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KENMOTSU MANIFOLDS COUPLED WITH η-ρ-EINSTEIN SOLITONS ADMITTING AN EXTENDED $\mathcal{M}$-PROJECTIVE CURVATURE TENSOR

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Abstract. The object of the present paper is to study some curvature conditions on Kenmotsu manifolds. Initially, we analyze the condition ξ-$\mathcal{M}^e$ projective flat and φ-$\mathcal{M}^e$ semi-symmetric on Kenmotsu manifolds coupled with an η-ρ-Einstein soliton. Subsequently, we elaborate the conditions $\mathcal{M}^e \cdot \mathcal{R}=0$, $\mathcal{M}^e \cdot \mathcal{M}^e=0$ and $\mathcal{M}^e \cdot \mathcal{Q}=0$ on Kenmotsu manifolds in view of an η-ρ-Einstein soliton, where $\mathcal{M}^e$ is the extended $\mathcal{M}$-projective curvature tensor. In addition, we verify the results with a concrete example.

Keywords: Kenmotsu manifolds, Extended $\mathcal{M}$-projective curvature tensor $\mathcal{M}^e$, ξ-$\mathcal{M}^e$ projectively flat, φ-$\mathcal{M}^e$ semi symmetric, η-ρ-Einstein soliton and η-Einstein manifolds.

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1. Introduction

The product of an almost contact manifold $\mathbb{M}$ and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes $\mathbb{M}$ to be an almost contact metric manifold and supposes that the product metric $\mathbb{G}$ on $\mathbb{M} \times \mathbb{R}$ is Kaehlerian, then the structure on $\mathbb{M}$ is cosymplectic [15] and not Sasakian. On the other hand Oubina [18] pointed out that if the conformally related metric $e^{2t}\mathbb{G}, t$ being the coordinate on $\mathbb{R}$, is Kaehlerian, then $\mathbb{M}$ is Sasakian and conversely. In [22], S. Tanno classified connected almost contact metric

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manifolds whose automorphism groups possess the maximum dimension. For such a manifold $\mathbb{M}$, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, $\mathbb{M}$ is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, $\mathbb{M}$ is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, $\mathbb{M}$ is a warped product space $\mathbb{R} \times_f \mathcal{C}^n$. In 1972, Kenmotsu studied a class of contact Riemannian manifolds that satisfy specific conditions [17]. We call it Kenmotsu manifold. If a Kenmotsu manifold satisfies the condition $\mathcal{R}(X,Y)\cdot\mathcal{R}$=0, it must have a constant curvature of -1, where $\mathcal{R}$ denotes the Riemannian curvature tensor and $\mathcal{R}(X,Y)$ refers to the tensor algebra derivation at each point in the tangent vectors X,Y. Kenmotsu manifolds have been studied by many authors such as (see, [3], [4], [20], [12], [19], [8], [9], [21], [13],[11],[10],[24],[31]) and many others. The metric g on $(\mathbb{M}, g)$ is called a ρ-Einstein soliton if there is a smooth vector field $\mathbb{V}$ such that [2]:

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_{\mathbb{V}} g = (\gamma_1 + \rho r)g, \tag{1.1}$$

where $\mathcal{L}_{\mathbb{V}}$ and r denote the Lie derivative and Ricci scalar respectively, where $\rho \neq 0$, $\gamma_1 \in \mathbb{R}$. As usual ρ-Einstein soliton is steady for γ_1=0, shrinking for $\gamma_1 > 0$ and expanding for $\gamma_1 < 0$. A new type of soliton called η-ρ -Einstein soliton which is a generalization of ρ-Einstein soliton given by

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_{\mathbb{V}} g = (\gamma_1 + \rho r)g + \gamma_2 \eta \otimes \eta, \tag{1.2}$$

where γ_1, $\gamma_2 \in \mathbb{R}$. Analogous to equation (1.2), we recall η-ρ-Einstein soliton and so equation (1.2) takes the form

$$\mathcal{S} + Hess(\psi) = (\gamma_1 + \rho r)g + \gamma_2 \eta \otimes \eta. \tag{1.3}$$

As η-ρ-Einstein soliton (or gradient η-ρ-Einstein soliton) can be classified as (i) ρ-Einstein soliton (or gradient ρ-Einstein soliton) [2] if γ_2=0, (ii) η-Einstein soliton (or gradient η-Einstein soliton) [14] if ρ=$\frac{1}{2}$, (iii) η-traceless Ricci soliton (or gradient η-traceless Ricci soliton) if ρ=$\frac{1}{2n+1}$, (iv) η-Schouten soliton (or gradient η-Schouten soliton) [23] if ρ=$\frac{1}{4n}$. In this sequel many authors have been studied Kenmostu manifold with reference to different type of solitons (see,[5], [6], [7], [27], [26], [29],[28],[32],[30]) and many others.

More specific, the Lie derivative $(\mathcal{L}_\xi g)(H_1, H_2)$ given by

$$(\mathcal{L}_\xi g)(H_1, H_2) = g(\nabla_{H_1}\xi, H_2) + g(H_1, \nabla_{H_2}\xi). \tag{1.4}$$

The work of the paper is organized as follows: After the introduction, in section 2, we carried out the basic exposition on Kenmotsu manifold. In section 3, we analyze ξ-$\mathcal{M}^e$

projectively flat Kenmotsu manifold and deduce the interesting result coupled with an η-ρ-Einstein soliton. In section 4 we take up φ-$\mathcal{M}^e$ semi-symmetric in Kenmotsu manifold admitting an η-ρ-Einstein soliton. Again in section 5, 6 and 7 we discuss the some curvature conditions namely, $\mathcal{M}^e \cdot \mathcal{R}$=0, $\mathcal{M}^e \cdot \mathcal{M}^e$=0 and $\mathcal{M}^e \cdot \mathcal{Q}$=0 on such manifold and we verifies the results by suitable example. The conclusion of the work is given in the last section 8.

2. Preliminaries

Let $(\mathbb{M}^{2n+1}, \varphi, \xi, \eta, g)$ be an $(2n+1)$-dimensional almost contact metric manifold, where φ is a $(1,1)$-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that the (φ, ξ, η, g) structure satisfies the conditions [1]:

$$\varphi^2 H_1 = -H_1 + \eta(H_1)\xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \tag{2.5}$$

$$g(H_1, \xi) = \eta(H_1), \ \eta(\varphi H_1) = 0, \tag{2.6}$$

$$g(\varphi H_1, \varphi H_2) = g(H_1, H_2) - \eta(H_1)\eta(H_2), \tag{2.7}$$

$$g(\varphi H_1, H_2) = -g(H_1, \varphi H_2), \tag{2.8}$$

for any $H_1, H_2 \in \chi(\mathbb{M})$. If moreover

$$(\nabla_{H_1}\varphi)H_2 = g(\varphi H_1, H_2)\xi - \eta(H_2)\varphi H_1, \tag{2.9}$$

$$\nabla_{H_1}\xi = H_1 - \eta(H_1)\xi, \tag{2.10}$$

where ∇ denotes the Levi-Civita connection on $(\mathbb{M}^{2n+1}, g)$, then $(\mathbb{M}^{2n+1}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold. In this case, it is well known that [17]:

$$\mathcal{R}(H_1, H_2)H_3 = g(H_1, H_3)H_2 - g(H_2, H_3)H_1, \tag{2.11}$$

$$\mathcal{R}(H_1, H_2)\xi = \eta(H_1)H_2 - \eta(H_2)H_1, \tag{2.12}$$

$$\mathcal{R}(H_1, \xi)H_3 = g(H_1, H_3)\xi - \eta(H_3)H_1, \tag{2.13}$$

$$\mathcal{R}(\xi, H_2)H_3 = \eta(H_3)H_2 - g(H_2, H_3)\xi, \tag{2.14}$$

$$\mathcal{S}(\varphi X, \varphi Y) = \mathcal{S}(X, Y) + 2n\eta(X)\eta(Y), \tag{2.15}$$

$$\mathcal{S}(H_1, \xi) = -2n\eta(H_1), \tag{2.16}$$

$$\mathcal{S}(\xi, \xi) = -2n, \tag{2.17}$$

$$\mathcal{Q}\xi = -2n\xi, \tag{2.18}$$

$\forall\ H_1, H_1 \in \chi(\mathbb{M})$. According to [25], the $\mathcal{M}$-projective curvature tensor and the extended $\mathcal{M}$-projective curvature tensor $\mathcal{M}^e$ on $(\mathbb{M}^{2n+1}, g)$ are defined by

$$\begin{aligned}\mathcal{M}(H_1,H_2)H_3 &= \mathcal{R}(H_1,H_2)H_3 - \frac{1}{4n}[\mathcal{S}(H_2,H_3)H_1 - \mathcal{S}(H_1,H_3)H_2 \\ &+ g(H_2,H_3)\mathcal{Q}H_1 - g(H_1,H_3)\mathcal{Q}H_2],\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathcal{M}^e(H_1,H_2)H_3 &= \mathcal{M}(H_1,H_2)H_3 - \eta(H_1)\mathcal{M}(\xi,H_2)H_3 \\ &- \eta(H_2)\mathcal{M}(H_1,\xi)H_3 - \eta(H_3)\mathcal{M}(H_1,H_2)\xi,\end{aligned} \tag{2.20}$$

for any $H_1,\ H_2,\ H_3 \in \chi(\mathbb{M})$. Now using the Eq. (2.12), (2.13),(2.14),(2.16), (2.17) and (2.18) we get from (2.19) that

$$\begin{aligned}\mathcal{M}(H_1,H_2)\xi &= \eta(H_1)H_2 - \eta(H_2)H_1 \\ &- \frac{1}{4n}[2n\eta(H_1)H_2 - 2n\eta(H_2)H_1 + \eta(H_2)\mathcal{Q}H_1 \\ &- \eta(H_1)\mathcal{Q}H_2],\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathcal{M}(\xi,H_2)H_3 &= \eta(H_3)H_2 - g(H_2,H_3)\xi \\ &- \frac{1}{4n}[\mathcal{S}(H_2,H_3)\xi + 2n\eta(H_3)H_2 - 2ng(H_2,H_3)\xi \\ &- \eta(H_3)\mathcal{Q}H_2],\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mathcal{M}(H_1,\xi)H_3 &= g(H_1,H_3)\xi - \eta(H_3)H_1 \\ &- \frac{1}{4n}[-2n\eta(H_3)H_1 - \mathcal{S}(H_1,H_3)\xi + \eta(H_3)\mathcal{Q}H_1 \\ &+ 2ng(H_1,H_3)\xi].\end{aligned} \tag{2.23}$$

Also, taking $H_3{=}\xi$ in (2.20) we yield

$$\mathcal{M}^e(H_1,H_2)\xi = -\eta(H_1)\mathcal{M}(\xi,H_2)\xi - \eta(H_2)\mathcal{M}(H_1,\xi)\xi. \tag{2.24}$$

For fix $H_1{=}\xi$ in (2.21) along with (2.5) and (2.18), we get

$$\mathcal{M}(\xi,H_2)\xi = \frac{1}{2}H_2 + \frac{1}{4n}\mathcal{Q}H_2. \tag{2.25}$$

Again by substituting $H_2{=}\xi$ in (2.21) and using (2.5) and (2.18), we have

$$\mathcal{M}(H_1,\xi)\xi = -\frac{1}{2}H_1 - \frac{1}{4n}\mathcal{Q}H_1. \tag{2.26}$$

Using (2.25) and (2.26) in (2.24), we obtain

$$\mathcal{M}^e(H_1,H_2)\xi = -\frac{1}{2}\eta(H_1)H_2 - \frac{1}{4n}\eta(H_1)\mathcal{Q}H_2 + \frac{1}{2}\eta(H_2)H_1 + \frac{1}{4n}\eta(H_2)\mathcal{Q}H_1. \tag{2.27}$$

Taking H_1=ξ in (2.27) and using (2.5) and (2.18), we get

$$\mathcal{M}^e(\xi,H_2)\xi = -\frac{1}{2}H_2 - \frac{1}{4n}\mathcal{Q}H_2. \tag{2.28}$$

Again taking H_1=ξ in (2.20) and using (2.5), we obtain

$$\mathcal{M}^e(\xi,H_2)H_3 = -\eta(H_2)\mathcal{M}(\xi,\xi)H_3 - \eta(H_3)\mathcal{M}(\xi,H_2)\xi. \tag{2.29}$$

For fix, H_2=ξ in (2.22) and using (2.6), (2.16) and (2.18), we yield

$$\mathcal{M}(\xi,\xi)H_3 = 0. \tag{2.30}$$

With the help of (2.25) and (2.30), Eq.(2.29) reduces to

$$\mathcal{M}^e(\xi,H_2)H_3 = -\frac{1}{2}\eta(H_3)H_2 - \frac{1}{4n}\eta(H_3)\mathcal{Q}H_2. \tag{2.31}$$

Similarly, one can get

$$\mathcal{M}^e(H_1,\xi)H_3 = \frac{1}{2}\eta(H_3)H_1 + \frac{1}{4n}\eta(H_3)\mathcal{Q}H_1. \tag{2.32}$$

Definition 2.1. *An almost contact manifold* $(\mathbb{M}^{2n+1},g)$ *is said to be an* η*-Einstein if its Ricci tensor* $\mathcal{S}$ *has the form*

$$\mathcal{S} = \mathcal{A}g + \mathcal{B}\eta\otimes\eta, \tag{2.33}$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *are constants. If* $\mathcal{B}$=0*, then it is identified as Einstein and if* $\mathcal{A}$=0*, it is know as special type of* η*-Einstein.*

3. ξ-$\mathcal{M}^e$-projectively flat Kenmotsu Manifolds

Definition 3.1. *An* $(2n+1)$*-dimensional manifold is said to be* ξ-$\mathcal{M}^e$ *projectively flat if it fulfills the condition*

$$\mathcal{M}^e(H_1,H_2)\xi = 0, \tag{3.34}$$

for all $H_1,H_2\in\chi(\mathbb{M})$.

Theorem 3.1. *A* ξ-$\mathcal{M}^e$ *projectively flat Kenmotsu manifold* $(\mathbb{M}^{2n+1},g)$ *is an Einstein manifold.*

Proof. Let $(\mathbb{M}^{2n+1}, g)$ be ξ-$\mathcal{M}^e$ projectively flat. Then from (2.20), we have

$$\eta(H_1)\mathcal{M}(\xi, H_2)\xi + \eta(H_2)\mathcal{M}(H_1, \xi)\xi = 0. \tag{3.35}$$

Using (2.23) and $H_3{=}\xi$ in (2.22), we obtain from (3.35) that

$$\begin{aligned}\eta(H_1)[\eta(\xi)H_2 \quad &- \quad g(H_2,\xi)\xi - \frac{1}{4n}\{\mathcal{S}(H_2,\xi)\xi + 2n\eta(\xi)H_2 - 2ng(H_2,\xi)\xi - \eta(\xi)\mathcal{Q}H_2\}] \\ &+ \quad \eta(H_2)[g(H_1,\xi)\xi - \eta(\xi)H_1 - \frac{1}{4n}\{-\mathcal{S}(H_1,\xi)\xi \\ &- \quad 2nH_1 + \eta(\xi)\mathcal{Q}H_1 + 2ng(H_1,\xi)\xi\}] = 0. \end{aligned}\tag{3.36}$$

With the help of (2.5), (2.6) and (2.16), Eq. (3.36), reduces to

$$\frac{1}{2}\{\eta(H_1)H_2 - \eta(H_2)H_1\} - \frac{1}{4n}\{\eta(H_2)\mathcal{Q}H_1 - \eta(H_1)\mathcal{Q}H_2\} = 0. \tag{3.37}$$

Taking $H_2{=}\xi$ in (3.37) we yield

$$\mathcal{Q}H_1 = -2nH_1, \tag{3.38}$$

which implies

$$\mathcal{S}(H_1, H_4) = -2ng(H_1, H_4). \tag{3.39}$$

□

Thus the Theorem 3.1 is completed.

Theorem 3.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady, or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Proof. Also from (1.2), we have

$$\mathcal{S}(H_1, H_2) + \frac{1}{2}(\mathcal{L}_{\mathbb{V}}g)(H_1, H_2) = (\gamma_1 + \rho r)g(H_1, H_2) + \gamma_2\eta(H_1)\eta(H_2). \tag{3.40}$$

Taking trace after putting $H_1{=}H_2{=}e_i$, $1 \leq i \leq 2n+1$ in (3.40), we get

$$\mathcal{S}(e_i, e_i) + \frac{1}{2}(\mathcal{L}_{\mathbb{V}}g)(e_i, e_i) = (\gamma_1 + \rho r)g(e_i, e_i) + \gamma_2\eta(e_i)\eta(e_i). \tag{3.41}$$

Using (3.39) in (3.41), we obtain

$$\mathrm{div}\mathbb{V} = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2. \tag{3.42}$$

If $\mathbb{V}$ is solenoidal, i.e., $\mathrm{div}\mathbb{V}{=}0$, then (3.42) implies that

$$\gamma_1 = -[2n + \frac{\gamma_2}{(2n+1)} + \rho r]. \tag{3.43}$$

□

So the proof of Theorem 3.2 is finished. Utilizing the Theorem 3.2, we state the following Corollary.

Corollary 3.1. *If a ξ-$\mathcal{M}^e$ protectively flat Kenmotsu manifold admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 3.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady, or shrinking as $r < -4n$, $r = -4n$, or $r > -4n$.*

Corollary 3.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady, or reducing as $r < -2n(2n+1)$, $r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 3.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is growing, steady, or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Again, if $\mathbb{V}$=grad(f), where f is a smooth function on $(\mathbb{M}^{2n+1}, g)$. Then from equation (3.42) we yield the following result.

Theorem 3.3. *If the metric g of a $(\mathbb{M}^{2n+1}, g)$ satisfies an η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n + 1) + \gamma_2.$$

4. φ-$\mathcal{M}^e$ semi-symmetric on Kenmotsu manifold

Definition 4.1. *An $(2n+1)$-dimensional manifold is said to be φ-$\mathcal{M}^e$ semi-symmetric if it fulfills the criterion*

$$\mathcal{M}^e(H_1, H_2) \cdot \varphi = 0, \tag{4.44}$$

for all $H_1, H_2 \in \chi(\mathbb{M})$.

Theorem 4.1. *A φ-$\mathcal{M}^e$ semi-symmetric Kenmotsu manifold $(\mathbb{M}^{2n+1}, g)$ is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e(H_1,H_2)\cdot\varphi$=0 on $(\mathbb{M}^{2n+1},g)$ from (2.20) implies that

$$(\mathcal{M}^e(H_1,H_2)\cdot\varphi)H_3=\mathcal{M}^e(H_1,H_2)\varphi H_3-\varphi\mathcal{M}^e(H_1,H_2)H_3=0. \tag{4.45}$$

for any vector fields $H_1,H_2,H_3\in\chi(\mathbb{M})$.

Since form (2.20) we have

$$\begin{aligned}\mathcal{M}^e(H_1,H_2)\varphi H_3 &= \mathcal{M}(H_1,H_2)\varphi H_3-\eta(H_1)M(\xi,H_2)\varphi H_3\\ &- \eta(H_2)\mathcal{M}(H_1,\xi)\varphi H_3-\eta(\varphi H_3)\mathcal{M}(H_1,H_2)\xi.\end{aligned} \tag{4.46}$$

Using (2.6), (2.11), (2.19), (2.22), and (2.23) in (4.46), we get

$$\begin{aligned}\mathcal{M}^e(H_1,H_2)\varphi H_3 &= g(H_1,\varphi H_3)H_2-g(H_2,\varphi H_3)H_1\\ &- \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)H_1-\mathcal{S}(H_1,\varphi H_3)H_2+\eta(H_2)\mathcal{Q}H_1-\eta(H_1)\mathcal{Q}H_2\}\\ &- \eta(H_1)[-g(H_2,\varphi H_3)\xi-\frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)\xi-2ng(H_2,\varphi H_3)\xi\}]\\ &- \eta(H_2)[g(H_1,\varphi H_3)\xi-\frac{1}{4n}\{-\mathcal{S}(H_1,\varphi H_3)\xi+2ng(H_1,\varphi H_3)\xi\}].\end{aligned} \tag{4.47}$$

Again,

$$\begin{aligned}\varphi\mathcal{M}^e(H_1,H_2)H_3 &= \varphi\mathcal{M}(H_1,H_2)H_3-\eta(H_1)\varphi\mathcal{M}(\xi,H_2)H_3\\ &- \eta(H_2)\varphi\mathcal{M}(H_1,\xi)H_3-\eta(H_3)\varphi\mathcal{M}(H_1,H_2)\xi.\end{aligned} \tag{4.48}$$

Using (2.5), (2.11), (2.12), (2.19), (2.21), (2.22), and (2.23) in (4.48), we have

$$\begin{aligned}\varphi\mathcal{M}^e(H_1,H_2)H_3 &= g(H_1,H_3)\varphi H_2-g(H_2,H_3)\varphi H_1\\ &-\frac{1}{4n}\{\mathcal{S}(H_2,H_3)\varphi H_1-\mathcal{S}(H_1,H_3)\varphi H_2+g(H_2,H_3)\mathcal{Q}\varphi H_1-g(H_1,H_3)\mathcal{Q}\varphi H_2\}\\ &-\eta(H_1)[\eta(H_3)\varphi H_2-\frac{1}{4n}\{2n\eta(H_3)\varphi H_2-\eta(H_3)\mathcal{S}\varphi H_2\}]\\ &-\eta(H_2)[-\eta(H_3)\varphi H_1-\frac{1}{4n}\{-2n\eta(H_3)\varphi H_1+\eta(H_3)\mathcal{Q}\varphi H_1\}]\\ &-\eta(H_3)[\eta(H_1)\varphi H_2-\eta(H_2)\varphi H_1]\\ &+\frac{\eta(H_3)}{4n}\{2n\eta(H_1)\varphi H_2-2n\eta(H_2)\varphi H_1+\eta(H_2)\mathcal{Q}\varphi H_1-\eta(H_1)\mathcal{Q}\varphi H_2\}.\end{aligned} \tag{4.49}$$

Using (4.47) and (4.48) in (4.45), we get

$$\begin{aligned}
g(H_1,\varphi H_3)H_2 - g(H_2,\varphi H_3)H_1 \\
&- \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)H_1 - \mathcal{S}(H_1,\varphi H_3)H_2 + \eta(H_2)\mathcal{Q}H_1 - \eta(H_1)\mathcal{Q}H_2\} \\
&- \eta(H_1)[-g(H_2,\varphi H_3)\xi - \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)\xi - 2ng(H_2,\varphi H_3)\xi\}] \\
&- \eta(H_2)[g(H_1,\varphi H_3)\xi - \frac{1}{4n}\{-\mathcal{S}(H_1,\varphi H_3)\xi + 2ng(H_1,\varphi H_3)\xi\}] \\
&- [g(H_1,H_3)\varphi H_2 - g(H_2,H_3)\varphi H_1] \\
&+ \frac{1}{4n}[\mathcal{S}(H_2,H_3)\varphi H_1 - \mathcal{S}(H_1,H_3)\varphi H_2] \\
&+ \frac{1}{4n}[g(H_2,H_3)\mathcal{Q}\varphi H_1 - g(H_1,H_3)\mathcal{Q}\varphi H_2] \\
&+ \eta(H_1)[\eta(H_3)\varphi H_2 - \frac{1}{4n}\{2n\eta(H_3)\varphi H_2 - \eta(H_3)\mathcal{Q}\varphi H_2\}] \\
&+ \eta(H_2)[-\eta(H_3)\varphi H_1 - \frac{1}{4n}\{-2n\eta(H_3)\varphi H_1 + \eta(H_3)\mathcal{Q}\varphi H_1\}] \\
&+ \eta(H_3)[\eta(H_1)\varphi H_2 - \eta(H_2)\varphi H_1] \\
&- \frac{\eta(H_3)}{4n}[2n\eta(H_1)\varphi H_2 - 2n\eta(H_2)\varphi H_1 + \eta(H_2)\mathcal{Q}\varphi H_1 - \eta(H_1)\mathcal{Q}\varphi H_2] = 0.
\end{aligned} \tag{4.50}$$

Taking $H_2{=}\xi$ in (4.50) and using (2.5), (2.6), (2.16), (2.18), we have

$$\begin{aligned}
\frac{1}{4n}\{2\mathcal{S}(H_1,\varphi H_3)\xi &- 2ng(H_1,\varphi H_3)\xi - 2n\eta(H_3)\varphi H_1 + \eta(H_3)\mathcal{Q}\varphi H_1\} \\
&+ \eta(H_3)\varphi H_1 = 0.
\end{aligned} \tag{4.51}$$

For fix, $H_3 = \xi$ in (4.51) and using (2.5), we obtain

$$\mathcal{Q}\varphi H_1 = -2n\varphi H_1. \tag{4.52}$$

Replacing H_1 by φH_1 in (4.52) and using (2.5), (2.18), one can get

$$\mathcal{Q}H_1 = -2nH_1, \tag{4.53}$$

which implies that

$$\mathcal{S}(H_1,H_4) = -2ng(H_1,H_4). \tag{4.54}$$

□

Therefore, the Theorem 4.1 is completed.

Like wise section 3, we reflect the following result:

Theorem 4.2. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-*ρ*-Einstein soliton on* $(\mathbb{M}^{2n+1}, g)$*. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, *or* $r > \frac{-2n}{\rho}$.

Corollary 4.1. *If a* φ*-*$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold admits an* η*-*ρ*-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 4.2. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-Einstein soliton on* φ*-*$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -4n,\ r = -4n$, *or* $r > -4n$.

Corollary 4.3. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-traceless soliton on* φ*-*$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if it is expanding, steady or shrinking as* $r < -2n(2n+1),\ r = -2n(2n+1)$, *or* $r > -2n(2n+1)$.

Corollary 4.4. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-Schouten soliton on* φ*-*$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -8n^2$, $r = -8n^2$, *or* $r > -8n^2$.

Theorem 4.3. *If the metric* g *of a* $(2n+1)$*-dimensional* φ*-*$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold admits* η*-*ρ*-Einstein soliton* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$*, where* $\mathbb{V}$ *is gradient of smooth function* f*, then the Laplace equation satisfied by* f *is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

5. Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0

Theorem 5.1. *If a* $(2n+1)$*-dimensional Kenmotsu manifold satisfying the condition* $\mathcal{M}^e \cdot \mathcal{R}$=0*, then* $(\mathbb{M}^{2n+1}, g)$ *is an Einstein manifold.*

Proof. Let $(\mathbb{M}^{2n+1}, g)$ satisfies the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then from [11], we have

$$\begin{aligned} \mathcal{M}^e(\xi, U)\mathcal{R}(H_1, H_2)H_3 \ &- \ \mathcal{R}(\mathcal{M}^e(\xi, U)H_1, H_2)H_3 \\ &- \ \mathcal{R}(H_1, \mathcal{M}^e(\xi, U)H_2)H_3 \\ &- \ \mathcal{R}(H_1, H_2)\mathcal{M}^e(\xi, U)H_3 = 0. \end{aligned} \tag{5.55}$$

Taking H_3=ξ in (5.55) and using (2.12), we get

$$\eta(\mathcal{M}^e(\xi, U)H_1)H_2 - \eta(\mathcal{M}^e(\xi, U)H_2)H_1 + \mathcal{R}(H_1, H_2)\mathcal{M}^e(\xi, U)\xi = 0. \tag{5.56}$$

Using (2.28), (2.31) in (5.56) and then using (2.6), (2.11), (2.16), we obtain

$$\frac{1}{2}\{g(H_1, U)H_2 - g(H_2, U)H_1\} + \frac{1}{4n}\{\mathcal{S}(H_1, U)H_2 - \mathcal{S}(H_2, U)H_1\} = 0. \tag{5.57}$$

Replacing H_2=ξ in (5.57), using (2.6) and (2.16), we yield

$$\mathcal{S}(H_1, U)\xi + 2ng(H_1, U)\xi = 0. \tag{5.58}$$

Taking the inner product of (5.58) with ξ and using (2.5), we obtain

$$\mathcal{S}(H_1, U) = -2ng(H_1, U). \tag{5.59}$$

□

So, the proof of the Theorem 5.1 is completed.

Therefore, as section 4, we state that

Theorem 5.2. *If $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Corollary 5.1. *If a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0 admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 5.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -4n$, $r = -4n$, or $r > -4n$.*

Corollary 5.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady or shrinking as $r < -2n(2n+1)$, $r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 5.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Theorem 5.3. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0 admits η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

6. Kenmotsu Manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e=0$

Theorem 6.1. *If a $(2n+1)$-dimensional Kenmotsu manifold satisfies the condition $\mathcal{M}^e \cdot \mathcal{M}^e=0$, then $(\mathbb{M}^{2n+1}, g)$ is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e \cdot \mathcal{M}^e=0$ on $(\mathbb{M}^{2n+1}, g)$ implies that

$$\begin{aligned}\mathcal{M}^e(\xi,U)\mathcal{M}^e(H_1,H_2)H_3 &- \mathcal{M}^e(\mathcal{M}^e(\xi,U)H_1,H_2)H_3\\ &- \mathcal{M}^e(H_1,\mathcal{M}^e(\xi,U)H_2)H_3\\ &- \mathcal{M}^e(H_1,H_2)\mathcal{M}^e(\xi,U)H_3=0. \qquad (6.60)\end{aligned}$$

Taking $H_3=\xi$ in (6.60), we get

$$\begin{aligned}\mathcal{M}^e(\xi,U)\mathcal{M}^e(H_1,H_2)\xi &- \mathcal{M}^e(\mathcal{M}^e(\xi,U)H_1,H_2)\xi\\ &- \mathcal{M}^e(H_1,\mathcal{M}^e(\xi,U)H_2)\xi\\ &- \mathcal{M}^e(H_1,H_2)\mathcal{M}^e(\xi,U)\xi=0. \qquad (6.61)\end{aligned}$$

Using (2.27), (2.28) and (2.31) in (6.61), we have

$$\begin{aligned}-\frac{1}{2}\eta(H_1)\mathcal{M}^e(\xi,U)H_2 &- \frac{1}{4n}\eta(H_1)\mathcal{M}^e(\xi,U)\mathcal{Q}H_2+\frac{1}{2}\eta(H_2)\mathcal{M}^e(\xi,U)H_1\\ &+ \frac{1}{4n}\eta(H_2)\mathcal{M}^e(\xi,U)\mathcal{Q}H_1+\frac{1}{2}\eta(H_1)\mathcal{M}^e(U,H_2)\xi\\ &+ \frac{1}{4n}\eta(H_1)\mathcal{M}^e(\mathcal{Q}U,H_2)\xi+\frac{1}{2}\eta(H_2)\mathcal{M}^e(H_1,U)\xi\\ &+ \frac{1}{4n}\eta(H_2)\mathcal{M}^e(H_1,\mathcal{Q}U)\xi+\frac{1}{2}\mathcal{M}^e(H_1,H_2)U\\ &+ \frac{1}{4n}\mathcal{M}^e(H_1,H_2)\mathcal{Q}U=0. \qquad (6.62)\end{aligned}$$

Taking $H_2=\xi$ in (6.62) and using (2.5), (2.18), (2.31) and (2.32), we get

$$\begin{aligned}-\frac{1}{8n}\eta(\mathcal{Q}H_1)U &- \frac{1}{16n^2}\eta(\mathcal{Q}H_1)\mathcal{Q}U+\frac{1}{4}\eta(H_1)U+\frac{1}{8n}\eta(H_1)\mathcal{Q}U\\ &+ \frac{1}{2}\eta(U)H_1+\frac{1}{4n}\eta(U)\mathcal{Q}H_1+\frac{1}{4n}\eta(\mathcal{Q}U)H_1\\ &+ \frac{1}{8n^2}\eta(\mathcal{Q}U)\mathcal{Q}H_1=0, \qquad (6.63)\end{aligned}$$

which implies that $\eta(H_1)\neq 0$, therefore equation (6.63) turns into

$$\mathcal{S}(U,H_4)=-2ng(U,H_4). \qquad (6.64)$$

□

Thus the proof of the Theorem 6.1 is completed.

As per section 5, we reflect the outcome

Theorem 6.2. *If $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Corollary 6.1. *If an $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0 admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 6.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -4n$, $r = -4n$, or $r > -4n$.*

Corollary 6.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady or shrinking as $r < -2n(2n+1)$, $r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 6.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Theorem 6.3. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0 admits η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

7. Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0

Theorem 7.1. *If a $(2n+1)$ dimensional Kenmotsu manifold satisfies the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0, then $(\mathbb{M}^{2n+1}, g)$ is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 on $(\mathbb{M}^{2n+1}, g)$ implies that

$$\mathcal{M}^e(H_1, H_2)\mathcal{Q}H_3 - \mathcal{Q}(\mathcal{M}^e(H_1, H_2)H_3) = 0. \tag{7.65}$$

Taking H_2=ξ in (7.65), we get

$$\mathcal{M}^e(H_1, \xi)\mathcal{Q}H_3 - \mathcal{Q}(\mathcal{M}^e(H_1, \xi)H_3) = 0. \tag{7.66}$$

Using (2.32) in (7.66), we have

$$\frac{1}{2}\eta(\mathcal{Q}H_3)H_1 + \frac{1}{4n}\eta(\mathcal{Q}H_3)\mathcal{Q}H_1 - \mathcal{Q}[\frac{1}{2}\eta(\mathcal{Q}H_3)H_1 + \frac{1}{4n}\eta(\mathcal{Q}H_3)\mathcal{Q}H_1] = 0. \tag{7.67}$$

By virtue of (2.16), we get from (7.67) that

$$n\eta(H_3)H_1 + \eta(H_3)\mathcal{Q}H_1 + \mathcal{Q}(\frac{1}{4n}\eta(H_3)\mathcal{Q}H_1) = 0, \tag{7.68}$$

which implies that

$$n\eta(H_3)H_1 + \frac{1}{2}\eta(H_3)\mathcal{Q}H_1 = 0. \tag{7.69}$$

Now, taking the inner product of (7.69) with H_4, we obtain

$$n\eta(H_3)g(H_1, H_4) + \frac{1}{2}\eta(H_3)\mathcal{S}(H_1, H_4) = 0, \tag{7.70}$$

which implies that $\eta(H_3) \neq 0$, thus from (7.70) we yield

$$\mathcal{S}(H_1, H_4) = -2ng(H_1, H_4). \tag{7.71}$$

□

Thus the Theorem 7.1 is finished.

Following Section 6, we derive:

Theorem 7.2. *If $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Corollary 7.1. *If an $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 7.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -4n,\ r = -4n$, or $r > -4n$.*

Corollary 7.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady or shrinking as $r < -2n(2n+1),\ r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 7.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Theorem 7.3. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 admits η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n + 1) + \gamma_2.$$

8. An Example

The notion of Ricci η-parallelity for Sasakian manifolds was introduced by M. Kon [16]. In [8] the authors proved that a three-dimensional Kenmotsu manifold has η-parallel Ricci tensor if and only if it is of constant scalar curvature. So, we verify the theorem obtained in [8] by a concrete example.

Let a 3-dimensional manifold $\mathbb{M} = \{(h_1, h_2, h_3) \in \mathbb{R}^3 : h_3 \neq 0\}$, where (h_1, h_2, h_3) are the standard coordinates and the linearly independent vector fields in $\mathbb{R}^3$ as follows

$$p_1 = e^{h_3}\frac{\partial}{\partial h_1}, \quad p_2 = e^{h_3}\frac{\partial}{\partial h_2}, \quad p_3 = -\frac{\partial}{\partial h_3}.$$

We defined the Riemannian metric g by

$$g(p_i, p_j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let φ be a $(1,1)$ tensor field defined by

$$\varphi(p_1) = -p_2, \quad \varphi(p_2) = p_1, \quad \varphi(p_3) = 0.$$

If η denote the 1-form defined by $\eta(H_1) = g(H_1, p_3)$ for any $H_1 \in \mathcal{X}(M)$. Then we have

$$\varphi^2 H_1 = -H_1 + \eta(H_1)p_3, \ \eta(p_3) = 1,$$

$$g(\varphi H_1, \varphi H_2) = g(H_1, H_2) - \eta(H_1)\eta(H_2),$$

for any $H_2 \in \chi(\mathbb{M})$. Then for p_3=ξ, the structure (φ, ξ, η, g) establish an almost contact metric structure on $\mathbb{M}^3$.

Let ∇ be the Levi-Civita connection with respect to g. We have

$$[p_1, p_2] = 0, \quad [p_2, p_3] = p_2, \quad [p_3, p_1] = -p_1.$$

Using Koszul's formula, we can obtain

$$\nabla_{p_1}p_1 = -p_1, \quad \nabla_{p_2}p_1 = 0, \quad \nabla_{p_3}p_1 = 0,$$

$$\nabla_{p_1}p_2 = 0, \quad \nabla_{p_2}p_2 = -p_3, \quad \nabla_{p_3}p_2 = 0,$$

$$\nabla_{p_1}p_3 = p_1, \quad \nabla_{p_2}p_3 = p_2, \quad \nabla_{p_3}p_3 = 0.$$

As per above consequence for p_3=ξ, the manifold satisfies $\nabla_{H_1}\xi$=$H_1 - \eta(H_1)\xi$. Therefore, it can be classified as a Kenmotsu manifold.

Now, the components of curvature tensor $\mathcal{R}$ are as follows

$$\mathcal{R}(p_1,p_2)p_3 = 0, \quad \mathcal{R}(p_2,p_3)p_3 = -p_2, \quad \mathcal{R}(p_1,p_3)p_3 = -p_1,$$

$$\mathcal{R}(p_1,p_2)p_2 = -p_1, \quad \mathcal{R}(p_2,p_3)p_2 = -p_3, \quad \mathcal{R}(p_1,p_3)p_2 = 0,$$

$$\mathcal{R}(p_1,p_2)p_1 = 0, \quad \mathcal{R}(p_2,p_3)p_1 = 0, \quad \mathcal{R}(p_1,p_3)p_1 = p_1.$$

Also the Ricci tensor $\mathcal{S}$, one can get

$$\mathcal{S}(p_1,p_1) = \mathcal{S}(p_2,p_2) = \mathcal{S}(p_3,p_3) = -2.$$

Again, we can easily verify the following

$$\nabla_{H_1}\mathcal{S}(\varphi p_1,\varphi p_2) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_2,\varphi p_3) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_1,\varphi p_1) = 0,$$

$$\nabla_{H_1}\mathcal{S}(\varphi p_1,\varphi p_3) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_3,\varphi p_1) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_2,\varphi p_2) = 0,$$

$$\nabla_{H_1}\mathcal{S}(\varphi p_2,\varphi p_1) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_3,\varphi p_2) = 0,\ \nabla_{H_1}\mathcal{S}(\varphi p_3,\varphi p_3) = 0.$$

Therefore, we conclude that $\nabla_{H_1}\mathcal{S}(\varphi H_2,\varphi H_3) = 0$, for all $H_1, H_2, H_3 \in \chi(\mathbb{M})$.

So, the Ricci tensor is η-parallel. Also, the scalar curvature of the manifold is -6, then the Theorems $3.1, 4.1, 5.1, 6.1$ and 7.1 are effectively satisfied by this example.

9. Conclusion

As a generalization of ρ-Einstein soliton [2], we study a new type soliton is called an η-ρ-Einstein soliton and gradient η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold admitting extended $\mathcal{M}$-Projective curvature tensor.The study of such new types of solitons is of significant interest from different fields due to its wide applications in general relativity, cosmology, quantum field theory, string theory, thermodynamics, mathematical physics, etc. That is why, we depict some geometrical properties of an η-ρ-Einstein soliton and gradient η-ρ-Einstein soliton on such manifold.

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