇ is a metric connection on N, if and only if for each $Z \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$, we have*

(i) $\nabla_Z\bar{J}\xi + \nabla_{\bar{J}\xi}Z \in \Gamma\bar{J}(Rad(TN))$,
(ii) $\nabla_{\bar{J}\xi}tZ - A_{nZ}\bar{J}\xi \in \Gamma(Rad(TN))$,
(iii) $Bh(Z, \bar{J}\xi) = 0$.

Proof. Consider $Z \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$. Now, by Eqs. (2.13), we have

$$\overline{\nabla}_Z\xi = -\overline{\nabla}_Z\bar{J}^2\xi = -\bar{J}\overline{\nabla}_Z\bar{J}\xi + \overline{\nabla}_{\bar{J}\xi}\bar{J}Z - \bar{J}\overline{\nabla}_{\bar{J}\xi}Z.$$

Then using Eqs. (2.3), (3.15) and (3.16), we get

$$-\bar{J}(\nabla_Z \bar{J}\xi + \nabla_{\bar{J}\xi} Z) + \nabla_{\bar{J}\xi} tZ + h(\bar{J}\xi, tZ) - A_{nZ}\bar{J}\xi + \nabla^t_{\bar{J}\xi} nZ - 2Bh(Z, \bar{J}\xi) - 2Ch(Y, \bar{J}\xi)$$
$$= \nabla_Z \xi + h(Z, \xi),$$

Comparing tangential components on both sides of above equation, we derive

$$\nabla_Z \xi = -t(\nabla_Z \bar{J}\xi + \nabla_{\bar{J}\xi} Z) + \nabla_{\bar{J}\xi} tZ - A_{nZ}\bar{J}\xi - 2Bh(Z, \bar{J}\xi).$$

Hence, $\nabla_Z \xi \in \Gamma(Rad(TN))$, if and only if the conditions (i), (ii) and (iii) are satisfied. □

Lemma 3.3. *Let N be a **pw.bi-s.l.s.** of $\bar{N}$ and $(\theta_j)_p$, $(j = 1, 2)$ be the slant angle. Then, for a unit vector $Z \in \Gamma(D_j)_p$, we have*

$$tZ = \cos(\theta_j)_p(Z)\bar{Z} \tag{3.55}$$

where $\bar{Z}$ represents a unit vector in $(D_j)_p$ such that $g(\bar{Z}, Z) = 0$.

Proof. Let $Z \in \Gamma(D_j)_p$, $(j = 1, 2)$ such that $g(Z, Z) = 1$, then we have

$$\cos(\theta_j)_p(Z) = \frac{|tZ|}{|\bar{J}Z|} = \frac{|tZ|}{|Z|} = |tZ|. \tag{3.56}$$

Now, define $\bar{Z} = \dfrac{tZ}{|tZ|}$, then clearly $|\bar{Z}| = 1$ and $tZ = |tZ|\bar{Z}$. Next from Eq. (3.56), we have

$$tZ = \cos(\theta_j)_p(Z)\bar{Z}.$$

We know that for an indefinite nearly Kähler manifold, $g(\bar{J}Z, Z) = 0$. Using Lemma(3.1) and Eq.(3.15), we get $g(tZ, Z) = 0$. Further, we have $g(\bar{Z}, Z) = g\left(\dfrac{tZ}{|tZ|}, Z\right) = \dfrac{1}{|tZ|} g(tZ, Z) = 0$, which proves the lemma.

□

Definition 3.2. *A q-lightlike submanifold N of an indefinite nearly Kähler manifold $\bar{N}$ with index $2q$ is said to be a bi-slant lightlike submanifold if*

(i) *$\bar{J}(Rad(TN))$ is a distribution on N such that $\bar{J}(Rad(TN)) \cap Rad(TN)$=$\{0\}$.*

(ii) *There exists non-degenerate orthogonal distributions D_1 and D_2 on N such that $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.*

(iii) *$\bar{J}D_1 \perp D_2$ and $\bar{J}D_2 \perp D_1$.*

(iv) *The distribution D_j is slant with slant angle θ_j (for $j = 1, 2$) i.e, for each $p \in N$ and non-zero tangent vector field $Z \in (D_j)_p$, the angle $(\theta_j)_p$ between $\bar{J}Z$ and the vector space $(D_j)_p$ is independent of choice of $Z \in \Gamma(D_j)_p$ and $p \in N$.*

If $D_j \neq \{0\}$ and $\theta_j \neq 0, \pi/2$, then, the bi-slant lightlike submanifold is said to be proper.

Next, we provide conditions for proper **pw.bi-s.l.s.** N of $\bar{N}$ to be a bi-slant lightlike submanifold.

Theorem 3.8. *A proper **pw.bi-s.l.s.** N of $\bar{N}$ is a bi-slant lightlike submanifold of $\bar{N}$, if and only if,*

$$g(A_{nZ_j}Y, \bar{Z}_j) + g(A_{nY}Z_j, \bar{Z}_j) = 2g(A_{n\bar{Z}_j}Y, Z_j) + g((\nabla_{Z_j}t)Y, \bar{Z}_j),$$

where for $p \in U \subset N$, $Z_j \in \Gamma(D_j)_p$ is a unit vector field, $\bar{Z}_j \in \Gamma(D_j)_p$ is a unit vector field such that $g(\bar{Z}_j, Z_j) = 0$ for $j = 1, 2$ and $Y \in \Gamma(TN)$.

Proof. Assume N be a proper **pw.bi-s.l.s.** of $\bar{N}$ and $Z_j \in \Gamma(D_j)_p$ for $p \in U \subset N$, be unit vector field. For $Y \in \Gamma(TN)$, using Eqs. (3.15), (2.4), (2.5), (2.7) and (3.55), we have

$$\begin{aligned}\overline{\nabla}_Y \bar{J} Z_j = &- sin(\theta_j)_p(Z) Y((\theta_j)_p(Z))\bar{Z}_j + cos(\theta_j)_p(Z)(\nabla_Y \bar{Z}_j + h^l(Y, \bar{Z}_j) + h^s(Y, \bar{Z}_j)) \\ &- A_{nZ_j}Y + \nabla^s_Y nZ_j + D^l(Y, nZ_j) \qquad (3.57)\end{aligned}$$

aand

$$\begin{aligned}\overline{\nabla}_{Z_j} \bar{J} Y = &\nabla_{Z_j} tY + h^l(Z_j, tY) + h^s(Z_j, tY) - A_{nY}Z_j + \nabla^l_{Z_j} L(nY) + \nabla^s_{Z_j} S(nY) \\ &+ D^l(Z_j, S(nY)) + D^s(Z_j, L(nY)). \qquad (3.58)\end{aligned}$$

Again, using Eqs. (2.4), (3.15) ans (3.16), we get

$$\begin{aligned}\bar{J}(\overline{\nabla}_{Z_j} Y) + \bar{J}(\overline{\nabla}_Y Z_j) = &t\nabla_{Z_j} Y + t\nabla_Y Z_j + n\nabla_{Z_j} Y + n\nabla_Y Z_j + 2Bh^l(Z_j, Y) \\ &+ 2Bh^s(Z_j, Y) + 2Ch^s(Z_j, Y). \qquad (3.59)\end{aligned}$$

As $\bar{N}$ is an indefinite nearly kähler manifold, therefore, using Eq. (2.13), we get

$$\overline{\nabla}_Y \bar{J} Z_j + \overline{\nabla}_{Z_j} \bar{J} Y = \bar{J}(\overline{\nabla}_{Z_j} Y) + \bar{J}(\overline{\nabla}_Y Z_j).$$

Using Eqs. (3.57), (3.58) and (3.59) , comparing tangential parts of resulting equation and taking inner product with respect to $\bar{Z}_j \in \Gamma(D_j)_p$, we have

$$\begin{aligned}&- sin(\theta_j)_p(Z) Y((\theta_j)_p(Z)) + cos(\theta_j)_p(Z) g(\nabla_Y \bar{Z}_j, \bar{Z}_j) - g(A_{nZ_j}Y, \bar{Z}_j) - g(A_{nY}Z_j, \bar{Z}_j) \\ &+ g(\nabla_{Z_j} tY, \bar{Z}_j) = g(t\nabla_{Z_j} Y, \bar{Z}_j) + g(t\nabla_Y Z_j, \bar{Z}_j) + 2g(Bh^l(Z_j, Y), \bar{Z}_j) + 2g(Bh^s(Z_j, Y), \bar{Z}_j).\end{aligned}$$

(3.60)

Next from Eq. (2.12), we have $(\nabla_Y g)(\bar{Z}_j, \bar{Z}_j) = 0$, which gives $g(\nabla_Y \bar{Z}_j, \bar{Z}_j) = 0$. Then consider,

$$\begin{aligned} g(t\nabla_Y Z_j, \bar{Z}_j) &= g\left(t\nabla_Y Z_j, \frac{tZ_j}{|tZ_j|}\right) \\ &= \frac{1}{|tZ_j|} g(t\nabla_Y Z_j, tZ_j) \\ &= \frac{1}{|tZ_j|} \cos^2(\theta_j)_p(Z) g(\nabla_Y Z_j, Z_j) \\ &= 0. \end{aligned} \tag{3.61}$$

Also, from Eq. (2.8), we have

$$\begin{aligned} 2g(Bh^s(Y, Z_j), \bar{Z}_j) = -2\bar{g}(h^s(Y, Z_j), \bar{J}\bar{Z}_j) &= -2\bar{g}(h^s(Y, Z_j), n\bar{Z}_j) \\ &= -2g(A_{n\bar{Z}_j} Y, Z_j). \end{aligned} \tag{3.62}$$

Now, using (3.61) and (3.62) along with the fact that $g(\nabla_Y \bar{W}, \bar{W}) = 0$ in (3.60), we have

$$\begin{aligned} -sin(\theta_j)_p(Z) Y((\theta_j)_p(Z)) =& g(A_{nZ_j} Y, \bar{Z}_j) + g(A_{nY} Z_j, \bar{Z}_j) - g((\nabla_{Z_j} t)Y, \bar{Z}_j) \\ &- 2g(A_{n\bar{Z}_j} Y, Z_j), \end{aligned} \tag{3.63}$$

As N is proper **pw.bi.s.l.s.** of $\bar{N}$, N is a bi-slant lightlike submanifold iff $Y((\theta_j)_p(Z)) = 0$ i.e, θ_j is independent of choice of $p \in N$ which proves the theorem. □

Theorem 3.9. *Assume N be a proper* ***pw.bi-s.l.s.*** *of $\bar{N}$. If*

(i) *there exists $tr(TN)$ which is parallel along TN with respect to metric connection $\overline{\nabla}$.*

(ii) *t is parallel with respect to induced connection ∇ on N.*

Then, N becomes a bi-slant lightlike submanifold of $\bar{N}$.

Proof. Assume that $Y \in \Gamma(TN)$, $Z_j \in \Gamma(D_j)_p$ for $j = 1, 2$, where $p \in U \subset N$. Then, using Lemma (3.1), $\{nZ_j,\ n\bar{Z}_j\} \in \Gamma(S(TN^{\perp})) \subset \Gamma(tr(TN))$. Since $tr(TN)$ is parallel along TN with respect to metric connection $\overline{\nabla}$, we have, $\{\overline{\nabla}_Y nZ_j,\ \overline{\nabla}_Y n\bar{Z}_j\} \in \Gamma(tr(TN))$ which implies $A_{nZ_j} Y = A_{n\bar{Z}_j} Y = 0$. Similarly, using the fact that $tr(TN)$ is parallel along TN with respect to metric connection $\overline{\nabla}$ and Eq.(2.5), we get $A_{nY} Z_j = 0$. Also, by condition *(ii)*, $(\nabla_{Z_j} t)Y = 0$. As N is proper **pw.bi.s.l.s.** of $\bar{N}$, from Eq. (3.63), $Y((\theta_j)_p(Z)) = 0$ i.e, θ_j is independent of choice of $p \in N$ which proves the theorem. □

Definition 3.3. [7] *A lightlike submanifold (N, g) of a semi-Riemannian $(\bar{N}, \bar{g})$ is called totally umbilical, if there exist a transversal curvature vector field $H \in \Gamma(tr(TN))$ on N such* *that*

$$h(Z_1, Z_2) = H\bar{g}(Z_1, Z_2), \tag{3.64}$$

for $Z_1, Z_2 \in \Gamma(TN)$. *Using Eqs. (2.4) and (2.7), clearly* N *is totally umbilical, if and only if there exist smooth vector fields* $H^l \in \Gamma(ltr(TN))$ *and* $H^s \in \Gamma(S(TN^{\perp}))$ *such that*

$$h^l(Z_1, Z_2) = H^l g(Z_1, Z_2), \; h^s(Z_1, Z_2) = H^s g(Z_1, Z_2), \; D^l(Z_1, V) = 0, \tag{3.65}$$

for $Z_1, Z_2 \in \Gamma(TN)$ *and* $V \in \Gamma(S(TN^{\perp}))$.

Theorem 3.10. *Assume that* N *be a totally umbilical proper* ***pw.bi-s.l.s.*** *of* $\bar{N}$. *Then* N *becomes a bi-slant lightlike submanifold of* $\bar{N}$, *if* $H^s \in \Gamma(\mu_p)$.

Proof. Let $Z \in \Gamma(D_j)_p$ for some $j = 1, 2$ and $p \in U \subset N$. Then, using Eq. (3.64) and Corollary (3.1), we have

$$\overline{\nabla}_{tZ} tZ = \nabla_{tZ} tZ + cos^2(\theta_j)_p g(Z, Z) H.$$

Now, applying $\bar{J}$ on both sides of above equation and using Eqs.(2.7), (3.15) and Theorem (3.1), we get,

$$\begin{aligned} sin2(\theta_j)_p tZ(\theta_j)_p Z - cos^2(\theta_j)_p \nabla_{tZ} Z - A_{ntZ} tZ + \nabla^s_{tZ} ntZ + D^l(tZ, ntZ) = \\ cos^2(\theta_j)_p g(Z, Z)(\bar{J}H^l + \bar{J}H^s) + t\nabla_{tZ} tZ + n\nabla_{tZ} tZ. \end{aligned} \tag{3.66}$$

Comparing transversal components of above equation and taking the inner product of resulting expression with ntZ, we have

$$cos^2(\theta_j)_p g(Z, Z)\bar{g}(CH^s, ntZ) + \bar{g}(n\nabla_{tZ} tZ, ntZ) = \bar{g}(\nabla^s_{tZ} ntZ, ntZ). \tag{3.67}$$

Using the fact that $\bar{\nabla}$ is a metric connection on $\bar{N}$ with respect to $\bar{g}$ along with Eq.(2.7) and Corollary (3.1), we have

$$\bar{g}(\nabla^s_{tZ} ntZ, ntZ) = \frac{1}{2}(sin2(\theta_j)_p g(tZ, tZ) tZ(\theta_j)_p + sin^2(\theta_j)_p \overline{\nabla}_{tZ} \bar{g}(tZ, tZ)). \tag{3.68}$$

Further using Eq.(3.68) in Eq. (3.67) along with Corollary (3.1) and hypothesis that $H^s \in \Gamma(\mu_p)$, we acquire

$$sin^2(\theta_j)_p \bar{g}(\nabla_{tZ} tZ, tZ) = \frac{1}{2}(sin2(\theta_j)_p g(tZ, tZ) tZ(\theta_j)_p + sin^2(\theta_j)_p \overline{\nabla}_{tZ} \bar{g}(tZ, tZ)). \tag{3.69}$$

As $\overline{\nabla}$ is metric connection on $\bar{N}$ with respect to $\bar{g}$, thus we have

$$\overline{\nabla}_{tZ} \bar{g}(tZ, tZ) = 2\bar{g}(\overline{\nabla}_{tZ} tZ, tZ). \tag{3.70}$$

Next using Eq.(3.70) in Eq.(3.69), we get,

$$sin2(\theta_j)_p g(tZ,tZ)tZ(\theta_j)_p = 0.$$

Since g is non-degenerate on $\Gamma(D_j)_p$ and N is proper **pw.bi-s.l.s.**, thus we conclude that $tZ(\theta_j)_p$=0; this shows that θ_j is independent of choice of $p \in N$ which proves the result. □

Definition 3.4. *A lightlike submanifold* (N, g, ∇) *of* $(\bar{N}, \bar{g}, \overline{\nabla})$ *is called totally geodesic if any geodesic of* N *is a geodesic of* $\bar{N}$. *Using Eq.(2.4),* (N, g, ∇) *is totally geodesic in* $(\bar{N}, \bar{g}, \overline{\nabla})$ *if and only if the second fundamental form vanishes on* N *i.e,* $h^l(Z_1, Z_2) = h^s(Z_1, Z_2) = 0$ $\forall Z_1, Z_2 \in \Gamma(TN)$.

Theorem 3.11. *Assume* N *is a totally umbilical* ***pw.bi-s.l.s.*** *of* $\bar{N}$ *with* $H^s \in \Gamma(\mu)$ *and* $\overline{\nabla}^s_Z V \in \Gamma(\mu)$ *for* $V \in \Gamma(S(TN^\perp))$ *and* $Z \in \Gamma(D_j)_p$ *for j=1,2. Then,* N *is totally geodesic in* $\bar{N}$.

Proof. Let $Z \in \Gamma(D_j)_p$ for some $j = 1, 2$ and $p \in N$. Then from Eq. (2.13), we have $\bar{\nabla}_Z \bar{J}Z = \bar{J}\bar{\nabla}_Z Z$. Further using Eqs. (2.4), (2.7), (3.15) and (3.16), we obtain

$$\begin{aligned}\nabla_Z tZ + h^l(Z, tZ) + h^s(Z, tZ) - A_{nZ}Z + D^l(Z, nZ) + \nabla^s_Z nZ = \\ t\nabla_Z Z + n\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) + Ch^s(Z, Z). \qquad (3.71)\end{aligned}$$

On comparing the tangential components on both sides of above equation and using Eq.(3.65), we get

$$\nabla_Z tZ - A_{nZ}Z = t\nabla_Z Z + g(Z, Z)BH^l + g(Z, Z)BH^s.$$

taking inner product with $\bar{J}\xi \in \Gamma(Rad(TN))$, where $\xi \in \Gamma(Rad(TN))$, we get

$$\begin{aligned}\bar{g}(\nabla_Z tZ, \bar{J}\xi) - \bar{g}(A_{nZ}Z, \bar{J}\xi) =& \bar{g}(t\nabla_Z Z, \bar{J}\xi) + g(Z, Z)\bar{g}(BH^l, \bar{J}\xi) \\ &+ g(Z, Z)\bar{g}(BH^s, \bar{J}\xi). \qquad (3.72)\end{aligned}$$

Now, using Eqs.(2.13), (3.15), (2.12) and (3.16), we have

$$\bar{g}(t\nabla_Z Z, \bar{J}\xi) = 0 = \bar{g}(BH^s, \bar{J}\xi) = \bar{g}(A_{nZ}Z, \bar{J}\xi). \qquad (3.73)$$

Also,

$$\begin{aligned}\bar{g}(\nabla_Z tZ, \bar{J}\xi) = \bar{g}(\overline{\nabla}_Z tZ, \bar{J}\xi) =& -\bar{g}(\overline{\nabla}_Z \bar{J}tZ, \bar{J}\xi)\\ &= -\bar{g}(\overline{\nabla}_Z t^2Z, \xi) - \bar{g}(\overline{\nabla}_Z ntZ, \xi)\\ &= \bar{g}(h^l(Z, t^2Z), \xi)\\ &= cos^2\theta_p(Z)g(Z,Z)\bar{g}(H^l, \xi). \end{aligned} \tag{3.74}$$

Using Eqs.(3.16) and (2.13), we get

$$\bar{g}(BH^l, \bar{J}\xi) = \bar{g}(H^l, \xi). \tag{3.75}$$

Now, using Eqs.(3.73), (3.74) and (3.75) in (3.72), we have

$$g(Z,Z)\bar{g}(H^l, \xi)(1 + cos^2(\theta_j)_p(Z)) = 0.$$

As g is non-degenerate on $\Gamma(D_j)_p$, therefore one has $\bar{g}(H^l, \xi) = 0$ which further implies that

$$H^l = 0. \tag{3.76}$$

Secondly, On comparing the transversal components of Eq.(3.71) and then considering the inner product of resulting part with $\bar{J}H^s$, we get

$$\bar{g}(\nabla^s_Z nZ, \bar{J}H^s) = g(Z,Z)\bar{g}(H^s, H^s). \tag{3.77}$$

As $\overline{\nabla}$ is a metric connection, we have $(\overline{\nabla}_Z\bar{g})(nZ, \bar{J}H^s) = 0)$, which on using Eq. (2.7) together with hypothesis $H^s \in \Gamma(\mu)$ and $\overline{\nabla}^s_Z V \in \Gamma(\mu)$, for $V \in \Gamma(S(TN^\perp))$ yields that

$$\bar{g}(\nabla^s_Z nZ, \bar{J}H^s) = 0. \tag{3.78}$$

Then using Eq.(3.78) in Eq.(3.77), we get

$$g(Z,Z)\bar{g}(H^s, H^s) = 0.$$

As the slant distribution is non-degenerate, therefore,

$$H^s = 0. \tag{3.79}$$

Thus, the proof follows. □

4. Totally Geodesic Foliations Determined by Distributions

In this section, we investigate the conditions for foliations determined by distributions $Rad(TN), D_1$ and D_2 to be totally geodesic.

Theorem 4.1. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, $Rad(TN)$ defines a totally geodesic foliation if and only if*

$$g(A_{n\phi_4 W}Z_1 + A_{n\phi_5 W}Z_1, tZ_2) = g(\nabla_{Z_1}tW + \nabla_{Z_1}tW, tZ_2),$$

for $Z_1, Z_2 \in \Gamma(Rad(TN))$ and $W \in \Gamma(S(TN))$.

Proof. In order to show that $Rad(TN)$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_{Z_1}Z_2 \in \Gamma(Rad(TN))$ for $Z_1, Z_2 \in \Gamma(Rad(TN))$. Using the fact that $\overline{\nabla}$ is a metric connection along with Eq.(2.13) and (2.4), for $Z_1, Z_2 \in \Gamma(Rad(TN))$ and $W \in \Gamma(S(TN))$, we get

$$g(\nabla_{Z_1}Z_2, W) = -\bar{g}(\overline{\nabla}_{Z_1}\bar{J}W, \bar{J}Z_2) - \bar{g}(\overline{\nabla}_W \bar{J}Z_1, \bar{J}Z_2) + \bar{g}(\bar{J}\overline{\nabla}_W Z_1, \bar{J}Z_2). \tag{4.80}$$

Moreover, using Eqs.(3.18) and (3.22) in Eq.(4.80), we get

$$g(\nabla_{Z_1}Z_2, W) = \bar{g}(\nabla_{Z_1}tW + \nabla_W tZ_1 - A_{n\phi_4 W}Z_1 - A_{n\phi_5 W}Z_1, tZ_2),$$

which proves the theorem. □

Theorem 4.2. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, D_1 defines a totally geodesic foliation if and only if*

- (i) *$\overline{\nabla}_{Z_1}\bar{J}W + \overline{\nabla}_W \bar{J}Z_1$ has no components along D_1, $S(TN^{\perp})$ and $\nabla_W Z_1$ has no component along D_1.*
- (ii) *$A_N Z_1$ has no component along D_1.*
- (iii) *$\nabla_{Z_1}W'$ has no component along D_1.*
- (iv) *$\nabla_{Z_1}V$ has no component along D_1.*

for $Z_1 \in \Gamma(D_1)$, $N \in \Gamma(ltr(TN))$, $W' \in \Gamma\bar{J}(ltr(TN))$, $V \in \Gamma\bar{J}(Rad(TN))$ and $W \in \Gamma(D_2)$.

Proof. Assume $Z_1, Z_2 \in \Gamma(D_1)$. To show that $\nabla_{Z_1}Z_2 \in \Gamma(D_1)$, it is sufficient to show that $\nabla_{Z_1}Z_2$ has no components along $Rad(TN)$, $\bar{J}Rad(TN)$, $\bar{J}ltr(TN)$ and D_2. For $W \in \Gamma(D_2)$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4) and (2.13), we have

$$\bar{g}(\nabla_{Z_1}Z_2, W) = -\bar{g}(\overline{\nabla}_{Z_1}\bar{J}W + \overline{\nabla}_W \bar{J}Z_1, \bar{J}Z_2) + \bar{g}(\nabla_W Z_1, Z_2). \tag{4.81}$$

For $N \in \Gamma(ltr(TN))$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4) and (2.6), we have

$$\bar{g}(\nabla_{Z_1} Z_2, N) = \bar{g}(A_N Z_1, Z_2). \tag{4.82}$$

Also, for $W' \in \Gamma(\bar{J}ltr(TN))$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4), we have

$$\bar{g}(\nabla_{Z_1} Z_2, W') = -\bar{g}(\nabla_{Z_1} W', Z_2). \tag{4.83}$$

Now, consider $V \in \Gamma\bar{J}Rad(TN)$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4), we have

$$\bar{g}(\nabla_{Z_1} Z_2, V) = -\bar{g}(\nabla_{Z_1} V, Z_2). \tag{4.84}$$

hence, the result follows from Eqs.(4.81), (4.82), (4.83) and (4.84). □

Following the same procedure as above, it can easily be shown that

Theorem 4.3. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{\nabla}_{Z_1}\bar{J}W + \overline{\nabla}_W \bar{J}Z_1$ *has no components along D_2, $S(TN^{\perp})$ and $\nabla_W Z_1$ has no component along D_2.*

(ii) $A_N Z_1$ *has no component along D_2.*

(iii) $\nabla_{Z_1} W'$ *has no component along D_2.*

(iv) $\nabla_{Z_1} V$ *has no component along D_2.*

for $Z_1 \in \Gamma(D_2)$, $N \in \Gamma(ltr(TN))$, $W' \in \Gamma\bar{J}(ltr(TN))$, $V \in \Gamma\bar{J}(Rad(TN))$ and $W \in \Gamma(D_1)$.

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References

[1] Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., & Fernandez, M. (2000). Slant submanifolds in Sasakian manifolds. Glasg. Math. J., 42, 125–138.

[2] Carriazo, A. (2002). New developments in slant submanifolds theory. Narosa Publishing House.

[3] Carriazo, A. (2000). Bi-slant immersion. In Proc. ICRAMS (pp. 88–97). Kharagpur, India.

[4] Chen, B. Y. (1990). Slant immersions. Bull. Austral. Math. Soc., 41, 135–147.

[5] Chen, B. Y. (1990). Geometry of slant submanifolds. Katholieke University, Leuven.

[6] Chen, B. Y., & Garay, O. (2012). Pointwise slant submanifolds in almost Hermitian manifolds. Turk. J. Math., 36, 630–640.

[7] Duggal, K. L., & Jin, D. H. (2003). Totally umbilical lightlike submanifolds. Kodai Math. J., 26, 49–68.

[8] Duggal, K. L., & Bejancu, A. (1996). Lightlike submanifolds of semi-Riemannian manifolds and applications (Mathematics and its Applications, Vol. 364). Kluwer Academic Publishers.

[9] Etayo, F. (1998). On quasi-slant submanifolds of an almost Hermitian manifold. Publ. Math. Debrecen, 53, 217–223.

[10] Gray, A. (1970). Nearly Kaehler manifolds. J. Diff. Geom., 4, 283–309.

[11] Gupta, G., Sachdeva, R., Kumar, R., & Nagaich, R. (2018). Pointwise slant lightlike submanifolds of indefinite Kaehler manifolds. Medterr. J. Math., 15(3).

[12] Karmakar, P. (2024). Totally contact umbilical screen-slant and screen-transversal lightlike submanifolds of indefinite Kenmotsu manifold. Mathematica Bohemica, 149(4), 603–613.

[13] Kumar, T., & Kumar, S. (2016). Screen bi-slant lightlike submanifolds of indefinite Kähler manifolds. Balkan J. Geom. Appl., 145, 30–36.

[14] Kumar, T., Pruthi, M., Kumar, S., & Kumar, P. (2021). Geometric characteristics of screen slant lightlike submanifolds of indefinite nearly Kähler manifolds. Balkan J. Geom. Appl., 26, 44–54.

[15] Lotta, A. (1996). Slant submanifolds in contact geometry. Bull. Math. Soc. Roumanie, 39, 183–198.

[16] Lotta, A. (1998). Three-dimensional slant submanifolds of K-contact manifolds. Balkan J. Geom. Appl., 3, 37–51.

[17] Pruthi, M., & Kumar, S. (2023). Pointwise bi-slant lightlike submanifolds and their warped products. J. Geom. Phys., 191(2).

[18] Papaghiuc, N. (1994). Semi-slant submanifolds of Kählerian manifold. Annal. St. Univ. Iasi., 9, 55–61.

[19] Sahin, B. (2009). Warped product submanifolds of a Kähler manifold with a slant factor. Ann. Pol. Math., 95, 207–226.

[20] Sahin, B. (2008). Slant lightlike submanifolds of indefinite Hermitian manifolds. Balkan J. Geom. Appl., 13, 107–119.

[21] Sahin, B. (2012). Slant lightlike submanifolds of indefinite Sasakian manifolds. Filomat, 26, 277–287.

[22] Sahin, B. (2009). Screen slant lightlike submanifolds. Int. Electron. J. Geom., 2, 41–54.

[23] Shukla, S. S., & Yadav, A. (2015). Semi-slant lightlike submanifolds of indefinite Kähler manifolds. Revista De La, 56, 21–37.

[24] Shukla, S. S., & Yadav, A. (2016). Screen pseudo-slant lightlike submanifolds of indefinite Kähler manifolds. Novi Sad J. Math., 46, 147–158.

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