International Journal of Maps in Mathematics

*Volume **8**, Issue **2**, 2025, Pages:**534-566***

E-ISSN: 2636-7467

www.simadp.com/journalmim

DOMINANCE INDEX- A NEW PERSPECTIVE FOR DECISION-MAKING USING DUAL HESITANT FUZZY SOFT SETS

DEEPA V, ATHIRA T M *, MANJUNATH A S, AND SUNIL JACOB JOHN

Abstract. The dual hesitant fuzzy soft set (DHFSS), a hybrid structure of a dual hesitant fuzzy set and a soft set, is highly effective in handling membership and non-membership values using a set of possible values. This article explores an entirely different application of DHFSS for representing preliminary data involved in decision-making problems. Moreover, an innovative measure for comparing DHFSSs, namely the Dominance Index, which determines the dominance of one DHFSS over another, is presented. Furthermore, a linear algebraic approach, integrated with the Dominance Index of a dual hesitant fuzzy element, is proposed for solving decision-making problems. Finally, a real-life decision-making problem involving the evaluation of mobile tower work sites based on the performance of their workers is presented and solved using the proposed method to demonstrate its applicability.

Keywords: Decision-Making, Dual Hesitant Fuzzy Soft Set, Dominance Index, Ranking, MCDM.

2020 Mathematics Subject Classifications: 03E72, 90Bxx.

1. Introduction

The main objectives of research on hesitant fuzzy sets and their related hybrid structures are the construction of methods for solving Multi-Criteria Decision-Making(MCDM) problems. Since Torra [1] proposed the hesitant fuzzy set in 2010, many MCDM problems

Received: 2024.11.26 *Revised: 2025.03.23* *Accepted: 2025.05.09*

* Corresponding author

Deepa V ⋄ deepasunil33@gmail.com ⋄ https://orcid.org/0000-0001-6257-1368

Athira T M ⋄ athira.tm@manipal.edu ⋄ https://orcid.org/0000-0001-5358-233X

Manjunath A S ⋄ manjunadem@rit.ac.in ⋄ https://orcid.org/0000-0002-0853-4328

Sunil Jacob John ⋄ sunil@nitc.ac.in ⋄ https://orcid.org/0000-0002-6333-2884.

have been solved using structures such as hesitant fuzzy sets [2, 3, 4, 5], dual hesitant fuzzy sets [6, 7, 8], hesitant fuzzy soft sets [9, 10] and dual hesitant fuzzy soft sets [11, 12, 13]. MCDM problems have an inevitable place in most real-life situations. It is understood that an MCDM problem deals with the evaluation of a set of alternatives based on a set of decision criteria. This paper provides an innovative method for presenting data of an MCDM problem using a dual hesitant fuzzy soft set and also develops a method for processing that data, and thereby arriving at a reliable decision. MCDM problems can be categorized into three types based on the nature of its criteria, as (i) all categories are crisp, (ii) all are fuzzy, and (iii) mixed type. Among these, this paper focuses is of the second type because the criteria of the problem presented here exhibit some hesitancy.

Among those structures handling fuzziness and uncertainty, the dual hesitant fuzzy soft set seems to be a promising tool in MCDM due to its ability to simultaneously handle fuzziness, parametrization, hesitancy, and non-membership. As a starting point in developing the concept of the dual hesitant fuzzy soft set, Y. He [11] developed a method to encompass and solve decision-making problems using a dual hesitant fuzzy soft set. Further, he also proposed a technique for ranking the alternatives in the problem. Following this, several studies [14, 16] have been conducted in the area of including the introduction of process such as the proposal of concepts like distance [17], similarity [17], aggregation operators [18], and correlation coefficients [12] for comparing two dual hesitant fuzzy soft sets. Recently, studies on weighted hesitant fuzzy soft sets [31] have been developed, where a weight vector is assigned to all possible membership degrees of each element.

Decision-making problems are often associated with uncertainty and imprecision that cannot be effectively solved using classical fuzzy set models alone. Dual hesitant fuzzy sets extend hesitant fuzzy sets by considering multiple membership and non-membership values, thus providing a more comprehensive representation of uncertainty. However, in many real-world scenarios, decision-making problems involve multiple parameters that must be evaluated simultaneously. Therefore, considering more flexible and adaptive models is essential. Soft-set theory offers a parameterized approach that provides a systematic and effective mechanism for dealing with uncertainty in decision-making. Integrating soft sets into the dual hesitant fuzzy model enables more efficient modeling of multi-criteria decision problems, concurrent treatment of multiple attributes and their associated uncertainties, and improving the flexibility and adaptability of decision models to better reflect the complexity of the real world.

Usually, a decision-making problem involves several alternatives, their parameters, and evaluations corresponding to different parameters of each alternative. The goal is to rank these alternatives based on assessments. However, in real-life situations, each alternative may consist of a structure that includes other types of alternatives and their parameters, making the ranking process more complex. The dominance index is a widely used measure in decision-making frameworks to compare and evaluate alternatives, particularly in fuzzy and hesitant fuzzy environments. It determines the extent to which one alternative dominates another. Early work by Zadeh [32] on fuzzy sets laid the foundation for dominance-based comparisons, which were later extended to hesitant fuzzy sets and dual hesitant fuzzy sets, a more refined representation of uncertainty was achieved, leading to the development of the dominance index for comparing DHFSS elements.

Fuzzy sets [32] and soft set [33] frameworks have made significant progress, leading to the development of various generalized models that extend traditional approaches to solve more complex decision-making problems. (2,1)-fuzzy sets [34] introduce a more refined approach by incorporating weighted aggregate operators, enhancing their applicability in multi-criteria decision-making (MCDM) methods. Similarly, (3,2)-fuzzy sets [35] extend this concept to higher dimensions and find application in topology and optimal choice theory, enabling more sophisticated modeling of uncertainty in decision systems. A further generalization is provided by (m, n)-fuzzy set [36, 37], which establish a generalized framework for orthopair fuzzy sets and provide a robust framework for addressing MCDM problems. Furthermore, (a, b)-fuzzy soft sets [38] represent a new class of fuzzy soft sets that consider multiple attributes, thus improving the decision-making process by incorporating a broader range of evaluations. Finally, K_m^n-Rung picture fuzzy sets extend traditional fuzzy models by including multiple degrees of membership, non-membership, and hesitation. This makes them suitable for capturing complex uncertainties in real-world problems. These contributions pave the way for more flexible and powerful tools in decision-making and significantly enrich the theoretical foundations of fuzzy and soft-set frameworks.

After depicting the data using dual hesitant fuzzy soft sets, which are the building blocks of the problem, the next challenge was to compare the dual hesitant fuzzy elements efficiently. Here, the authors made use of the fact that a dual hesitant fuzzy set is an extension of hesitant fuzzy set. After exploring various approaches for comparing hesitant fuzzy elements like aggregation method [19, 20], entropy method [5, 21] distance and similarity measure method [22, 23] etc., the authors concluded inclusion measure approach is the most suitable one for this purpose. Speaking of the inclusion measure, it has a long history. It has originated from the so-called relation subset-hood. The inclusion measure is a relation that can be seen as the fuzzification of the crisp inclusion relation. It is a very useful tool for comparing objects in a wide range of fields such as fuzzy sets [24], intuitionistic fuzzy sets [25, 26], hesitant fuzzy sets [27], interval neutrosophic sets [28], etc. Using the techniques of inclusion measure, the authors have developed a method to quantify the dominance of one object over another. Since this dominance index fails to satisfy the transitivity condition, only a pairwise comparison is possible. Here, the authors have modified this approach in accordance with their purpose. The endogenous cardinalization [29] provided by this approach enables researchers to quantify each object's achievement in addition to merely ranking them. In this paper, the researchers have also depicted an evaluation problem to illustrate the practicability of their approach.

This paper is organized as follows: The first section discusses some concepts that are needed for the further sequel. The second and third sections introduces the dual hesitant fuzzy Maclaurin symmetric mean and the weighted dual hesitant fuzzy Maclaurin symmetric mean. A partial order and hybrid monotonic inclusion measure for dual hesitant fuzzy elements are presented in the fourth section. After that, we move on to discussing the methodology for ranking the objects in an evaluation problem. In the final section, we present a real-life problem to demonstrate the efficacy of the proposed method.

2. Preliminaries

This section provides essential definitions and background concepts that serve as the foundation for the remainder of this article. Also, throughout this paper, HFS, DHFS, DHFSS and DHFE stands for hesitant fuzzy set, dual hesitant fuzzy set, dual hesitant fuzzy soft set and dual hesitant fuzzy element respectively.

2.1. **HFS, DHFS, and DHFSS:** The following are definitions, associated concepts and supporting examples for HFS, DHFS, and DHFSS.

Definition 2.1. [1] *Let X be a reference set, a hesitant fuzzy set (HFS) E on X is defined in terms of a function* h *that when applied to X returns a subset of* $[0, 1]$.

To be easily understood, Xu and Xia [15] *expressed an HFS by the following mathematical form:*

$$\mathrm{E} = \{< \mathrm{x}, \mathrm{h}(\mathrm{x}) > / \mathrm{x} \in \mathrm{X}\},$$

where $h(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xu and Xia [15] called $h(x)$ a hesitant fuzzy element (HFE).

Definition 2.2. [2] *Let X be a fixed set, then a dual hesitant fuzzy set* (DHFS) D *on X is described as:*

$$\mathtt{D} = \{< \mathtt{x}, \mathtt{h}(\mathtt{x}), \mathtt{g}(\mathtt{x}) >, \mathtt{x} \in \mathtt{X}\},$$

in which $\mathtt{h}(\mathtt{x})$ *and* $\mathtt{g}(\mathtt{x})$ *are two sets of some values in* $[0,1]$ *denoting the possible membership degrees and nonmembership degrees of the element* $\mathtt{x} \in \mathtt{X}$ *to the set* D, *respectively with the conditions:*

$$0 \leq \gamma, \eta \leq 1, \quad 0 \leq \gamma^+ + \eta^+ \leq 1,$$

where $\gamma \in \mathtt{h}(\mathtt{x}), \eta \in \mathtt{g}(\mathtt{x}), \gamma^+ \in \mathtt{h}^+(\mathtt{x}) = \cup_{\gamma \in \mathtt{h}(\mathtt{x})} \mathtt{max}\{\gamma\}$ *and* $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} max\{\eta\}$, *for all $x \in X$. For convenience, the pair* $\mathtt{d}(\mathtt{x}) = (\mathtt{h}(\mathtt{x}), \mathtt{g}(\mathtt{x}))$ *is called a dual hesitant fuzzy element(*DHFE*), denoted by* $\mathtt{d} = (\mathtt{h}, \mathtt{g})$.
Denote by DHFS(U), *the set of all Dual Hesitant fuzzy sets over* U.

Definition 2.3. [3] *Let* (U, E) *be a soft universe and* $\mathtt{A} \subseteq \mathtt{E}$. *A pair* $Ç = (\tilde{\mathtt{F}}, \mathtt{A})$ *is called a Dual hesitant fuzzy soft set (DHFSS) over* U, *where* $\tilde{\mathtt{F}}$ *is a mapping given by* $\tilde{\mathtt{F}} : \mathtt{A} \to \mathtt{DHFS}(\mathtt{U})$. *In general* $\tilde{\mathtt{F}}(\mathtt{e})$ *can be written as,*

$$\tilde{\mathtt{F}}(\mathtt{e}) = \{< \mathtt{x}, \mathtt{h}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x}), \mathtt{g}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x}) > / \mathtt{x} \in \mathtt{U}\},$$

where $\mathtt{h}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x})$ and $\mathtt{g}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x})$ *are two sets of some values in* $[0,1]$, *denoting the possible membership degrees and non membership degrees that object* x *holds on parameter* e, *respectively.*

To represent dual hesitant fuzzy soft sets concisely, Y.He [3] proposed a tabular representation, which is depicted in the following example in detail.

Example 2.1. [3] *Let* U *be a set of four participants performing dance program, which is denoted by* $\mathtt{U} = \{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4\}$. *Let* E *be a parameter set, where*

$$\mathtt{E} = \{\mathtt{e}_1, \mathtt{e}_2, \mathtt{e}_3\} = \{\mathtt{confident}, \mathtt{creative}, \mathtt{graceful}\}.$$

Suppose that there are three judges who are invited to evaluate the membership degrees and non-membership degrees of a candidate $\mathtt{x}_\mathtt{j}$ *to a parameter* $\mathtt{e}_\mathtt{i}$ *with several possible values in* $[0,1]$. *Then the tabular representation of dual hesitant fuzzy soft set* $Ç = (\tilde{\mathtt{F}}, \mathtt{A})$ *defined as below by Table 2.1 gives the evaluation of the performance of candidates by three judges.*

Table 2.1. Tabular Representation of dual hesitant fuzzy soft set $Ģ = (\tilde{F}, A)$

U	e_1	e_2	e_3
x_1	{.6,.7,.8}{.3,.2,.1}	{.5,.6,.4}{.4,.3,.2}	{.4,.4,.3}{.7,.6,.6}
x_2	{.4,.5,.6}{.3,.2,.1}	{.5,.4,.3}{.5,.3,.3}	{.5,.7,.7}{.3,.2,.2}
x_3	{.8,.7,.7}{.2,.1,.1}	{.7,.8,.8}{.2,.2,.1}	{.5,.6,.7}{.3,.2,.1}
x_4	{.3,.4,.4}{.6,.5,.4}	{.5,.6,.6}{.4,.3,.2}	{.7,.6,.8}{.2,.1,.1}

To compare the DHFEs, Zhu et al.[2] introduced the following comparison laws:

Definition 2.4. [2] *The score and accuracy function of a* DHFE $d = (h, g)$ *are*

$$s_d = (1/\#h) \sum_{\gamma \in h} \gamma - (1/\#g) \sum_{\eta \in g} \eta$$

and

$$p_d = (1/\#h) \sum_{\gamma \in h} \gamma + (1/\#g) \sum_{\eta \in g} \eta$$

respectively, where $\#h$ *and* $\#g$ *are the number of elements in* h *and* g *respectively, then*

i. *if* $s_{d_1} > s_{d_2}$, *then* d_1 *is superior to* d_2

ii. *if* $s_{d_1} = s_{d_2}$, *then*

   1. *if* $p_{d_1} = p_{d_2}$, *then* d_1 *is equivalent to* d_2, *denoted by* $d_1 \sim d_2$
   2. *if* $p_{d_1} > p_{d_2}$, *then* d_1 *is superior than* d_2, *denoted by* $d_1 \succ d_2$

In [2], Zhu et al. proposed the following operational laws for DHFEs :

Definition 2.5. [2] *Let* $d = (h, g)$, $d_1 = (h_1, g_1)$ *and* $d_2 = (h_2, g_2)$ *be three* DHFEs, *then*

(1) $d_1 \oplus d_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \{\eta_1\eta_2\}\}$

(2) $d_1 \otimes d_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\{\gamma_1\gamma_2\}, \{\eta_1 + \eta_2 - \eta_1\eta_2\}\}$

(3) $nd = \bigcup_{\gamma \in h, \eta \in g} \{\{1 - (1-\gamma)^n\}, \{\eta^n\}\}$

(4) $d^n = \bigcup_{\gamma \in h, \eta \in g} \{\{\gamma^n\}, \{1 - (1-\eta)^n\}\}$

The following assumptions are made in the rest of the paper:

* Elements of h and g are arranged in increasing order.
* $\mathbb{H}$ denote the set of all finite subsets of $[0, 1]$ whose elements are arranged in increasing order.
* $d = (h, g; l, l')$ represents a dual hesitant fuzzy element $(h, g) \in \mathbb{H} \times \mathbb{H}$, with $l(h) = l$ and $l(g) = l'$.

2.2. **The Maclaurin Symmetric Mean.** Due to its ability to capture the inter-relationship among the multi-input arguments, the Maclaurin symmetric mean (MSM), introduced by Maclaurin [4], has a prominent place in the list of aggregation operators. The MSM is defined as follows:

Definition 2.6. [4] *Let $a_i, \{i = 1, 2, ..., n\}$ be a collection of non-negative real numbers, and $k \in \{1, 2, \ldots, n\}$. If*

$$\mathtt{MSM}^{(k)}(a_1, a_2, ..., a_n) = \left(\frac{\sum\limits_{1 \le i_1 < i_2 < ... < i_k \le n} \left(\prod\limits_{j=1}^{k} a_{i_j} \right)}{C_n^k} \right)^{1/k},$$

then $\mathtt{MSM}^{(k)}$ is called the Maclaurin symmetric mean (MSM), where $(i_1, i_2, ..., i_k)$ traverse through all the k-tuples combinations of $(1, 2, \ldots, n)$, and C_n^k is the binomial coefficient.

In 2015, Quin et al.[5] extend the notion of MSM to hesitant fuzzy environment and defined hesitant fuzzy Maclaurin symmetric mean (HFMSM) as follows:

Definition 2.7. [4] *Let $\mathtt{h_i}, (\mathtt{i} = 1, 2, ..., \mathtt{n})$ be a collection of HFEs and $k = 1, 2, ..., n$. If*

$$\mathtt{HFMSM}^{(k)}(h_1, h_2, ..., h_n) = \left(\frac{\bigoplus\limits_{1 \le i_1 < i_2 < ... < i_k \le n} \left(\bigotimes\limits_{j=1}^{k} h_{i_j} \right)}{C_n^k} \right)^{1/k},$$

then $\mathtt{HFMSM}^{(k)}$ is called the hesitant fuzzy Maclaurin symmetric mean (HFMSM) operator.

3. The Dual Hesitant Fuzzy Maclaurin Symmetric Mean

The evaluation problem we discussed in this paper has dual hesitant fuzzy soft framework. Meanwhile, we need an aggregation operator that reflect the inter-relationship among the arguments. As Maclaurin symmetric mean is a right candidate for this purpose, we define dual hesitant fuzzy Maclaurin symmetric mean in this section as follows.

Definition 3.1. *Let $\mathtt{d_j} = (\mathtt{h_j}, \mathtt{g_j}), (\mathtt{j} = 1, 2, ..., \mathtt{n})$ be a group of DHFEs and $\mathtt{k} \in \{1, 2, ..., \mathtt{n}\}$. If*

$$\mathtt{DHFMSM}^{(k)}(\mathtt{d_1}, \mathtt{d_2}, ..., \mathtt{d_n}) = \left(\frac{\bigoplus\limits_{1 \le i_1 < i_2 < ... < i_k \le n, i_j \in \mathbb{Z},\ \forall j=1 \text{ to } k} \left(\bigotimes\limits_{j=1}^{k} \mathtt{d_{i_j}} \right)}{C_n^k} \right)^{1/k},$$

where C_n^k denote the number of combinations of n things taken k at a time. Then $\mathtt{DHFMSM}^{(k)}$ *is called the dual hesitant fuzzy Maclaurin symmetric mean (*DHFMSM*) operator. Here* $\overset{k}{\underset{j=1}{\otimes}} \mathtt{d}_{\mathtt{i}_\mathtt{j}}$ *reflects the interrelationship among* $\mathtt{d}_{\mathtt{i}_1}, \mathtt{d}_{\mathtt{i}_2}, ..., \mathtt{d}_{\mathtt{i}_\mathtt{k}}$.

The following theorem exhibits a nice representation for the DHFMSM operator.

Theorem 3.1. Let $\mathtt{d_j} = (\mathtt{h_j}, \mathtt{g_j}), (\mathtt{j} = 1, 2, ..., \mathtt{n})$ be a collection of DHFEs and $\mathtt{k} \in \{1, 2, ..., \mathtt{n}\}$, then the aggregated value of $\mathtt{d_j}, \mathtt{j} = 1, 2, ..., \mathtt{n}$ using the proposed DHFMSM operator is again a DHFE, given by

$$\mathtt{DHFMSM}^{(k)}(\mathtt{d}_1, \mathtt{d}_2, ..., \mathtt{d}_\mathtt{n}) = (\bar{\mathtt{h}}, \bar{\mathtt{g}}),$$

where,

$$\bar{\mathtt{h}} = \bigcup_{\gamma_1 \in \mathtt{h}_1, ..., \gamma_\mathtt{n} \in \mathtt{h}_\mathtt{n}} \left\{ \left(1 - \left[\prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left(1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} \gamma_{\mathtt{i}_\mathtt{j}} \right) \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

and

$$\bar{\mathtt{g}} = \bigcup_{\eta_1 \in \mathtt{g}_1, ..., \eta_\mathtt{n} \in \mathtt{g}_\mathtt{n}} \left\{ 1 - \left(1 - \left[\prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left[1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} (1 - \eta_{\mathtt{i}_\mathtt{j}}) \right] \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

where $\mathtt{S} = \{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} / 1 \leq \mathtt{i}_1 < \mathtt{i}_2 < ... < \mathtt{i}_\mathtt{k} \leq \mathtt{n}\}$ and C_n^k denote the number of combinations of n things taken k at a time.

Proof. Using the operations of DHFEs given by definition 2.6(1-4), we have

$$\overset{k}{\underset{j=1}{\otimes}} \mathtt{d}_{\mathtt{i}_\mathtt{j}} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j} \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \prod_{\mathtt{j}=1}^{\mathtt{k}} \gamma_{\mathtt{i}_\mathtt{j}} \right\}, \left\{ 1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} (1 - \eta_{\mathtt{i}_\mathtt{j}}) \right\} \right\}$$

Eventually, we have

$$\bigoplus_{1 \leq \mathtt{i}_1 < \mathtt{i}_2 < ... < \mathtt{i}_\mathtt{k} \leq \mathtt{n}} \overset{k}{\underset{j=1}{\otimes}} \mathtt{d}_{\mathtt{i}_\mathtt{j}} = \bigcup_{\gamma_\mathtt{i} \in \mathtt{h}_\mathtt{i}, \eta_\mathtt{i} \in \mathtt{g}_\mathtt{i}} \left\{ \left\{ 1 - \prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left(1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} \gamma_{\mathtt{i}_\mathtt{j}} \right) \right\}, \left\{ \prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left(1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} (1 - \eta_{\mathtt{i}_\mathtt{j}}) \right) \right\} \right\}$$

and

$$\frac{\bigoplus_{1 \leq \mathtt{i}_1 < \mathtt{i}_2 < ... < \mathtt{i}_\mathtt{k} \leq \mathtt{n}} \overset{k}{\underset{j=1}{\otimes}} \mathtt{d}_{\mathtt{i}_\mathtt{j}}}{C_n^k} = \bigcup_{\gamma_\mathtt{i} \in \mathtt{h}_\mathtt{i}, \eta_\mathtt{i} \in \mathtt{g}_\mathtt{i}} \left\{ \left\{ 1 - \left[\prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left(1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} \gamma_{\mathtt{i}_\mathtt{j}} \right) \right]^{1/C_n^k} \right\}, \right.$$

$$\left\{\prod_{(i_1,i_2,\ldots,i_k)\in S}\left[1-\prod_{j=1}^{k}(1-\eta_{i_j})\right]\right\}^{1/C_n^k}\Bigg\}$$

Therefore,

$$\left(\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\ \overset{k}{\underset{j=1}{\otimes}} d_{i_j}}{C_n^k}\right)^{1/k}=\bigcup_{\gamma_i\in h_i,\eta_i\in g_i}\left\{\left\{\left(1-\left[\prod_{(i_1,\ldots,i_k)\in S}\left(1-\prod_{j=1}^{k}\gamma_{i_j}\right)\right]^{\frac{1}{C_n^k}}\right)^{1/k}\right\},\right.$$

$$\left.\left\{1-\left(1-\left[\prod_{(i_1,\ldots,i_k)\in S}\left[1-\prod_{j=1}^{k}(1-\eta_{i_j})\right]\right]^{\frac{1}{C_n^k}}\right)^{1/k}\right\}\right\}.$$

This completes the proof. □

The weighted dual hesitant fuzzy Maclaurin symmetric mean

In DHFMSM operator, every DHFE receives the same importance. But real-life decision-making situations demand different priorities for parameters and categories. So we have to incorporate the concept of weights in DHFMSM operator. Therefore, in this section, we shall propose the weighted dual hesitant fuzzy Maclaurin symmetric mean operator, which is defined as follows:

Definition 3.2. *Let $d_j=(h_j,g_j),(j=1,2,...,n)$ be a collection of DHFEs and $k\in\{1,2,...,n\}$. Let $w=(w_1,w_2,...,w_n)^T$ is the weight vetor, where w_j indicates the degree of importance of d_j, satisfying $w_j\in[0,1],\ j=1,2,...,n$ and $\sum_{j=1}^{n}w_j=1$. If*

$$\mathrm{WDHFMSM}_w^{(k)}(d_1,d_2,...,d_n)=\left(\frac{\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\left(\bigotimes_{j=1}^{k}(d_{i_j})^{w_{i_j}}\right)}{C_n^k}\right)^{1/k},$$

then WDHFMSM $^{(k)}$ is called the weighted dual hesitant fuzzy Maclaurin symmetric mean (WDHFMSM) operator.

According to the operations of DHFEs exhibited in section 2, we can derive the following theorem.

Theorem 3.2. Let $d_j=(h_j,g_j),(j=1,2,...,n)$ be a collection of DHFEs and $k\in\{1,2,...,n\}$, then the aggregated value of $d_j, j=1,2,...,n$ using the WDHFMSM operator is also a DHFE, given by

$$\mathtt{WDHFMSM}_{w}^{(k)}(d_1, d_2, ..., d_n) = (\bar{h}, \bar{g}),$$

where,

$$\bar{h} = \bigcup_{\gamma_1 \in h_1, ..., \gamma_n \in h_n} \left\{ \left(1 - \prod_{(i_1, i_2, ..., i_k) \in S} \left[1 - \prod_{j=1}^{k} \gamma_{i_j}^{w_{i_j}} \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

and

$$\bar{g} = \bigcup_{\eta_1 \in g_1, ..., \eta_n \in g_n} \left\{ 1 - \left(1 - \prod_{(i_1, i_2, ..., i_k) \in S} \left[1 - \prod_{j=1}^{k} (1 - \eta_{i_j})^{w_{i_j}} \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

where $S = \{(i_1, i_2, ..., i_k) \in \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} / 1 \leq i_1 < i_2 < ... < i_k \leq n\}$.
Here $\bigotimes_{j=1}^{k} (d_{i_j})^{w_{i_j}}$ reflects the inter-relationship among $d_{i_1}, d_{i_2}, ..., d_{i_k}$.

4. A novel partial order and hybrid monotonic inclusion measures for DHFEs

As previously mentioned, the framework of our evaluation problem is based on dual hesitant fuzzy soft sets. To effectively compare such sets, it is necessary to define an order relation on the set of DHFEs. In [12], Zhang et al. proposed a partial order for hesitant fuzzy elements (HFEs) using disjunctive semantic interpretation. In this work, we extend their approach to the dual hesitant fuzzy context.

Definition 4.1. *Let* $d_1 = (h_1, g_1; l_1, l_1^{'}), d_2 = (h_2, g_2; l_2, l_2^{'})$ *be two* DHFEs. *We define an order relation* $\leq^{\S}$ *between* d_1 *and* d_2 *as follows:*

$$d_1 \leq^{\S} d_2 \quad \textit{iff} \quad \begin{cases} h_1^i \leq h_2^i, \ \forall\, i = 1 \cdots l_1 \quad \textit{if} \quad l_1 \leq l_2 \\ \\ h_1^{l_1 - l_2 + i} \leq h_2^i, \ \forall\, i = 1 \cdots l_2 \quad \textit{Otherwise} \\ \\ \qquad \textit{and} \\ \\ g_1^j \geq g_2^j, \ \forall\, j = 1 \cdots l_2{}' \quad \textit{if} \quad l_1^{'} \geq l_2^{'} \\ \\ g_1^j \geq g_2^{l_2^{'} - l_1^{'} + j}, \ \forall\, j = 1 \cdots l_1^{'} \quad \textit{Otherwise} \end{cases}$$

For any two DHFS A and B on X, $A \subseteq^{\S} B$ iff $d_A(x) \leq^{\S} d_B(x), \forall x \in X$. The ordered set is denoted by $(\mathtt{DHFS}(X), \subseteq^{\S})$. We can easily prove the following theorem.

Theorem 4.1. $(\mathbb{H}\times \mathbb{H}, \leq^{\S})$ is a partially ordered set. Moreover, $\subseteq^{\S}$ is a partial order on $\mathtt{DHFS(X)}$.

Proof.

(1) Reflexive: Clearly the reflexive property hold for $\leq^{\S}$.

(2) Antisymmetric: Let $d_1 \leq^{\S} d_2$ and $d_2 \leq^{\S} d_1$, where $d_1 = (h_1, g_1; l_1, l_1')$ and $d_2 = (h_2, g_2; l_2, l_2')$. Now, the anti-symmetry of $\leq^{\S}$ can be easily proved using the monotonicity property of h as well as g and using the definition of $\leq^{\S}$. To prove $d_1 = d_2$, we have to consider the following four cases:

(i): $l_1 \leq l_2$ and $l_1' \geq l_2'$.

(ii): $l_1 \leq l_2$ and $l_1' \leq l_2'$.

(iii): $l_1 \geq l_2$ and $l_1' \leq l_2'$.

(iv): $l_1 \geq l_2$ and $l_1' \geq l_2'$.

Case(i): From $d_1 \leq^{\S} d_2, l_1 \leq l_2\ l_1' \geq l_2'$, we get, $h_1^i \leq h_2^i, \forall i = 1 \cdots$. l_1 and $g_1^j \geq g_2^j, \forall j = 1 \cdots l_2'$. Also, from $d_2 \leq^{\S} d_1$, we get $h_2^{l_2 - l_1 + i} \leq h_1^i,\ \forall\ i = 1 \cdots l_1$, and $g_2^j \geq g_1^{l_1' - l_2' + j},\ \forall\ j = 1 \cdots l_2'$. By increasing property of h_2, it follows that $h_1^i \leq h_2^i \leq h_2^{l_2 - l_1 + i} \leq h_1^i,\ \forall\ i = 1 \cdots l_1$.
From this, it is evident that $l_1 = l_2$ and $h_1^i = h_2^i,\ \forall\ i = 1 \cdots l_1$.
Again by increasing property of g_1, we get $g_1^j \geq g_2^j \geq g_1^{l_1' - l_2' + j} \geq g_1^j,\ \forall\ j = 1 \cdots l_2'$. From the above inequalities, it is clear that $l_1' = l_2'$ and $g_1^j = g_2^j,\ \forall\ \ j = 1 \cdots l_1'$. Thus we prove $d_1 = d_2$. Here we depict only one of the four above cases; others can be proved similarly.

(3) Transitive: Let $d_1 \leq^{\S} d_2$ and $d_2 \leq^{\S} d_3$, where $d_1 = (h_1, g_1; l_1, l_1')$, $d_2 = (h_2, g_2; l_2, l_2')$, $d_3 = (h_3, g_3; l_3, l_3')$. We claim that $d_1 \leq^{\S} d_3$. We can easily prove our claim using the transitivity property of $\leq$, monotonicity property of h as well as g and the definition of $\leq^{\S}$. Here we have to consider several cases, but it is only a routine calculations. So we demonstrate only one case and others are left to the reader. Suppose $l_3 \leq l_1 \leq l_2$ and $l_3' \leq l_1' \leq l_2'$. From $d_1 \leq^{\S} d_2, l_1 \leq l_2$ and $l_1' \leq l_2'$, we get, $h_1^i \leq h_2^i,\ \forall\ i = 1 \cdots l_1$ and $g_1^j \geq g_2^{l_2' - l_1' + j},\ \forall\ j = 1 \cdots l_1'$. Also, from $d_2 \leq^{\S} d_3, l_3 \leq l_2$ $l_3' \leq l_2'$, we get, $h_2^{l_2 - l_3 + i} \leq h_3^i,\ \ \forall\ i = 1 \cdots l_3$ and $g_2^j \geq g_3^j,\ \forall\ j = 1 \cdots l_3'$. Applying the increasing property of h_2 and g_2 together with the inequality $l_2 - l_3 \geq l_1 - l_3$, we get
$h_1^{l_1 - l_3 + i} \leq h_2^{l_1 - l_3 + i} \leq h_2^{l_2 - l_3 + i} \leq h_3^i,\ \forall\ i = 1 \cdots l_3$ and

$\mathtt{g}_1^{\mathtt{j}} \geq \mathtt{g}_2^{\mathtt{l}_2'-\mathtt{l}_1'+\mathtt{j}} \geq \mathtt{g}_2^{\mathtt{j}} \geq \mathtt{g}_3^{\mathtt{j}}$, $\forall\ \mathtt{j} = 1 \cdots \mathtt{l}_3'$. From the observations $\mathtt{l}_1 \geq \mathtt{l}_3$ and $\mathtt{l}_1' \geq \mathtt{l}_3'$, transitivity is obvious. Hence the proof.

□

It is well known that a partially ordered set generally contains elements that are not mutually comparable. The presence of such elements naturally leads to the need for an inclusion measure. Therefore, we define an inclusion measure on the partially ordered set $(\mathbb{H}\times\mathbb{H}, \leqslant^{\S})$. In the following, we first provide an axiomatic definition of the inclusion measure. According to H. Y. Zhang [41], hybrid monotonicity is essential for a rational generalization of inclusion measures. Accordingly, we define a hybrid monotonic inclusion measure on $(\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$.

Definition 4.2. *Let* $\mathtt{d}_1, \mathtt{d}_2 \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$. *A real number* $\mathfrak{Inc}(\mathtt{d}_1, \mathtt{d}_2) \in [0, 1]$ *is called an HM inclusion measure between* $\mathtt{d}_1$ *and* $\mathtt{d}_2$, *if* $\mathfrak{Inc}(\mathtt{d}_1, \mathtt{d}_2)$ *satisfies the following properties.*

- **(ID1):** $\mathfrak{Inc}(\mathtt{d}_1, \mathtt{d}_2)$ *=1 if and only if* $\mathtt{d}_1 \leq^{\S} \mathtt{d}_2$
- **(ID2):** *If* $\mathtt{d} = \overline{1}$, *then* $\mathfrak{Inc}(\mathtt{d}, \mathtt{d}^c)$ *= 0, where* $\overline{1} = (\{1\}, \{0\})$
- **(ID3):** *If* $\mathtt{d}_1 \leq^{\S} d_2$, *then for any* $\mathtt{d}_3 \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$, $\mathfrak{Inc}(d_3, d_1) \leq \mathfrak{Inc}(d_3, d_2)$, $\mathfrak{Inc}(d_2, d_3) \leq \mathfrak{Inc}(d_1, d_3)$

To study the structure of an inclusion measure, axiomatic approach is the best choice. But, our aim is to use inclusion measure in a decision making problem. So we are more interested in constructive approach. In the following section, we present a concrete example for inclusion measure which satisfies our proposed axioms.

Proposition 4.1. *For* $\mathtt{d}_1 = (\mathtt{h}_1, \mathtt{g}_1; \mathtt{l}_1, \mathtt{l}_1')$ *and* $\mathtt{d}_2 = (\mathtt{h}_2, \mathtt{g}_2; \mathtt{l}_2, \mathtt{l}_2') \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$, *let*

$$\mathfrak{Inc}(d_1, d_2) = \mathtt{s}_{(\ \mathtt{d}_{\mathcal{L}}(d_1, d_2)\ )}$$

where $\mathtt{s}_\mathtt{d}$ *is the score function of the* DHFE $\mathtt{d}$ *and* $\mathtt{d}_{\mathcal{L}}(d_1, d_2) = (\mathtt{h}, \mathtt{g})$ *is a* DHFE, *called* $\mathcal{L}$-*subsethood index of* $\mathtt{d}_1$ *and* $\mathtt{d}_2$, *where,*

$$\mathtt{h} = \begin{cases} \bigcup_{\mathtt{i}=1}^{\mathtt{l}_1} \mathfrak{I}_{\mathtt{L}}(\mathtt{h}_1^{\mathtt{i}}, \mathtt{h}_2^{\mathtt{i}}), & \textit{if} \quad \mathtt{l}_1 \leq \mathtt{l}_2 \\ \bigcup_{\mathtt{i}=1}^{\mathtt{l}_2} \mathfrak{I}_{\mathtt{L}}(\mathtt{h}_1^{\mathtt{l}_1-\mathtt{l}_2+\mathtt{i}}, \mathtt{h}_2^{\mathtt{i}}), & \textit{Otherwise} \end{cases}$$

and

$$\mathtt{g} = \begin{cases} \bigcup_{\mathtt{j}=1}^{\mathtt{l}_1'} \mathfrak{I}_{\mathtt{L}}(\mathtt{g}_2^{\mathtt{l}_2'-\mathtt{l}_1'+\mathtt{j}}, \mathtt{g}_1^{\mathtt{j}}), & \textit{if} \quad \mathtt{l}_1' \leq \mathtt{l}_2' \\ \bigcup_{\mathtt{j}=1}^{\mathtt{l}_2'} \mathfrak{I}_{\mathtt{L}}(\mathtt{g}_2^{\mathtt{j}}, \mathtt{g}_1^{\mathtt{j}}), & \textit{Otherwise} \end{cases}$$

Here, $\mathfrak{I}_{\mathtt{L}}(\mathtt{x},\mathtt{y}) = \mathtt{min}(1,\ 1-\mathtt{x}+\mathtt{y})$ *is the well-known R-implicator based on Lukasiewicz t-norm, viz., Lukasiewicz implicator, proposed by* [40]. *Then* $\mathfrak{Inc}(d_1, d_2)$ *is an HM-inclusion measure for* DHFE.

Proof. We can easily verify the axiomatic requirements (ID1), (ID2) and (ID3) of HM-inclusion measure for DHFE. Hence Q.E.D. □

The concept of HM-inclusion measure was defined in this section intending to use it in our evaluation problem, but there we want a DHFE. We know HM-inclusion measure is not a DHFE. So we have decided to use the $\mathcal{L}$-subset hood index instead of HM-inclusion measure in the evaluation problem. The necessity of DHFE into the evaluation problem had led us to take this decision.

In the following section, we discuss our evaluation problem and develop a methodology for ranking the objects in the problem by make use of the proposed definitions in this paper.

5. A novel methodology for ranking objects in an evaluation problem

The decision-making problem is being described as follows. $\mathtt{q}$ objects $\mathtt{F}_1, \mathtt{F}_2, ..., \mathtt{F}_\mathtt{q}$ shall be compared in our problem. Here each $\mathtt{F}_\mathtt{i}$ may be a branch of a company or a project under a vendor. Further to this, each of the q objects will be characterized as the parameterized collection of subsets of the universal set U, where U consists of categories of workers belonging to the object $\mathtt{F}_\mathtt{i}$. Let $\mathtt{U} = \Big\{x_1, x_2, ..., x_n.\Big\}$ be the universal set and $\mathtt{E} = \Big\{e_1, e_2, ..., e_m\Big\}$ be the parameter set. Here, E consists of parameters which are defined by experts in the relevant field. Moreover, the character of parameters of this problem is fuzzy. Also, the universal set U and the parameter set E are one and the same for all the q objects in this problem. Nevertheless, the number of workers in each category $\mathtt{x}_\mathtt{s}$ in different $\mathtt{F}_\mathtt{i}$ may vary. Also, note that the number of workers in distinct categories $\mathtt{x}_\mathtt{s},\ \mathtt{s} \in \{1, 2, ..., \mathtt{n}\}$ in the same object $\mathtt{F}_\mathtt{i}$ may be different. From these observations, we arrived at the conclusion that the evaluation of categories $\mathtt{x}_\mathtt{s},\ \mathtt{s} \in \{1, 2, ..., \mathtt{n}\}$ for the parameters $\mathtt{e}_\mathtt{r},\ \mathtt{r} \in \{1, 2, ..., \mathtt{m}\}$ in the object $\mathtt{F}_\mathtt{i},\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$ can be better presented by using an HFE or a DHFE. Since the provision for assigning negative mark is an added benefit for an assessment procedure, dual hesitant fuzzy element seems to be a better representative than hesitant fuzzy element. Thus, we constructed a dual hesitant fuzzy soft set $(\mathtt{F_i}, \mathtt{A}),\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$ for describing the object $\mathtt{F_i},\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$. Further, $(\mathtt{F_i}, \mathtt{A}),\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$ can be implicitly described as

$$(\mathtt{F_i}, \mathtt{A}) = \left\{ \mathtt{d}_{\mathtt{F_i(e_r)}}(\mathtt{x_s}) = \left(\mathtt{h}_{\mathtt{F_i(e_r)}}(\mathtt{x_s}), \mathtt{g}_{\mathtt{F_i(e_r)}}(\mathtt{x_s})\right) / \ \mathtt{r} = 1, ..., \mathtt{m} \text{ and } \mathtt{s} = 1, ..., \mathtt{n}. \right\}$$

and we denote $(\mathtt{F_i}, \mathtt{A}),\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$ by simply $\mathtt{F_i},\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$. Here $\mathtt{h}_{\mathtt{F_i(e_r)}}(\mathtt{x_s})$ and $\mathtt{g}_{\mathtt{F_i(e_r)}}(\mathtt{x_s})$ represent the sets of memberships and non-memberships of the workers in the category $\mathtt{x_s}$ to the set describing the parameter $\mathtt{e_r}$, respectively. We develop the following method for ranking these $\mathtt{F_i},\ \mathtt{i} = 1, 2, ..., \mathtt{q}$ by being motivated from the work of Herrero[29].

Step 1:: Consider two objects $\mathtt{F_i}$ and $\mathtt{F_j}$. Form the collection of $\mathcal{L}$-subsethood indexes, viz.,

$\{\mathtt{d}_{\mathcal{L}}\left(\mathtt{d}_{\mathtt{F_j(e_r)}}(\mathtt{x_s}), \mathtt{d}_{\mathtt{F_i(e_r)}}(\mathtt{x_s})\right) :\ \mathtt{r} = 1, ..., \mathtt{m}\ ;\ \mathtt{s} = 1, ..., \mathtt{n}\}$.

Step 2:: In this step, we fix $\mathtt{s}$. i.e., we consider the category $\mathtt{x_s}$. Here a weight vector for the parameters of this category must be defined by the decision makers, viz., $\mathtt{w}^{(\mathtt{s})}$ such that $\mathtt{w}^{(\mathtt{s})} = \left(\mathtt{w}_1^{(\mathtt{s})}, \mathtt{w}_2^{(\mathtt{s})}, ..., \mathtt{w}_\mathtt{m}^{(\mathtt{s})}\right)$, $\mathtt{w}_\mathtt{r}^{(\mathtt{s})} \in [0, 1]$ and $\sum_{\mathtt{r}=1}^{\mathtt{m}} \mathtt{w}_\mathtt{r}^{(\mathtt{s})} = 1$, where $\mathtt{w}_\mathtt{r}^{(\mathtt{s})}$ indicates the importance of the parameter $\mathtt{e_r}$ to the alternative $\mathtt{x_s}$. Then using the WDHFMSM operator and the weight vector $\mathtt{w}^{(\mathtt{s})}$, the $\mathcal{L}$-subsethood indexes are aggregated as follows:

$$\left(\frac{\bigoplus\limits_{1 \leq \mathtt{r}_1 < \mathtt{r}_2 < ... < \mathtt{r}_\mathtt{k} \leq \mathtt{m}} \left(\bigotimes\limits_{\mathtt{t}=1}^{\mathtt{k}} \left[\mathtt{d}_{\mathcal{L}}\left(\mathtt{d}_{\mathtt{F_j(e_{r_t})}}(\mathtt{x_s}), \mathtt{d}_{\mathtt{F_i(e_{r_t})}}(\mathtt{x_s})\right) \right]^{\mathtt{w}_{\mathtt{r_t}}} \right)}{\mathtt{C}_\mathtt{m}^\mathtt{k}} \right)^{1/\mathtt{k}},$$

which is a DHFE denoted by $\delta^{(\mathtt{s})}(\mathtt{F_j}, \mathtt{F_i})$.

Step 3:: Repeat step 2 for each $\mathtt{x_s},\ \mathtt{s} \in \{1, 2, ..., \mathtt{n}\}$ and the collection $\{\delta^{(\mathtt{s})}(\mathtt{F_j}, \mathtt{F_i}) :\ \mathtt{s} = 1, ..., \mathtt{n}\}$ are formed.

Step 4:: Before proceeding further, the weight vector deciding the importance of categories should be determined by the decision makers.

Let it be $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\mathtt{n})$, $\lambda_\mathtt{s} \in [0, 1]$ and $\sum_{\mathtt{s}=1}^{\mathtt{n}} \lambda_\mathtt{s} = 1$, where $\lambda_\mathtt{s}$ indicates the importance of the alternative $\mathtt{x_s}$. Note that this weight vector λ is same for all the objects $\mathtt{F_i},\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$.

The Weighted geometric aggregation mean operator, proposed by Xia[15], could be used here for final aggregation, viz., $\bigotimes\limits_{\mathtt{s}=1}^{\mathtt{n}} \left(\delta^{(\mathtt{s})}(\mathtt{F_j}, \mathtt{F_i})\right)^{\lambda_\mathtt{s}}$, which is again a DHFE denoted by $\delta(\mathtt{i}, \mathtt{j})$. In our problem, we want to pay more attention to arguments having too high or too low performance. It justifies our decision of choosing WGM operator.

Step 5:: Now we find out the status of the DHFE $\delta(\mathtt{i},\mathtt{j})$, i.e., $\mathtt{S}_{(\delta(\mathtt{i},\mathtt{j}))}$, and it is denoted by $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$. Since $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$ is ultimately derived from $\mathcal{L}$-subsethood index of $\mathtt{F_j}$ over $\mathtt{F_i}$, $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$ gives out the degree of dominance of $\mathtt{F_i}$ relative to $\mathtt{F_j}$. Hence it will be called dominance index of $(\mathtt{F_i},\mathtt{A})$ over $(\mathtt{F_j},\mathtt{A})$.

Step 6:: Repeat steps 1-5 for $\mathtt{i},\mathtt{j} = 1,2,3,...,\mathtt{q}$, $\mathtt{i} \neq \mathtt{j}$. Thus we find out all combinations of dominance index and let us denote this collection by P, viz., $P = \{\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) : i,j = 1,2,...,q \ ; \ i \neq j\}$. Then P can be viewed as a comprehensive form of our evaluation problem.

Here we discuss the following remarks about the dominance index.

Remark 5.1. (i) $0 \leq \mathcal{D}(\mathtt{F_i},\mathtt{F_j}) \leq 1$.

(ii) $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 1 \Rightarrow \mathtt{F_i}$ *is completely dominant with respect to* $\mathtt{F_j}$ *in all aspects.*

(iii) $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 0 \Rightarrow$ *Stunning performance by the first object* $\mathtt{F_i}$ *while no performance at all by the second object* $\mathtt{F_j}$. *In a real working site, this will never happen. So in this paper, without loss of generality, we assume* $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) > 0$.

(iv) *For a fixed j, if* $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 1, \forall \mathtt{i},\ \mathtt{i} \neq \mathtt{j}$, *then* $\mathtt{F_j}$ *is inferior to every other objects. In that case* $\mathtt{F_j}$ *can be eliminated from further evaluation process and can be given the last rank. This remark shall be used later in this paper.*

Besides the above-said properties, this dominance index could be used for the pairwise comparison of objects. i.e., $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) \leq \mathcal{D}(\mathtt{F_j},\mathtt{F_i}) \Rightarrow \mathtt{F_i} \leq \mathtt{F_j}$ or literally, $\mathtt{F_j}$ dominates $\mathtt{F_i}$.

We know inclusion measure doesn't satisfy the transitive relation, and also the dominance index is derived from inclusion measure. So that, this newly introduced measure 'dominance index' is not suitable for the comparison of more than two objects. If this measure is being utilized in our problem, we need to extend this into more general settings which involve more than two objects. Towards this aim, some definitions are proposed as follows.

Definition 5.1. *Relative dominance of* $\mathtt{F_i}$ *with respect to* $\mathtt{F_j}$ *is given by*

$$\mathtt{R}(\mathtt{i},\mathtt{j}) = \frac{\mathcal{D}(\mathtt{F_i},\mathtt{F_j})}{\sum_{k=1,k\neq i}^{q} \mathcal{D}(\mathtt{F_k},\mathtt{F_i})} \ , \qquad \mathtt{i},\mathtt{j} = 1,2,3,...,\mathtt{q} \ , \ \mathtt{i} \neq \mathtt{j}.$$

Now, the net dominance of $\mathtt{F_i}$ can be defined as the weighted average of its relative dominance.

Definition 5.2. *Net dominance of* $\mathtt{F_i}$ *is given by*

$$\mathtt{N}(\mathtt{i}) = \sum_{\mathtt{j}=1,\mathtt{j}\neq\mathtt{i}}^{\mathtt{q}} \mathtt{w_j}\mathtt{R}(\mathtt{i},\mathtt{j}), \quad \mathtt{i} = 1,2,3,...,\mathtt{q}.$$

where $\mathtt{w_j}$ *is a measure of the importance of the object* $\mathtt{F_j}, \mathtt{j} = 1, 2, 3, ..., \mathtt{q}, \mathtt{i} \neq \mathtt{j}$.

Here we wish to mention one thing. Both $\mathtt{N(i)}$ and $\mathtt{w_i}$ give out the rank of the object $\mathtt{F_i}$. Initially, both of them are unknown, and our aim is to find the rank of the object $\mathtt{F_i}$ which may be $\mathtt{N(i)}$ or $\mathtt{w_i}$. If there exist an invariant system of weights $(\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ satisfying $(\mathtt{N(1)}, \mathtt{N(2)}, ..., \mathtt{N(q)}) = (\mathtt{w_1}, \mathtt{w_2}, ..., \mathtt{w_q}) = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$, and $\mathtt{N(i)} = \sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}} \mathtt{w_j R(i,j)}, \mathtt{i} = 1, ..., \mathtt{q}$, then we are succeeded in this journey. This will be achieved by applying a little bit theories of linear algebra here. For that, a matrix $\mathtt{P}^* = [\mathtt{p_{ij}}]$ will be constructed as follows:

$$\mathtt{p_{ij}} = \begin{cases} \mathcal{D}(\mathtt{F_i}, \mathtt{F_j}), & i \neq j \\ (q-1) - \sum_{k=1,k\neq i}^{q} \mathcal{D}(\mathtt{F_k}, \mathtt{F_i}), & i = j \end{cases}$$

For the matrix $\mathtt{P}^*$, the following observations have been made.

- $\mathtt{P}^*$ is a positive matrix. This observation results from the properties of dominance index discussed earlier.
- Each column sum of $\mathtt{P}^*$ is q-1.
- $\mathtt{P}^*$ is an irreducible matrix.

Using the matrix $\mathtt{P}^*$, we can construct an eigenvalue problem $\mathtt{P}^*\mathtt{X} = \lambda \mathtt{X}$, $\mathtt{X} = \begin{bmatrix} w_1 & w_2 & ... & w_q \end{bmatrix}^{\mathtt{T}}$, which is equivalent to the comprehensive form of the problem $\mathtt{P}$. From now on, we consider this eigenvalue problem instead of $\mathtt{P}$. Such a transformation gives the benefit of solving the evaluation problem consistently and uniquely. From the characteristics of $\mathtt{P}^*$, it is clear that $\mathtt{q}-1$ is the unique dominant eigenvalue of $\mathtt{P}^*$. According to Perron-Frobenius theorem, the matrix $\mathtt{P}^*$ has a strictly positive eigenvector corresponding to the eigenvalue $\mathtt{q}-1$, viz., $\mathtt{V} = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ with $\mathtt{P}^*\mathtt{V} = (\mathtt{q}-1)\mathtt{V}$, where $\mathtt{v_i} = \dfrac{\sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}} (\mathtt{v_j} * \mathcal{D}(\mathtt{F_i}, \mathtt{F_j}))}{\sum_{\mathtt{k=1,k\neq i}}^{\mathtt{q}} \mathcal{D}(\mathtt{F_k}, \mathtt{F_i})}, \mathtt{i} = 1, 2, 3, ..., \mathtt{q}$. Also, we know that this eigen vector is unique up to scalar multiplication. So we can make this eigen vector unique by imposing the condition $\sum_{\mathtt{i=1}}^{\mathtt{n}} \mathtt{v_i} = \mathtt{q}$. Thus, a unique and consistent system of weights $(\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ satisfying $(\mathtt{N(1)}, \mathtt{N(2)}, ..., \mathtt{N(q)}) = (\mathtt{w_1}, \mathtt{w_2}, ..., \mathtt{w_q}) = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ and $\mathtt{N(i)} = \sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}} \mathtt{w_j R(i,j)}$, $\mathtt{i} = 1, 2, 3, ..., \mathtt{q}$ have been obtained.

This vector $\mathtt{V} = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ is called the worth vector [29] associated with our evaluation problem $\mathtt{P}$. Usually, in a ranking method, the decision-maker consider which one is better than the other; nevertheless, they need not calculate how much better it is. But, here we need this feature. We think that the differences between preferences are also important. Here, Herrero's worth vector provide this feature. Each component of this vector gives the worth associated with a respective object. In other words, it cardinalizes the objects. According to Herrero [29], the worth vector provides not only a complete ranking of the objects under consideration but also an endogenous cardinalization that allows a quantitative estimate of their differences. We can now use the following observations of Herrero [29].

- $v_i > v_j \Rightarrow$ the object F_i is dominant with respect to the object F_j.
- The condition $\sum_{i=1}^{q} v_i = q$ allows us to identify the objects which are above or below the average.
- There exist a consistent evaluation function f which associates an evaluation problem P to its worth vector.i.e., $f(P) = V$, where $V = (v_1, v_2, ..., v_q)$ satisfying $P^*V = (q-1)V$ and $\sum_{i=1}^{q} v_i = q$. This function f enable us to handle distinct evaluation problems consistently and uniquely.

For ranking, we adopt the following steps.

Step 1:: First sort out v_i's in descending order.

Step 2:: Let the sorted vectors be $u_1, u_2, ..., u_q$.

Step 3:: If $u_i = v_j$, then the rank of the object F_j is i. Also, its worth is v_j. Repeat this for every $i = 1, 2, 3, ..., q$. Thus our evaluation have been completed .

To assess the performance of the proposed method, in the following section, we depict a problem of continuous evaluation of workers in a mobile tower construction site.

Practical Example. SU Square Projects and Infrastructures (P) Ltd - an ISO 9001: 2008 certified company - is primarily engaged in the construction and maintenance of mobile communication towers for various passive infrastructure providers in the telecom sector in Kerala. The promoters of the company aim to provide optimal coverage to rural and mountainous areas throughout Kerala and southern India. To achieve this, they have developed several strategies to implement targeted and efficient actions.

As part of its business expansion, the company's promoters have decided to evaluate workers based on a predetermined performance package. The primary objective of this initiative is to minimize the time required to complete tower installations without compromising quality.

Through this evaluation, they aim to tap into the full potential of each employee. To foster healthy competition, they have decided to rank the various sites according to worker performance. In addition, gifts have been included in the package as incentives, directly linked to site rankings. This initiative has understandably generated interest and enthusiasm among the workers.

If this ranking can be cardinalized, that is, expressed in numerical terms, the distribution of perks can be carried out in a more consistent and objective manner. It is worth recalling here that our proposed ranking method allows for such cardinalization, thereby enhancing its practical applicability. As the next step, we proceed to analyze the compatibility of the problem's structure with the proposed framework.

Suppose $\mathtt{F}_1, \mathtt{F}_2, ..., \mathtt{F}_\mathtt{q}$ are $\mathtt{q}$ sites considered for evaluation. The selection of the appropriate assessment criteria is an inevitable part of the evaluation process. This selection should be made by experts in the respective fields. In the background of years of experience, the promoters select the appropriate parameters for the evaluation. The list of parameters and their descriptions are given in Table 5.2. Let the set of parameters be denoted by A, that is, $\mathtt{A} = \{\mathtt{TAT}, \mathtt{Quality}, \mathtt{Safety}, \mathtt{Costing}\}$.

Table 5.2. Parameters List

Parameters	**Descriptions**	**Notation**
Turnaround Time (TAT)	The time taken to complete a particular project	TAT
Quality of Work	Maintaining required quality	Quality
Complying with Safety Norms	Every workforce must comply, e.g., wearing safety helmets, safety shoes, using barricades, signboards, etc.	Safety
Project Costing	Costs incurred for a particular project, including material and labor costs	Costing

Next, our discussion turns to the workforce. Each tower-working-site would have needed different categories of workers. Those categories of workers provided by the promoters are as shown in Table 5.3.

Table 5.3. List of categories of workers together with their description

Categories of workers	Details about no. of workers	Notation	Weights
Civil Engineer	One engineer per site	CE	.2
Electrical Engineer	One engineer per site	EE	.25
Mason	Each site will have two, three or four masons	MN	.15
Helper (Mason)	The number of helpers depends upon the number of masons. A site requires six masons including helpers	HM	.05
Electrician	Each site will have three electricians.	EN	.1
Trainees(Electrician)	For helping electricians,there are two trainees.	ET	.05
Riggers	Normally 6 riggers per site, but for Roof Top Towers(RTT) it is 7	RS	.08
Head load workers for each site is 10	The number of head load workers	HL	.07
Concrete Labors	A site requires 20 concrete labors but RTT needs only 10	CL	.05

Here we would like to indicate some important points. From the description of categories of workers, it is clear that each category may have more than one members. So to get a better picture, we have to evaluate them individually. Also, note that different categories may contain a distinct number of deputies. Further, the number of employees belongs to the same category in two distinct sites may not be the same. These observations have led us to choose the hesitant fuzzy elements as an appropriate structure for representing the evaluation of a category based on a parameter. The provision for assigning negative marking is an added benefit for an assessment. So that, the dual hesitant fuzzy element seems to be the better representative rather than the hesitant fuzzy element. Thus we arrive at the conclusion that the dual hesitant fuzzy soft set is used for exhibiting the evaluation details of a mobile tower site. This demonstration shall be described as follows. Here, $\mathtt{U} = \{\mathtt{CE}, \mathtt{EE}, \mathtt{MN}, \mathtt{HM}, \mathtt{EN}, \mathtt{ET}, \mathtt{RS}, \mathtt{HL}, \mathtt{CL}\}$, the set of category names of workers, are taken as the universal set. The dual hesitant fuzzy soft set $(\mathtt{F}_1, \mathtt{A})$ represents the evaluation measurements of $\mathtt{F}_1$. This DHFSS can be briefly described as follows. $\mathtt{F}_1$ is a mapping given by $\mathtt{F}_1 : \mathtt{A} \rightarrow \mathtt{DHFS(U)}$. Here, $(\mathtt{F}_1, \mathtt{A}) = \{\mathtt{F}_1(\mathtt{TAT}), \mathtt{F}_1(\mathtt{Quality}), \mathtt{F}_1(\mathtt{Safety}), \mathtt{F}_1(\mathtt{Costing})\}$, where each $\mathtt{F}_1(.)$ is a dual hesitant fuzzy set. To get better clarification, we discuss the case of a particular $\mathtt{F}_1(.)$, viz., $\mathtt{F}_1(\mathtt{TAT})$. Here $\mathtt{F}_1(\mathtt{TAT})$ is a dual hesitant fuzzy set which assigns to each member of U a dual hesitant fuzzy element. For example, Riggers, $\mathtt{RS} \in \mathtt{U}$, the corresponding dual hesitant fuzzy element is $\mathtt{d}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS}) = (\mathtt{h}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS}), \mathtt{g}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS}))$ where $\mathtt{h}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS})$ is a finite subset of [0,1] consisting of either 6 or 7 entries which represents the evaluation given to Riggers working at $\mathtt{F}_1$ for TAT. In other words, $\mathtt{h}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS})$ gives the membership of Riggers to the set which describes TAT. Similarly, $\mathtt{g}_{\mathtt{F}_1(\mathtt{TAT})}(\mathtt{RS})$ provides the non-membership of Riggers to the set which describes TAT. We know elements of DHFEs are arranged in increasing order. Here also, the marks obtained by different Riggers working at $\mathtt{F}_1$ could be arranged in increasing order. Our evaluation is about sites and not about employees. So that, there is no ambiguity in arranging the marks in this manner.

In a similar manner, we construct $\mathtt{F}_1(\mathtt{Quality}), \mathtt{F}_1(\mathtt{Safety}), \mathtt{F}_1(\mathtt{Costing})$ and thus formed $(\mathtt{F}_1, \mathtt{A})$, denoted by $\mathtt{F}_1$ Likewise we build $(\mathtt{F}_2, \mathtt{A})$ for site $\mathtt{F}_2$, $(\mathtt{F}_3, \mathtt{A})$ for site $\mathtt{F}_3$, and $(\mathtt{F}_4, \mathtt{A})$ for site $\mathtt{F}_4$, which are denoted by $\mathtt{F}_2, \mathtt{F}_3, \mathtt{F}_4$ respectively. In this way, we have accommodated successfully all the information provided by the experts. Now, by all means, we have been convinced that the proposed method is the suitable method for this problem. Thus, we are moving onto solving the problem using the proposed method.

TABLE 5.4. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_1} = \left(\tilde{\mathtt{F_1}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.9} {.1}	{.98} {.101}	{.96} {.001}	{.97} {.12}
EE	{.85} {.1}	{.88} {.01}	{.89} {.11}	{.84} {.005}
MN	{.91,.92} {.2}	{.93,.98} {.1}	{..94,.95} {.22}	{.99,.992} {.101}
HM	{.85,.86,.868,.869} {.2}	{.886,.89,.895,.93} {.13}	{.92,.94,.949,.952} {.1}	{..981,.983,.985,.989} {.09}
EN	{.9,.92,.923} {.103}	{..941,.942,.948} {.141}	{.891,.892,.894} {.12}	{..86,.864,.865} {.2}
ET	{.92,.95} {.201}	{.93,.96} {.138}	{.91,.93} {.17}	{..98,.99} {.142}
RS	{.91,.92,925,.927, .93,.934} {..005}	{.96,.964,.967,.972, .974,.98} {.02}	{.81,.83,.836,.84, .847,.85} {.001,.003}	{.91,.913,.924,.926, .929,.93} {.1}
HL	{.71,.714,.719,.723, .725,.727,.738,.739, .74,.743} {.07}	{.732,.736,.74,.745, .749,.76,.762,.765, .769,.78} {.156}	{..813,.824,.83,.845, .86,.864,.869,.87, .881,.883} {.1}	{..91,.913,.915,.95, .98,.981,.982,.984, .985,.989} {.09}
CL	{..73,.734,.735,.738, .739,.741,.742,.746, .749,.75,.752,.753, .755,.757,.758,.76, .762,.763,.765,.768} {.102}	{.81,.814,.815,.82, .834,.836,.838,.841, .843,.845,.846,.849, .852,.856,.859,.862, .864,.868,.87,.89} {.18}	{.91,.913,.915,.921, .924,.926,.928,.929, .932,.934,.935,.937, .938,.94,.942,.943, .944,.945,.946,.95} {.21}	{.87,.876,.877,.88, .884,.886,.887,.89, .892,.893,.895,.9, .92,.93,.95,.97, .972,.975,.977,.979} {.08}

TABLE 5.5. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_2} = \left(\tilde{\mathtt{F_2}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.8} {.1}	{..72} {.2}	{.7} {.07}	{.8} {.11}
EE	{.6} {.18}	{.65} {.121}	{..61} {.109}	{.63} {.123}
MN	{.7,.72,.74,.746} {.28}	{..77,.792,.798,.81} {.105}	{.75,.756,.7567,.761} {.127}	{.791,.794,.81,.83} {.174}
HM	{..52,.54} {.108}	{.59,.61} {.113}	{..61,.63} {.101}	{.67,.69} {.161}
EN	{..73,.74,.75} {.28}	{..716,.723,.74} {.26}	{.74,.743,.745} {.23}	{..732,.735,.761} {.21}
ET	{.62,.65} {.102}	{.68,.685} {.195}	{..645,..672} {.138}	{.656,.692} {.124}
RS	{.634,.639,.642,.645, .649,.651,.657} {.101}	{.636,.654,.672,.675, .681,.692,.71} {.131}	{..621,.628,.634,.637, .639,.64,.642} {.159}	{..624,.628,.637,.645, .676,.684,.692} {.128}
HL	{.71,.718,.72,.723, .727,.734,.74,.749, .752,.76} {.197}	{..76,.762,.763,.771, .772,.78,.794,.799, .88,.89} {.111}	{.81,.83,.85,.872, .876,.88,.882,.884, .887,.89} {.17}	{..83,.832,.834,.847, .849,.852,.853,.855, .857,.86} {.09}
CL	{..61,.672,.689,.692, .694,.696,.698,.71, .72,.726} {.001}	{.52,.525,.529,.531, .535,.538,.542,.545, .549,.559} {.007}	{.61,.68,.694,.712, .724,.75,.758,.778, .79,.81} {.0012}	{.634,.691,.695,.724, .728,.729,.73,.739, .743,.745} {.0089}

TABLE 5.6. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_3} = \left(\tilde{\mathtt{F_3}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.8} {.12}	{.82} {.23}	{.9} {.116}	{.85} {.017}
EE	{.9} {.101}	{.92} {.119}	{.85} {.181}	{.93} {.192}
MN	{.73,.81,.84,.91} {.21}	{.82,.85,.87,.89} {.101}	{.61,.82,.84,.89} {.10001}	{.91,.935,.94,.942} {.002}
HM	{.812,.823} {.11}	{.71,.75} {.15}	{.52,.61} {.05}	{.92,.95} {.07}
EN	{.93,.941,.95} {.1002}	{.941,.945,.95} {.001}	{.95,.953,.96} {.023}	{.936,.939,.95} {.008}
ET	{.82,.85} {.071}	{.836,.851} {.082}	{.72,.75} {.14}	{.91,.95} {.21}
RS	{.71,.73,.79,.85, .88,.92,.95} {.106}	{.91,.912,.918,.923, .934,.94,.945} {.11}	{.81,.815,.819,.821, .828,.834,.836} {.105}	{.852,.854,.86,.865, .872,.874,.88} {.009}
HL	{.71,.712,.734,.745, .82,.832,.838,.84, .85,.9} {.006}	{.81,.812,.815,.832, .86,.865,.881,.92, .925,.928} {.001}	{.91,.915,.918,.923, .927,.932,.938,.941, .943,.95} {.006}	{.8,.82,.824,.83, .835,.839,.85,.853, .86,.868} {.08}
CL	{.9,.913,.917,.924, .931,.936,.939,.94, .948,.95} {.172}	{.82,.825,.828,.831, .833,.84,.852,.853, .864,.87} {.025}	{.71,.72,.724,.73, .738,.74,.742,.744, .75,.752} {.173}	{.78,.81,.85,.91, .92,.925,.93,.938, .942,.945} {.087}

Table 5.7. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_4} = \left(\tilde{\mathtt{F_4}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.6} {.102}	{.7} {.189}	{.6} {.076}	{.9} {.045}
EE	{.95} {.108}	{.98} {.009}	{.8} {.0004}	{.92} {.153}
MN	{.8,.85,.87} {.11}	{.76,.81,.9} {.137}	{.9,.91,.93} {.023}	{.84,.89,.9} {.001}
HM	{.6,.65,.67} {.1}	{.8,.89,.92} {.02}	{.92,.94,.96} {.03}	{.87,.88,.89} {.132}
EN	{.78,.79,.85} {.076}	{.82,.84,.87} {.23}	{.74,.78,.79} {.2}	{.91,.92,.95} {.13}
ET	{.65,.68} {.122}	{.85,.87} {.114}	{.91,.94} {.176}	{.92,.96} {.13}
RS	{.84,.87,.89,.92, .94,.956,.97} {.106}	{.91,.912,.919,.923, .928,.934,.95} {.12}	{.71,.73,.74,.752, .759,.76,.769,.928, .934,.95} {.104}	{.91,.918,.92,.924, .928,.93,.934} {.113}
HL	{.65,.676,.685,.694, .725,.738,.824,.839, .841,.852} {.21}	{.91,.934,.943,.952, .956,.959,.964,.968, .969,.97} {.008}	{.7,.75,.78,.791, .792,.798,.82,.83, .845,.852} {.2}	{.9,.923,.941,.949, .95,.954,.958,.961, .962,.97} {..089}
CL	{.918,.925,93,.934, .938,.942,.946,.948, .951,.953} {.06}	{.94,.941,.943,.947, .952,.954,.957,.961, .962,.964} {.087}	{.71,.78,.79,.794, .799,.81,.845,.848, .849,.852} {.1}	{.92,.94,.945,.947, .952,.953,.957,.959, .962,.963} {.01}

TABLE 5.8. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_5} = \left(\tilde{\mathtt{F_5}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.4} {.1}	{.32} {.07}	{.38} {.13}	{.42} {.12}
EE	{.2} {.12}	{.3} {.04}	{.12} {.021}	{.45} {.118}
MN	{.31,.342,.36,.41} {.1}	{.36,.378,.394} {.17}	{.24,.245,.253,.26} {.03}	{.41,.423,.445,.45} {.071}
HM	{.2,.45} {.01}	{.35,.42} {.12}	{.13,.32} {.13}	{.34,.39} {.232}
EN	{.42,.435,.44} {.16}	{.38,.382,.39} {.13}	{.24,.28,.3} {.11}	{.45,.49,.53} {.18}
ET	{.51,.56} {.22}	{.43,.47} {.14}	{.27,.292} {.16}	{.53,.58} {.103}
RS	{.23,.274,.282,.287, .29,.3,.31} {.006}	{.31,.312,.313,.317, .32,.325,.327,.329} {.102}	{.134,.178,.18,.193, .195,.198,.21} {.17}	{.42,.43,.436,.442, .445,.456,.458} {.023}
HL	{.21,.223,.23,.242, .245,.251,.267,.35, .4,.42} {.211}	{.31,.33,.37,.48, .51,.53,.57,.59,.62} {.108}	{.12,.124,.127,.132, .135,.182,.24,.29, .3,.34} {.12}	{.41,.414,.418,.423, .425,.43,.478,.48, .482,.485} {..019}
CL	{.34,.345,.348,.352, .354,.357,.389,.392, .395,.41} {.16}	{.43,.434,.437,.439, .448,.449,.452,.46, .47,.475} {.17}	{.21,.213,.215,.237, .297,.299,.32,.33, .375,.39} {.012}	{.13,.17,.19,.21, .214,.218,.24,.248, .25,.27} {.11}

Table 5.9. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_6} = \left(\tilde{\mathtt{F_6}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.1} {.001}	{.01} {.0007}	{.2} {.103}	{.12} {.012}
EE	{.05} {.012}	{.03} {.004}	{.13} {.0021}	{.15} {.107}
MN	{.04,.07,.09,.11} {.009}	{.02,.03,.07,.08} {.107}	{.04,.05,.09,.11} {.13}	{.1,.11,.13,.14} {.061}
HM	{.13,.18} {.009}	{.15,.19} {.106}	{.13,.17} {.036}	{.14,.17} {.152}
EN	{.132,.135,.137} {.034}	{.048,.05,.08} {.15}	{.14,.148,.152} {.196}	{.136,.145,.178} {.007}
ET	{.046,.078} {.002}	{.021,.042} {.001}	{.091,.098} {.109}	{.34,.41} {.121}
RS	{.12,.124,.183,.24, .29,.34,.42} {.016}	{.042,.09,.098,.13, .139,.14,.172} {.202}	{.012,.017,.019,.034, .039,.052} {.107}	{.02,.026,.031,.038, .042,.043,.047} {.003}
HL	{.013,.016,.018,.021, .023,.026,.031,.034, .035,.039} {.101}	{.024,.026,.028,.029, .032,.034,.036,.038, .039,.043} {.1}	{.049,.051,.053,.054, .061,.08,.123,.137, .139,.152} {.0012}	{.52,.525,.585,.592, .61,.618,.624,.63, .631,.637} {.19}
CL	{.14,.145,.1452,.151, .159,.162,.168,.169, .171,.172} {.1006}	{.18,.183,.184,.192, .199,.23,.234,.24, .26,.263} {.1007}	{.04,.045,.053,.078, .092,.098,.13,.132, .153,.157} {.0124}	{.2,.22,.234,.247, .249,.25,.253,.257, .259,.261} {.011}

TABLE 5.10. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_7} = \left(\tilde{\mathtt{F_7}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.95} {.01}	{.99} {.00002}	{.97} {.0001}	{.99} {.0101}
EE	{.97} {.001}	{.99} {.004}	{.91} {.0501}	{.94} {.008}
MN	{.94,.95,.96} {.001}	{.97,.99,.999} {.107}	{.97,.98,.99} {.0103}	{.997,.998,.999} {.0401}
HM	{.91,.913,.914} {.0071}	{.912,.913,.915} {.0102}	{.96,.967,.98} {.103}	{.991,.993,.995} {.0232}
EN	{.961,.964,.978} {.016}	{.97,.973,.975} {.019}	{.965,.969,.972} {.011}	{.961,.964,.967} {.018}
ET	{.952,.96} {.022}	{.961,.972} {.014}	{.948,.952} {.016}	{.993,.997} {.0103}
RS	{.967,.968,.969,.97, .972,.974,.976} {.003}	{.981,.982,.983,.985, .987,.989,.99} {.105}	{.86,.87,.89,.9, .92,.93,.94} {.007}	{.964,.966,.967,.97, .971,.973,.98} {.0203}
HL	{.88,.882,.885,.887, .89,.892,.895,.897, .91,.93} {.011}	{.961,.963,.967,.969, .971,.973,.975,.977, .981,.983} {.008}	{.951,.953,.955,.957, .958,.96,.963,.965, .967,.969} {.0102}	{.986,.988,.989,.991, .992,.994,.995,.997, .998,.999} {.0019}
CL	{.951,.953,954,.955, .956,.958,.959,.961, .962,.964} {.00409}	{.962,.963,.964,.965, .967,.968,.969,.97, .971,.972} {.0024}	{.951,.952,.953,.954, .955,.956,.957,.959, .962,.967} {.0014}	{.981,.9823,.983,.9834, .9835,.984,.9842,.9844, .9846,.985} {.0001}

Table 5.11. Weight vectors which decide the importance of parameters in each category.

U/E	TAT	Quality	Safety	Costing
Civil Engineer	.35	.2	.1	.35
Electrical Engineer	.35	.2	.1	.35
Mason	.3	.25	.15	.3
Helper(Mason)	.35	.1	.2	.35
Electrician	.3	.2	.2	.3
Trainees(Electrician)	.3	.2	.2	.3
Riggers	.3	.25	.25	.2
Headload workers	.3	.1	.2	.4
Concrete labors	.3	.15	.15	.4

The tabular representation of F1, F2, F3, ..., F7 respectively, formed from the information provided by the promoters, are shown in tables 4, 5, 6, ..., 10. Recall that the weight vector for categories is shown in the last column of Table 5.3.
i.e., $\lambda = (0.2, 0.25, 0.15, 0.05, 0.1, 0.05, 0.08, 0.07, 0.05)$.

Table 5.11 provides the weight assigned by experts for the parameters in each category. In this table, each row represents the weight vector for the respective category in that row. For example, the first row corresponds to the weight vector $w^{(1)} = (0.35, 0.2, 0.1, 0.35)$ for `civilengineer`. That is, for the `civilengineer`, 0.35 weight is given for `TAT`, 0.2 for `Quality`, 0.1 for `Safety`, and 0.35 for `Costing`. Similarly, the fifth row gives the weight vector $w^{(5)} = (0.3, 0.2, 0.2, 0.3)$ for `Electrician`, the seventh row provides the weight vector $w^{(7)} = (0.3, 0.25, 0.25, 0.2)$ for the `Riggers`, and so on.

Thus the building blocks of the evaluation, namely, weights and evaluations, were obtained. Now the authors proceeded to construct P, the comprehensive form of the problem. For that passes steps 1 through 6 and calculated $D(F_i, F_j)$, $i, j = 1, 2, 3, 4$ $i \neq j$. Further, the matrix P^* can be constructed by using the comprehensive problem P, which is as shown below.

$$
\mathrm{P}^* = \begin{bmatrix}
.7263177 & .981574 & .977197 & .976024 & 1 & .981677 & .968049 \\
.949087 & .4481468 & .958576 & .960466 & 1 & 1 & .93128 \\
.986 & 1 & .6628048 & .998439 & 1 & 1 & .977028 \\
.9175843 & .9348083 & .9285823 & .5822471 & .9372493 & .9372493 & .9093323 \\
.7509 & .8298589 & .7658639 & .7717499 & .1227487 & .9283619 & .7206219 \\
.670111 & .805612 & .706976 & .711074 & .940002 & .1527118 & .614004 \\
1 & 1 & 1 & 1 & 1 & 1 & .8796848
\end{bmatrix}
$$

P^* instead of P is demonstrated because of limited space. The authors created an eigen-value problem $P^*X = \lambda X$, using this P^*. From the previous discussion, it is obvious that 6 (that is., $q - 1$) is the dominant eigenvalue. The objective is to determine an eigen-vector of this dominant eigenvalue. Here the authors are looking for the unique eigen vector $(v_1, v_2, ..., v_q)$ satisfying the condition $\sum_{i=1}^{q} v_i = q$. There are numerous methods and corresponding softwares available in the literature for finding out the eigen vector of an eigen-value problem. Since the authors needed eigenvector corresponding to the dominant eigenvalue, they adopted the power method and developed a C++ program for generating the required unique eigenvector associated with the dominant eigenvalue 6. The normalised eigen-vector, namely, the worth vector, calculated by this program is given as $(1.0968442, 1.0348668, 1.09787431.0221596, 0.8260924, 0.7783165, 1.143846)$. Then, went through the ranking procedure and obtained the ranking as

$$F7 > F3 > F1 > F2 > F4 > F5 > F6.$$

The ranking of sites together with their worth is exhibited in Table 5.12:

TABLE 5.12. Ranking of Sites

Site:	F_7	F_3	F_1	F_2	F_4	F_5	F_6
WORTH:	1.1438	1.0979	1.0968	1.0349	1.0222	0.8261	0.7783
RANK:	1	2	3	4	5	6	7

This information will be equipped promoters to distribute perks based on the pre-announced package (that is, distribute perk based on their worth) and which will improve the work quality of employees positively in subsequent.

Table 6.13. Ranking of Sites after removing Site F_7

Site:	F_3	F_1	F_2	F_4	F_5	F_6
WORTH:	1.104812	1.097866	1.080376	1.023899	0.878530	0.805518
RANK:	1	2	3	4	5	6

6. Discussion

Let us examine the significance of non-membership values in this problem. To do so, we exclude all non-membership values and recalculate the worth vector, which now becomes $(1.089, 1.011452, 1.07204, 1.06576, 0.8835705, 0.7626358, 1.1159)$. Previously, the worth of F_5 was 0.8261, but after the omission, it is increased to 0.8835705 indicating that non-membership values contribute to a decrease in worth. These observations clearly highlight the impact of non-membership values on the overall ranking. Since perks are awarded in proportion to worth, it is the collective responsibility of all employees at the site to ensure that no one engages in actions that contribute to non-membership values. Such vigilance helps minimize the risk of violating strictly prohibited rules.

Next, the authors discuss Remark 5.1. From P^*, we get $D(F_7, F_i) = 1,\ \forall i \neq 7$, which implies that F_7 is completely dominant with respect to $F_i,\ \forall i \neq 7$ in all aspects. By our earlier calculations, the rank of F_7 is one. This result coincides with remark 5.1(iii). To verify the second statement of remark 5.1 (iv), the authors eliminate F_7 and calculate the worth vector. The new ranking is as shown in Table 6.13. If the rank of each of the above six sites is incremented by one position and site F_7 is assigned the first rank, then it can be seen that this will coincide with the previous ranking; this verifies the remark 5.1(iv). However, if one needs the worth of F_7 in addition to just ranking, this site must be included in the ranking procedure. Another noteworthy thing is that the omission of F_7 increases the worth of other sites.

7. Conclusion

In this paper, the authors have developed an innovative method based on Linear Algebra, for solving a real-life decision-making problem. By choosing the dual hesitant fuzzy soft set as the framework, the problem becomes quite handy. By implementing the eigen-value concepts, the solution becomes more reliable and precise. This method is suitable for the evaluation of unrelated data. The authors have also presented a practical application for their proposed method which necessarily depicts the effectiveness of the method.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

References

[1] Torra, V. (2010). Hesitant fuzzy sets. International Journal of Intelligent Systems, 25(6), 529–539.

[2] Zhu, B., Xu, Z., & Xia, M. (2012). Dual hesitant fuzzy sets. Journal of Applied Mathematics, 2012(1), 879629.

[3] He, Y. P. (2016). An approach to dual hesitant fuzzy soft set based on decision making. In Fuzzy Systems & Operations Research and Management (pp. 339–349). Springer International Publishing.

[4] Maclaurin, C. (1729). A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra. Philosophical Transactions of the Royal Society of London, Series A, 1729(36), 59–96.

[5] Qin, J., Liu, X., & Pedrycz, W. (2015). Hesitant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision making. International Journal of Fuzzy Systems, 17, 509–520.

[6] Yu, D., Li, D. F., & Merigó, J. M. (2016). Dual hesitant fuzzy group decision making method and its application to supplier selection. International Journal of Machine Learning and Cybernetics, 7, 819–831.

[7] Ye, J. (2014). Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making. Applied Mathematical Modelling, 38(2), 659–666.

[8] Xu, Y., Rui, D., & Wang, H. (2015). Dual hesitant fuzzy interaction operators and their application to group decision making. Journal of Industrial and Production Engineering, 32(4), 273–290.

[9] Wang, F., Li, X., & Chen, X. (2014). Hesitant fuzzy soft set and its applications in multi-criteria decision making. Journal of Applied Mathematics, 2014(1), 643785.

[10] Babitha, K. V., & John, S. J. (2013). Hesitant fuzzy soft sets. Journal of New Results in Science, 2(3).

[11] He, Y. P. (2016). An approach to dual hesitant fuzzy soft set based on decision making. In Fuzzy Systems & Operations Research and Management (pp. 339–349). Springer International Publishing.

[12] Arora, R., & Garg, H. (2018). A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making. Engineering Applications of Artificial Intelligence, 72, 80–92.

[13] Garg, H., & Arora, R. (2018). Dual hesitant fuzzy soft aggregation operators and their application in decision-making. Cognitive Computation, 10(5), 769–789.

[14] Zhang, H., & He, Y. (2018). A rough set-based method for dual hesitant fuzzy soft sets based on decision making. Journal of Intelligent & Fuzzy Systems, 35(3), 3437–3450.

[15] Xu, Z., & Xia, M. (2011). Distance and similarity measures for hesitant fuzzy sets. Information Sciences, 181(11), 2128–2138.

[16] Zhang, H., & Yang, S. (2016). Inclusion measure for typical hesitant fuzzy sets, the relative similarity measure and fuzzy entropy. Soft Computing, 20, 1277–1287.

[17] Garg, H., & Arora, R. (2017). Distance and similarity measures for dual hesitant fuzzy soft sets and their applications in multicriteria decision making problem. International Journal for Uncertainty Quantification, 7(3).

[18] Garg, H., & Arora, R. (2020). Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set. Journal of Ambient Intelligence and Humanized Computing, 11, 375–410.

[19] Yu, D., Zhang, W., & Xu, Y. (2013). Group decision making under hesitant fuzzy environment with application to personnel evaluation. Knowledge-Based Systems, 52, 1–10.

[20] Qin, J., Liu, X., & Pedrycz, W. (2016). Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. Applied Soft Computing, 41, 428–452.

[21] Hu, J., Yang, Y., Zhang, X., & Chen, X. (2018). Similarity and entropy measures for hesitant fuzzy sets. International Transactions in Operational Research, 25(3), 857–886.

[22] Li, D., Zeng, W., & Li, J. (2015). New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making. Engineering Applications of Artificial Intelligence, 40, 11–16.

[23] Zeng, W., Li, D., & Yin, Q. (2016). Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition. Pattern Recognition Letters, 84, 267–271.

[24] Zhang, H. Y., & Zhang, W. X. (2009). Hybrid monotonic inclusion measure and its use in measuring similarity and distance between fuzzy sets. Fuzzy Sets and Systems, 160(1), 107–118.

[25] Grzegorzewski, P. (2011). On possible and necessary inclusion of intuitionistic fuzzy sets. Information Sciences, 181(2), 342–350.

[26] Zhang, H. Y., Yang, S. Y., & Yue, Z. W. (2016). On inclusion measures of intuitionistic and interval-valued intuitionistic fuzzy values and their applications to group decision making. International Journal of Machine Learning and Cybernetics, 7, 833–843.

[27] Zhang, H., & Yang, S. (2016). Inclusion measure for typical hesitant fuzzy sets, the relative similarity measure and fuzzy entropy. Soft Computing, 20, 1277–1287.

[28] Sahin, R., & Karabacak, M. (2015). A multi-attribute decision making method based on inclusion measure for interval neutrosophic sets. International Journal of Engineering and Applied Sciences, 2(2), 13–15.

[29] Herrero, C., & Villar, A. (2013). On the comparison of group performance with categorical data. PLoS One, 8(12), e84784.

[30] Alcantud, J. C. R., de Andrés Calle, R., & Torrecillas, M. J. M. (2016). Hesitant fuzzy worth: An innovative ranking methodology for hesitant fuzzy subsets. Applied Soft Computing, 38, 232–243.

[31] Wen, X. (2023). Weighted hesitant fuzzy soft set and its application in group decision making. Granular Computing, 8(6), 1583–1605.

[32] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353.

[33] Molodtsov, D. (1999). Soft set theory—first results. Computers & Mathematics with Applications, 37(4–5), 19–31.

[34] Al-shami, T. M. (2023). (2,1)-Fuzzy sets: Properties, weighted aggregated operators and their applications to multi-criteria decision-making methods. Complex & Intelligent Systems, 9(2), 1687–1705.

[35] Ibrahim, H. Z., Al-Shami, T. M., & Elbarbary, O. G. (2021). (3,2)-Fuzzy sets and their applications to topology and optimal choices. Computational Intelligence and Neuroscience, 2021, 1272266.

[36] Al-shami, T. M., & Mhemdi, A. (2023). Generalized frame for orthopair fuzzy sets: (m,n)-fuzzy sets and their applications to multi-criteria decision-making methods. Information, 14(1), 56.

[37] Sivadas, A., John, S. J., & Athira, T. M. (2024). (p,q)-fuzzy aggregation operators and their applications to decision-making. The Journal of Analysis, 2024, 1–30.

[38] Al-shami, T. M., Alcantud, J. C. R., & Mhemdi, A. (2023). New generalization of fuzzy soft sets: (a,b)-Fuzzy soft sets. AIMS Mathematics, 8(2), 2995–3025.

[39] Ibrahim, H. Z., Al-shami, T. M., Arar, M., & Hosny, M. (2024). k_m^n-Rung picture fuzzy information in a modern approach to multi-attribute group decision-making. Complex & Intelligent Systems, 10(2), 2605–2625.

[40] Smets, P., & Magrez, P. (1987). Implication in fuzzy logic. International Journal of Approximate Reasoning, 1(4), 327–347.

[41] Yang, S. Y., Zhang, H. Y., & Yue, Z. W. (2013). Inclusion measure and its use in measuring similarity and distance measure between hesitant fuzzy sets. IEEE International Conference on Granular Computing (GrC). IEEE.

Department of Mathematics, Government College Mokeri, Calicut University, Kozhikode, Kerala India-673507

Department of Mathematics, Manipal institute of Technology, Manipal Academy of Higher Education, Karnataka, India-576104.

Department of Applied Science, RIT- Govt. Engineering College Kottayam, Velloor P O, Kerala, India-686501

Department of Mathematics, National institute of Technology calicut, Kerala, India-673601