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SUBMERSIONS OF CR-SUBMANIFOLDS WITH TOTALLY UMBILICAL FIBERS

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Abstract. The purpose of this paper is to study totally umbilical submersions of a CR-submanifolds M of a Kaehler manifolds $\bar{M}$ onto an almost Hermitian manifolds N such that characteristic vector field is vertical or horizontal vector field. We first show the totally geodesicity of fibers of such submersions. In addition, inequality in terms of O'Neill tensor fields is presented and new results regarding the equality sitation are obtained. Finally, the geodesicity of the fibers by submersion with parallel mean curvature vector field was investigated in complex space form.

Keywords: Kaehler manifolds, Riemannian submersions, Submersions, CR-submanifolds, Complex space form.
MSC (2020): 53C15, 53C42.

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1. Introduction

In 1966, O'Neill [18], introduced the concept of Riemannian submersion. In the seventies A. Bejancu [5] introduced the notion of a CR-submanifold as a natural generalization of both complex submanifolds and totally real submanifolds. In [14], S.Koboyashi observed the above similarity between the Riemannian submersion and CR-submanifold in terms of the distributions, thus he introduced the submersion of a CR-submanifold. It has been studied by many authors in different classes (see, [1], [2], [11], [13], [16], [17], [19], [22]). Also, the case of totally umbilic fibers was introduced by G.Baditoiu and S.Ianus in [3].

This paper focus on the geometry of the fibers, emphasing on the concept of totally umbilicity. The paper is organized in the following way: In section 2, we recall basic notions for later section. In section 3, we study the totally umbilical submersion of a CR-submanifolds M of a Kaehler manifolds $\bar{M}$ onto an almost Hermitian manifolds N. We show the totally geodesicity of fibers of such submersions. The relation between the bisectional curvatures of the fibers and the respective manifolds is given. In section 4, we prove that there exist no totally umbilical submersion of CR-submanifold with nonzero parallel mean curvature vector field in complex space form $c \neq 0$. In addition, the totally geodesic conditions of

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such submersions in the complex space form were examined and results related to the section curvatures were obtained.

2. Preliminaries

In this section, we give some definitions from [4], [5], [12], [18] and [23] for later sections. Let $\bar{M}$ be a Riemannian manifold with an almost complex structure J and Hermitian metric $g_{\bar{M}}$ satisfying

$$J^2 = -I, \qquad g_{\bar{M}}(JX, JY) = g_{\bar{M}}(X, Y). \tag{2.1}$$

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y$$

for any $X, Y \in T\bar{M}$, where $\Gamma(T\bar{M})$ is the Lie algebra of vector fields in $\bar{M}$, then the $(\bar{M}, g_{\bar{M}})$ is called an almost Hermitian manifold. If the almost complex structure J also satisfies

$$(\bar{\nabla}_X J)Y = 0 \tag{2.2}$$

for every $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be a Kaehler manifold [23].

In [18], B.O.Neill defined the concept of Riemannian submersions as follows: A surjective map $F : (M^m, g) \to (N^n, g)$ respectively m and n dimensional between Riemannian manifolds is called a *Riemannian submersion* if:

(S1) F has maximal rank, and

(S2) F_*, restricted to $(kerF_*)^\perp$, is a linear isometry.

Respectively, invariant tensors T and A are defined by

$$T_E F = \nu\nabla_{\nu E}\mathcal{H}F + \mathcal{H}\nabla_{\nu E}\nu F, \tag{2.3}$$

$$A_E F = \nu\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\nu F. \tag{2.4}$$

It is known that the tensors T and A satisfy that

$$T_{\mathcal{P}}\mathcal{Q} = T_{\mathcal{Q}}\mathcal{P}, \tag{2.5}$$

$$A_{\mathcal{R}}\mathcal{W} = -A_{\mathcal{W}}\mathcal{R} = \frac{1}{2}\nu[\mathcal{R}, \mathcal{W}] \tag{2.6}$$

for any $\mathcal{P}, \mathcal{Q} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{R}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$. From (2.3) and (2.4) equalities, we have

$$\nabla_{\mathcal{P}}\mathcal{Q} = T_{\mathcal{P}}\mathcal{Q} + \nu\nabla_{\mathcal{P}}\mathcal{Q}, \tag{2.7}$$

$$\nabla_{\mathcal{P}}\mathcal{R} = T_{\mathcal{P}}\mathcal{R} + \mathcal{H}\nabla_{\mathcal{P}}\mathcal{R}, \tag{2.8}$$

$$\nabla_{\mathcal{R}}\mathcal{P} = A_{\mathcal{R}}\mathcal{P} + \nu\nabla_{\mathcal{R}}\mathcal{P}, \tag{2.9}$$

$$\nabla_{\mathcal{R}}\mathcal{W} = \mathcal{H}\nabla_{\mathcal{R}}\mathcal{W} + A_{\mathcal{R}}\mathcal{W}. \tag{2.10}$$

For given a submanifold, M, of a Kaehler manifolds $(\bar{M}, J, g_{\bar{M}})$, in [7], Bejancu introduced the concept of CR-submanifolds as follows: D is invariant, $JD \subset D$ and $D^\perp$ is anti-invariant, $JD^\perp \subset TM^\perp$, where $TM^\perp$ is the normal space of M.

On the other hand, for given a CR submanifold M, an almost Hermitian manifold N, in [14], Kobayashi introduced the submersion of a CR-submanifold as follows: the submersion $F : M \to N$ such that (i) $D^\perp$ is the kernel of F_*, i.e., $F_* D^\perp = \{0\}$,

(ii) J interchanges $D^\perp$ and $TM^\perp$, i.e., $JD^\perp = TM^\perp$,

(iii) $F_* : D_p \to T_{F(p)}N$ is a complex isometry.

For a vector field $\mathcal{P}$ on M, we get

$$\mathcal{P} = H\mathcal{P} + V\mathcal{P}, \tag{2.11}$$

where H and V denote the horizontal and vertical part of $\mathcal{P}$.

Since the fibers of a Riemann submersion are submanifolds of the definition manifold, the concepts defined for submanifolds are also defined for fibers. Let $F : M \to N$ be a Riemannian submersion. Besse [8] defined the Riemannian curvatures of the M manifold, the N manifold and fibers as follows, R, R^* and $\bar{R}$, respectively

$$R(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}) = \bar{R}(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}) - g(T_{\mathcal{P}}\mathcal{W}, T_{\mathcal{Q}}\mathcal{R}) + g(T_{\mathcal{Q}}\mathcal{W}, T_{\mathcal{P}}\mathcal{R}), \tag{2.12}$$

$$R(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{U}) = g((\nabla_{\mathcal{P}}T)(\mathcal{Q},\mathcal{R}),\mathcal{U}) - g((\nabla_{\mathcal{Q}}T)(\mathcal{P},\mathcal{R}),\mathcal{U}), \tag{2.13}$$

$$\begin{aligned} R(\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T}) \;&=\; R^*(\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T}) + 2g(A_{\mathcal{U}}\mathcal{L}, A_{\mathcal{K}}\mathcal{T}) \\ &\quad -g(A_{\mathcal{L}}\mathcal{K}, A_{\mathcal{U}}\mathcal{T}) + g(A_{\mathcal{U}}\mathcal{K}, A_{\mathcal{L}}\mathcal{T}), \end{aligned} \tag{2.14}$$

$\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T} \in \mathcal{X}^{\mathcal{H}}(M)$.

In [20], the Gauss and Codazzi equations for submersions are given as follows

$$g(R(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Y}) = g(\bar{R}(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Y}) - g(T_{\mathcal{M}}\mathcal{Y}, T_{\mathcal{N}}\mathcal{H}) + g(T_{\mathcal{N}}\mathcal{Y}, T_{\mathcal{M}}\mathcal{H}), \tag{2.15}$$

$$g(R(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Z}) = g((\nabla_{\mathcal{M}}T)_{\mathcal{N}}\mathcal{H},\mathcal{Z}) - g((\nabla_{\mathcal{N}}T)_{\mathcal{M}}\mathcal{H}),\mathcal{Z}), \tag{2.16}$$

$\mathcal{M},\mathcal{N},\mathcal{H},\mathcal{Y} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{Z} \in \mathcal{X}^{\mathcal{H}}(M)$

Definition 2.1. [23] *Let c be constant holomorphic sectional curvature. In this case the curvature tensor of $\bar{M}(c)$ is given by*

$$\begin{aligned} R(\mathcal{K},\mathcal{L})\mathcal{M} \;&=\; \frac{c}{4}[g(\mathcal{K},\mathcal{M})\mathcal{L} - g(\mathcal{L},\mathcal{M})\mathcal{K} + g(J\mathcal{K},\mathcal{M})J\mathcal{L} \\ &\quad -g(J\mathcal{L},\mathcal{M})J\mathcal{K} + 2g(J\mathcal{K},\mathcal{L})J\mathcal{M}] \end{aligned} \tag{2.17}$$

$\mathcal{K},\mathcal{L},\mathcal{M} \in \bar{M}$.

Example 2.1. *For any $\theta \in (0, \frac{\pi}{2})$, let $F : R^4 \to R^2$ be a map defined by $F(x_1, x_2, x_3, x_4) = (x_1 \cos\theta + x_3 \sin\theta, x_2 \sin\theta + x_4 \cos\theta)$. Then, by direct calculations, we obtain the Jacobian matrix of F as:*

$$F_* = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

The rank of F_* is equal to 2. Thus, the map F is a submersion. After some computations, we obtain

$$ker F_* = \nu = span\{V_1 = (\sin\theta, 0, -\cos\theta, 0), V_2 = (0, \cos\theta, 0, -\sin\theta)\}$$

$$(ker F_*)^{\perp} = \mathcal{H} = span\{H_1 = (\cos\theta, 0, \sin\theta, 0), H_2 = (0, \sin\theta, 0, \cos\theta)\}.$$

3. SUBMERSIONS OF CR-SUBMANIFOLDS WITH TOTALLY UMBILICAL FIBERS IN KAEHLER MANIFOLDS

In this section, we investigate the integrability of horizontal distribution of the totally umbilical submersion of a CR-submanifold and study totally geodesicness. We obtain relations between the curvatures of the fibers and the respective manifolds.

Let (M, g_1), (N, g_2) be Riemannian manifolds and $F : (M, g_1) \to (N, g_2)$ a Riemannian submersion. Any fiber of a Riemannian submersion F is called totally umbilical if

$$T_{\mathcal{U}}\mathcal{V} = g_1(\mathcal{U}, \mathcal{V})H, \tag{3.18}$$

for $\mathcal{U}, \mathcal{V} \in \mathcal{X}^{\nu}(M)$, where H is the mean curvature vector field of the fiber [18].
Now, we assume that $F : (M, g_M) \to (B, g_B)$ be a totally umbilical submersion of a CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B.

Theorem 3.1. *Let F be a submersion with totally umbilical fibers. Then the fibers of the submersion F are totally geodesic, if vertical distribution ν is parallel.*

Proof. By using equation (2.7), we get

$$\nabla_P \mathcal{Q} = T_P \mathcal{Q} + \hat{\nabla}_P \mathcal{Q}$$

for the vector fields $P, \mathcal{Q} \in \mathcal{X}^{\nu}(M)$.
Since the vertical distribution ν is parallel, we obtain $T_P \mathcal{Q} = g(P, \mathcal{Q})H = 0$. Hence, the fibers of the submersion F are totally geodesic. □

Theorem 3.2. *Let F be a submersion with totally umbilical fibers. Then we have the horizontal distribution $\mathcal{H}$ is integrable if and onl if*

$$A_{\mathcal{R}} J\mathcal{W} = A_{\mathcal{W}} J\mathcal{R}$$

for the vector fields $\mathcal{R}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$.

Proof. By (2.2) and (2.10), we obtain

$$\mathcal{H}\nabla_{\mathcal{R}} J\mathcal{W} + A_{\mathcal{R}} J\mathcal{W} = J\nabla_{\mathcal{R}}\mathcal{W}.$$

For the vertical vector field $\mathcal{Z}$, we have

$$g(A_{\mathcal{R}} J\mathcal{W} - A_{\mathcal{W}} J\mathcal{R}, \mathcal{Z}) = g(J[\mathcal{R}, \mathcal{W}], \mathcal{Z}).$$

Thus the proof the theorem completes. □

Corollary 3.1. *Let F be a submersion with totally umbilical fibers. Then M is locally Riemannian product if and only if $A_{\mathcal{Z}} J\mathcal{W} = A_{\mathcal{W}} J\mathcal{Z}$ and fibres are totally geodesic.*

Proof. M is called locally Riemannian product if the horizontal distributions $\mathcal{H}$ and vertical distribution ν are parallel. In this case, $\nabla_{\mathcal{Z}}\mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$ or equivalently $\nabla_{\mathcal{K}}\mathcal{L} \in \mathcal{X}^{\nu}(M)$ for all $\mathcal{Z}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$ and $\mathcal{K}, \mathcal{L} \in \mathcal{X}^{\nu}(M)$.

By using (2.7), we obtain

$$\nabla_{\mathcal{K}}\mathcal{L} = T_{\mathcal{K}}\mathcal{L} + \hat{\nabla}_{\mathcal{K}}\mathcal{L}.$$

Since fibres are totally geodesic, from Theorem 3.1 vertical distribution ν is parallel.
On the other hand by (2.10) we get

$$\nabla_{\mathcal{Z}}\mathcal{W} = \mathcal{H}\nabla_{\mathcal{Z}}\mathcal{W} + A_{\mathcal{Z}}\mathcal{W}.$$

Since the horizontal distribution $\mathcal{H}$ is integrable for $\mathcal{Z}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$, we have $[\mathcal{Z}, \mathcal{W}] \in \mathcal{X}^{\mathcal{H}}(M)$. Therefore $\nu[\mathcal{Z}, \mathcal{W}] = 0$. Then from (2.6), we have $A_{\mathcal{Z}}\mathcal{W} = 0$. Thus horizontal distribution is parallel. □

Theorem 3.3. *Let F be a submersion with totally umbilical fibers. The following assertions are equivalent:*

(i) the horizontal distribution $\mathcal{H}$ is integrable and $A_{J\mathcal{Z}}J\mathcal{W}=0$ for any $\mathcal{Z},\mathcal{W}\in\mathcal{X}^{\mathcal{H}}(M)$,

(ii) the horizontal distribution $\mathcal{H}$ is totally geodesic in M.

Proof. $(i)\Rightarrow(ii)$ We have

$$A_{J\mathcal{Z}}J\mathcal{W}=0.$$

By using (2.2), (2.10) and (2.11), we get

$$\begin{aligned} A_{J\mathcal{Z}}J\mathcal{W} &= \nabla_{J\mathcal{Z}}J\mathcal{W}-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W} \\ &= J(\mathcal{H}\nabla_{J\mathcal{Z}}\mathcal{W}+A_{J\mathcal{Z}}\mathcal{W})-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W} \\ &= J\mathcal{H}\nabla_{J\mathcal{Z}}\mathcal{W}+HA_{J\mathcal{Z}}\mathcal{W}+VA_{J\mathcal{Z}}\mathcal{W}-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W}. \end{aligned}$$

By comparing vertical parts, we get $A_{J\mathcal{Z}}J\mathcal{W}=VA_{J\mathcal{Z}}\mathcal{W}=0$, that is, $A_{J\mathcal{Z}}\mathcal{W}=0$. Since the horizontal distribution $\mathcal{H}$ is integrable, from Theorem 3.2, we get

$$A_{J\mathcal{Z}}\mathcal{W}=A_{\mathcal{W}}J\mathcal{Z}=0.$$

By using (2.2) and (2.10), we obtain

$$\begin{aligned} g(A_{\mathcal{W}}J\mathcal{Z},\mathcal{K}) &= g(\nabla_{\mathcal{W}}J\mathcal{Z}-\mathcal{H}\nabla_{\mathcal{W}}J\mathcal{Z},\mathcal{K}) \\ &= g(\nabla_{\mathcal{W}}J\mathcal{Z},\mathcal{K}) \\ &= g(J\nabla_{\mathcal{W}}\mathcal{Z},\mathcal{K})=0 \end{aligned}$$

for any $\mathcal{K}\in\mathcal{X}^{\nu}(M)$. Hence, the horizontal distribution $\mathcal{H}$ is totally geodesic in M. □

Theorem 3.4. *Let F be a submersion with totally umbilical fibers. If the bisectional curvatures $\bar{K}$ and K of $\bar{M}$ and M, respectively, then for orthonormal vertical vector $\mathcal{P}$ and $\mathcal{Q}$ we have*

$$K(\mathcal{P},\mathcal{Q})=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2.$$

Proof. Since F is a totally umbilical submersion then by using (2.15), we have

$$\begin{aligned} g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W}) &= g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W})-g(\mathcal{P},\mathcal{W})g(\mathcal{Q},\mathcal{R})\|H\|^2 \\ &\quad +g(\mathcal{Q},\mathcal{W})g(\mathcal{P},\mathcal{R})\|H\|^2 \end{aligned} \tag{3.19}$$

for the vertical vector fields $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}\in\mathcal{X}^{\nu}(M)$. Taking $\mathcal{W}=\mathcal{P}$, $\mathcal{R}=\mathcal{Q}$ in (3.19), we have

$$\begin{aligned} g(R(\mathcal{P},\mathcal{Q})\mathcal{Q},\mathcal{P}) &= g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{Q},\mathcal{P})-g(\mathcal{P},\mathcal{P})g(\mathcal{Q},\mathcal{Q})\|H\|^2 \\ &\quad +g(\mathcal{Q},\mathcal{P})g(\mathcal{P},\mathcal{Q})\|H\|^2. \end{aligned}$$

Since orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$, we get

$$K(\mathcal{P},\mathcal{Q})=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2.$$

□

4. Submersions of CR-submanifolds with totally umbilical fibers in Complex Space Forms

In this section, we study a totally umbilical submersion of a CR submanifold in complex space form. Conditions of such submersions with mean curvature vector field was investigated and results related to the section curvatures were obtained.
Now, we assume that $F^{'} : (M, g_M) \to (B, g_B)$ be a totally umbilical submersion of a CR submanifold M of a complex space form $\bar{M}(c)$ onto an almost Hermitian manifold B.

Lemma 4.1. *Let* $F^{'}$ *be a submersion with totally umbilical fibers. Then* $H \in \mathcal{X}^{\nu}(M)$.

Proof. For $\mathcal{Z} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{U} \in \mathcal{X}^{\mathcal{H}}(M)$, from (2.2) and (2.8) we obtain

$$T_{\mathcal{Z}} J\mathcal{U} + \mathcal{H}\nabla_{\mathcal{Z}} J\mathcal{U} = J(T_{\mathcal{Z}}\mathcal{U} + \mathcal{H}\nabla_{\mathcal{Z}}\mathcal{U}).$$

For any $\mathcal{W} \in \mathcal{X}^{\nu}(M)$ we have

$$g(T_{\mathcal{Z}} J\mathcal{U}, \mathcal{W}) = g(T_{\mathcal{Z}}\mathcal{U}, \mathcal{W}).$$

$$g(T_{\mathcal{Z}} J\mathcal{U}, \mathcal{W}) = -g(T_{\mathcal{Z}}\mathcal{W}, \mathcal{U}).$$

Since F is totally umbilical submersion, we get

$$g(\mathcal{Z}, J\mathcal{U})g(H, \mathcal{W}) = -g(\mathcal{Z}, \mathcal{W})g(H, \mathcal{U})$$

which means $H \in \mathcal{X}^{\nu}(M)$. □

Theorem 4.1. *There is no* $F^{'}$ *totally umbilical submersion,* $c \neq 0$ *with nonzero parallel mean curvature vector field* H*.*

Proof. For $Y \in \mathcal{X}^{\nu}(M)$ from Lemma 4.1, (2.7) and (3.18) we get

$$\begin{aligned} \nabla_Y H &= T_Y H + \nu\nabla_Y H \\ &= -g(H, H)Y + \nu\nabla_Y H \end{aligned}$$

On the other hand by (2.7) we have

$$R(Y, \mathcal{Z})H = T_Y T_{\mathcal{Z}} H + \nabla_Y \nabla_{\mathcal{Z}} H - \nabla_Y T_{\mathcal{Z}} H - T_{\mathcal{Z}} T_Y H - \nabla_{\mathcal{Z}} \nabla_Y H + \nabla_{\mathcal{Z}} T_Y H - \nabla_{[Y,\mathcal{Z}]} H$$

for any vertical field $\mathcal{Z}$. Since $F^{'}$ is totally umbilical submersion and H is parallel along the fibers, we get

$$R(Y, \mathcal{Z})H = 0. \tag{4.20}$$

Hence, since $\bar{M}(c)$ is complex space form, for $\mathcal{W} \in \mathcal{X}^{\nu}(M)$, we have

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{W}) - g(\mathcal{Z}, H)g(Y, \mathcal{W})]. \tag{4.21}$$

Taking $\mathcal{Z} = \mathcal{W}$ in (4.21), we obtain

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{Z})]$$

for any orthonormal vectors Y and $\mathcal{Z}$. From (4.20) and (4.21) we get

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{Z})] = 0.$$

For $g(Y, H) \neq 0$ and $g(\mathcal{Z}, \mathcal{Z}) \neq 0$, we have $c = 0$. □

Theorem 4.2. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then the mean curvature vector H of M is parallel if and only if the fibers of submersion $F^{'}$ are totally geodesic.*

Proof. For any $\mathcal{P},\mathcal{Q},\mathcal{R}\in\mathcal{X}^{\nu}(M)$ and $\mathcal{W}\in\mathcal{X}^{\mathcal{H}}(M)$, by using the equation (2.17), we obtain

$$\begin{aligned} g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W}) &= \frac{c}{4}[g(\mathcal{P},\mathcal{R})g(\mathcal{Q},\mathcal{W})-g(\mathcal{Q},\mathcal{R})g(\mathcal{P},\mathcal{W})+g(J\mathcal{P},\mathcal{R})g(J\mathcal{Q},\mathcal{W}) \\ &\quad -g(J\mathcal{Q},\mathcal{R})g(J\mathcal{P},\mathcal{W})+2g(J\mathcal{P},\mathcal{Q})g(J\mathcal{R},\mathcal{W})] \\ &= 0. \end{aligned}$$

On the other hand by (2.16) we have

$$g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W}) = g((\nabla_{\mathcal{P}}T)_{\mathcal{Q}}\mathcal{R},\mathcal{W})-g((\nabla_{\mathcal{Q}}T)_{\mathcal{P}}\mathcal{R},\mathcal{W}). \tag{4.22}$$

From [15], since $F^{'}$ is totally umbilical submersion, we have

$$\begin{aligned} (\nabla_{\mathcal{P}}T)_{\mathcal{Q}}\mathcal{R} &= \nabla_{\mathcal{P}}T_{\mathcal{Q}}\mathcal{R}-T_{\nabla_{\mathcal{P}}\mathcal{Q}}\mathcal{R}-T_{\mathcal{Q}}\nabla_{\mathcal{P}}\mathcal{R} \\ &= \nabla_{\mathcal{P}}\{g(\mathcal{Q},\mathcal{R})H\}-g(\nabla_{\mathcal{P}}\mathcal{Q},\mathcal{R})H-T_{\mathcal{Q}}\{T_{\mathcal{P}}\mathcal{R}+\bar{\nabla}_{\mathcal{P}}\mathcal{R}\} \\ &= g(\mathcal{Q},\mathcal{R})\nabla_{\mathcal{P}}H-g(\nabla_{\mathcal{P}}\mathcal{Q},\mathcal{R})H-g(\mathcal{P},\mathcal{R})g(H,H)\mathcal{Q}. \end{aligned} \tag{4.23}$$

Similarly we may also prove that

$$(\nabla_{\mathcal{Q}}T)_{\mathcal{P}}\mathcal{R} = g(\mathcal{P},\mathcal{R})\nabla_{\mathcal{Q}}H-g(\nabla_{\mathcal{Q}}\mathcal{P},\mathcal{R})H-g(\mathcal{Q},\mathcal{R})g(H,H)\mathcal{P}. \tag{4.24}$$

Substituting (4.23) and (4.24) equation into (4.22), we have

$$g(\mathcal{Q},\mathcal{R})g(\nabla_{\mathcal{P}}H,\mathcal{W}) = g(\mathcal{P},\mathcal{R})g(\nabla_{\mathcal{Q}}H,\mathcal{W}). \tag{4.25}$$

By using (2.7) and taking $\mathcal{Q}=\mathcal{R}$ in (4.25) we find

$$g(\mathcal{Q},\mathcal{Q})g(T_{\mathcal{P}}H,\mathcal{W})=0$$

for any orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$. We obtain $T_{\mathcal{P}}H=0$. Thus the proof the theorem completes. □

Theorem 4.3. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then for any vertical vector fields $\mathcal{P},\mathcal{Q}$, we have*

$$\frac{c}{4}=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2.$$

Proof. From (2.12) and (2.17) we get

$$\begin{aligned} &\frac{c}{4}[g(\mathcal{P},\mathcal{R})g(\mathcal{Q},\mathcal{W})-g(\mathcal{Q},\mathcal{R})g(\mathcal{P},\mathcal{W})] \\ &= g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W})-g(\mathcal{P},\mathcal{W})g(\mathcal{Q},\mathcal{R})g(H,H)+g(\mathcal{Q},\mathcal{W})g(\mathcal{P},\mathcal{R})g(H,H) \end{aligned} \tag{4.26}$$

for any vertical vector fields $\mathcal{R},\mathcal{W}$. Taking $\mathcal{R}=\mathcal{Q}$ and $\mathcal{P}=\mathcal{W}$ in (4.26) equation, we obtain

$$\frac{c}{4}=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2$$

for any orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$. □

As an immediate consequence of Theorem 4.3, we have the following.

Corollary 4.1. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then for any vertical vector fields $\mathcal{P},\mathcal{Q}$, we have*

$$c\leq 4\bar{K}(\mathcal{P},\mathcal{Q}).$$

The equality holds if and only if the fibers of submersion $F^{'}$ are totally geodesic.

Corollary 4.2. *Let $F^{'}$ be a totally umbilical submersion of a CR submanifold of a non-negatively complex space form. Then curvatures of fibers are positively.*

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