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A SOLITONIC STUDY ON PARA-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC NONMETRIC CONNECTION

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Abstract. In this paper we have introduced a new semi-symmetric nonmetric connection (briefly, SSNM-connection) and established its existence on para-Sasakian manifold. We obtain Riemannian curvature tensor, Ricci tensor, scalar curvature etc. with respect to the SSNM-connection and studied the properties of para-Sasakian manifold with the help of this connection. We also study η-Einstein soliton on para-Sasakian manifolds with respect to this connection and prove that a para-Sasakian manifold admitting η-Einstein soliton with respect to the SSNM-connection is a generalized η-Einstein manifold. Further, we investigate η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{R}.\overline{S} = 0, \overline{S}.\overline{R} = 0$ and $\overline{R}.\overline{R} = 0$, where $\overline{R}$ and $\overline{S}$ are Riemannian curvature tensor and Ricci tensor with respect to the SSNM-connection, respectively. At last, some conclusions are made after observing all the results and an example of 3-dimensional para-Sasakian manifold admitting the SSNM-connection is given in which all the results can be verified easily.

Keywords: Para-Sasakian manifold, Semi-symmetric nonmetric connection, Einstein soliton, η-Einstein soliton.

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1. Introduction

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds were introduced by Sato and Matsumoto [26]. Later, Adati and Matsumoto investigate some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold have been studied by many authors. For instance, we see [2, 16, 17, 19, 21, 25, 28] and their references.

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In 1924, Friedmann and Schouten gave the notion of semi-symmetric connection on a differentiable manifold. A linear connection on a differentiable manifold M is said to be semi-symmetric if its torsion tensor T satisfies

$$T(\Lambda_1, \Lambda_2) = \pi(\Lambda_2)\Lambda_1 - \pi(\Lambda_1)\Lambda_2, \tag{1.1}$$

for all Λ_1, $\Lambda_2 \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M and π is a 1-form associated with the vector field P given by

$$\pi(\Lambda_1) = \delta(\Lambda_1, P),$$

where δ is a metric on M. In 1932, Hayden [14] introduced the semi-symmetric metric connection on a Riemannian manifold and later it was named as Hayden connection. A linear connection ∇ is said to be metric connection if

$$(\nabla_{\Lambda_1}\delta)(\Lambda_2, \Lambda_3) = 0, \tag{1.2}$$

otherwise it is nonmetric. A systematic study of semi-symmetric metric connection was initiated by Yano [31] in 1970. He proved that a Riemannian manifold with respect to the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. The study of semi-symmetric metric connection was further developed by Amur and Puzara [4], Binh [5], De [11], Ozgur and Sular [20], Singh and Pandey [27] and many others.

On the other hand, semi-symmetric nonmetric connetion whose torsion is given by (1.1) was introduced by Agashe and Chafle [3] in 1992. They showed that a Riemannian manifold is projectively flat if it's curvature tensor with respect to the SSNM-connection vanishes. This linear connection was further developed by many researchers such as Chaubey and Ojha [9], De and Kamilya [12], De, Han and Zhao [13], Prasad and Singh [22], Prasad and Verma [23] and many others. Recently, in [10], Chaubey and Yieldiz defined a new type of SSNM-connection on Remannian manifolds. They investigated various curvature properties of Riemannian manifold with respect to the SSNM-connection and studied Ricci soliton on Riemannian manifold with respect to this connection. Motivated by their studies, here the SSNM-connection has been introduced on para-Sasakian manifold to study some properties and explore η-Einstein soliton on this manifold.

R. S. Hamilton was the first who introduced the notion of Ricci flow in the early 1980s. His [15] observation on Ricci flow was that it is a tool by which the formation of a manifold can be simplified. It is the process which deforms the metric of a differentiable manifold by smoothing out the irregularities. The equation of Ricci flow is given by

$$\frac{\partial \delta}{\partial t} = -2S, \tag{1.3}$$

where δ is a Riemannian metric, S is Ricci curvature tensor and t being the time. The solitons for the Ricci flow is the self similar solutions of the above partial differential equation, where the metrices at various times differ by a diffeomorphism of the manifold. A triple (δ, V, λ) is used to represent a Ricci soliton regard to Ricci flow, where V is a smooth vector field and λ is a scalar, which satisfies the equation

$$L_V \delta + 2S + 2\lambda\delta = 0, \tag{1.4}$$

where $L_V \delta$ denotes the Lie derivative of δ along the vector field V. A Ricci soliton is said to be shrinking if $\lambda < 0$, steady if $\lambda = 0$ and expanding if $\lambda > 0$. The vector field V is called potential vector field and if it is a gradient of a differentiable function, then the Ricci soliton (δ, V, λ) is said to be a gradient Ricci soliton and the associated differentiable function is named as potential function. Ricci soliton was further studied by many researchers. For instance, we see [8, 18, 24, 29, 30] and their references.

Catino and Mazzieri [7] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure $(\phi, \varsigma, \eta, \delta)$ is said to have an Einstein soliton (δ, V, λ) if

$$L_V \delta + 2S + (2\lambda - r)\delta = 0, \tag{1.5}$$

holds, where r being the scalar curvature. The Einstein soliton (δ, V, λ) is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation given by

$$\frac{\partial \delta}{\partial t} = -2S + r\delta. \tag{1.6}$$

Again as a generalization of Einstein soliton, the η-Einstein soliton on a Riemannian manifold $M\,(\phi, \varsigma, \eta, \delta)$ was introduced by Blaga [6] and it is given by

$$L_V \delta + 2S + (2\lambda - r)\delta + 2\beta\eta \otimes \eta = 0, \tag{1.7}$$

where, β is some constant. When $\beta = 0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\beta \neq 0$, the data $(\delta, V, \lambda, \beta)$ is called proper

η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$.

Definition 1.1. *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(\Lambda_2, \Lambda_3) = l_1 \delta(\Lambda_2, \Lambda_3) + l_2 \eta(\Lambda_2) \eta(\Lambda_3),$$

for all $\Lambda_2, \Lambda_3 \in \chi(M)$, where l_1, l_2 are scalars.

Definition 1.2. *A para-Sasakian manifold M is called a generalized η-Einstein manifold if its Ricci tensor is of the form*

$$S(\Lambda_2, \Lambda_3) = k_1 \delta(\Lambda_2, \Lambda_3) + k_2 \eta(\Lambda_2) \eta(\Lambda_3) + k_3 \delta(\Lambda_2, \phi\Lambda_3),$$

for all $\Lambda_2, \Lambda_3 \in \chi(M)$, where k_1, k_2 and k_3 are scalars.

This paper is structured as follows:

First two sections of the paper has been kept for introduction and preliminaries. In **Section-3**, we introduce semi-symmetric nonmetric connection ($\overline{\nabla}$) on para-Sasakian manifolds. In **Section-4**, we study η-Einstein soliton on para-Sasakian manifold with respect to $\overline{\nabla}$. **Section-5** deals with η-Einstein soliton on para-Sasakian manifold satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$. **Section-6** concerns with η-Einstein soliton on para-Sasakian manifold satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$. **Section-7** contains η-Einstein soliton on para-Sasakian manifold satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{R} = 0$. **Section-8** contains a non trivial example of three dimensional para-Sasakian manifold admitting semi-symmetric non metric connection.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with structure (ϕ, ς, η), where η is a 1-form, ς is the structure vector field, ϕ is a $(1,1)$-tensor field satisfying [26]

$$\phi^2(\Lambda_1) = \Lambda_1 - \eta(\Lambda_1)\varsigma, \eta(\varsigma) = 1, \tag{2.8}$$

$$\phi(\varsigma) = 0, \eta \circ \phi = 0, \tag{2.9}$$

for all vector field Λ_1 on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ς, η) admits a pseudo-Riemannian metric δ such that [32]

$$\delta(\phi\Lambda_1, \phi\Lambda_2) = -\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2), \tag{2.10}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure $(\phi, \varsigma, \eta, \delta)$. From (2.10) one can deduce that

$$\begin{aligned} \delta\left(\Lambda_1, \phi\Lambda_2\right) &= -\delta\left(\phi\Lambda_1, \Lambda_2\right), & (2.11)\\ \delta(\Lambda_1, \varsigma) &= \eta(\varsigma). & (2.12) \end{aligned}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [32] if

$$\delta\left(\Lambda_1, \phi\Lambda_2\right) = d\eta(\Lambda_1, \Lambda_2),$$

for all vector fields Λ_1, Λ_2 on M, where

$$d\eta(\Lambda_1, \Lambda_2) = \frac{1}{2}\left\{\Lambda_1\eta(\Lambda_2) - \Lambda_2\eta(\Lambda_1) - \eta([\Lambda_1, \Lambda_2])\right\}.$$

The manifold M is called a para-Sasakian manifold if

$$\left(\nabla_{\Lambda_1}\phi\right)\Lambda_2 = -\delta\left(\Lambda_1, \Lambda_2\right)\varsigma + \eta\left(\Lambda_2\right)\Lambda_1, \quad (2.13)$$

for any smooth vector fields Λ_1, Λ_2 on M.

In a para-Sasakian manifold the following relations also hold [32]

$$\begin{aligned} \left(\nabla_{\Lambda_1}\eta\right)\Lambda_2 &= \delta\left(\Lambda_1, \phi\Lambda_2\right), \nabla_{\Lambda_1}\varsigma = -\phi\Lambda_1, & (2.14)\\ \eta\left(R\left(\Lambda_1, \Lambda_2\right)\Lambda_3\right) &= \delta(\Lambda_1, \Lambda_3)\eta\left(\Lambda_2\right) - \delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right), & (2.15)\\ R\left(\Lambda_1, \Lambda_2\right)\varsigma &= \eta\left(\Lambda_1\right)\Lambda_2 - \eta\left(\Lambda_2\right)\Lambda_1, & (2.16)\\ R(\varsigma, \Lambda_1)\Lambda_2 &= -\delta(\Lambda_1, \Lambda_2)\varsigma + \eta\left(\Lambda_2\right)\Lambda_1, & (2.17)\\ R(\Lambda_1, \varsigma)\Lambda_2 &= \delta(\Lambda_1, \Lambda_2)\varsigma - \eta\left(\Lambda_2\right)\Lambda_1, & (2.18)\\ R(\varsigma, \Lambda_1)\varsigma &= \Lambda_1 - \eta\left(\Lambda_1\right)\varsigma, & (2.19)\\ S\left(\Lambda_1, \varsigma\right) &= -\left(n-1\right)\eta\left(\Lambda_1\right), & (2.20)\\ S\left(\varsigma, \varsigma\right) &= -\left(n-1\right), Q\varsigma = -\left(n-1\right)\varsigma, & (2.21)\\ S\left(\phi\Lambda_1, \phi\Lambda_2\right) &= S\left(\Lambda_1, \Lambda_2\right) + (n-1)\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right), & (2.22) \end{aligned}$$

for any smooth vector fields Λ_1, Λ_2 and Λ_3 on M.

3. Semi-symmetric nonmetric connection on para-Sasakian manifolds

In this section we get the relation between SSNM-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SSNM-connection. We also establish here the first Bianchi identity with respect to SSNM-connection on M.

Let $M\ (\phi,\varsigma,\eta,\delta)$ be an n-dimensional para-Sasakian manifold equipped with Levi-Civita connection ∇ corresponding to the Riemannian metric δ. Let a linear connection $\overline{\nabla}$ on M be defined by

$$\overline{\nabla}_{\Lambda_1}\Lambda_2 = \nabla_{\Lambda_1}\Lambda_2 + \frac{1}{2}\left[\eta\left(\Lambda_2\right)\Lambda_1 - \eta\left(\Lambda_1\right)\Lambda_2\right], \tag{3.23}$$

for all $\Lambda_1,\ \Lambda_2 \in \chi\left(M\right).$

Using the fact that ∇ is a metric connection, we have from (3.23) that

$$\begin{aligned}\left(\overline{\nabla}_{\Lambda_1}\delta\right)\left(\Lambda_2,\Lambda_3\right) &= \frac{1}{2}\left[\delta\left(\Lambda_1,\Lambda_2\right)\eta\left(\Lambda_3\right)+\delta\left(\Lambda_1,\Lambda_3\right)\eta\left(\Lambda_2\right)\right]\\ &\quad -\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right),\end{aligned} \tag{3.24}$$

for all $\Lambda_1,\ \Lambda_2,\ \ \Lambda_3 \in \chi\left(M\right).$ Therefore $\overline{\nabla}$ is a nonmetric connection on M. The torsion tensor of $\overline{\nabla}$ is given by

$$\overline{T}\left(\Lambda_1,\Lambda_2\right) = \eta\left(\Lambda_2\right)\Lambda_1 - \eta\left(\Lambda_1\right)\Lambda_2. \tag{3.25}$$

Suppose that the connection $\overline{\nabla}$ defined on M is connected with the Levi-Civita connection ∇ by the relation

$$\overline{\nabla}_{\Lambda_1}\Lambda_2 = \nabla_{\Lambda_1}\Lambda_2 + \mathcal{H}\left(\Lambda_1,\Lambda_2\right), \tag{3.26}$$

where $\mathcal{H}\left(\Lambda_1,\Lambda_2\right)$ is a tensor field of type $(1,1)$. By definition of torsion tensor, we have

$$\overline{T}\left(\Lambda_1,\Lambda_2\right) = \mathcal{H}\left(\Lambda_1,\Lambda_2\right) - \mathcal{H}\left(\Lambda_2,\Lambda_1\right). \tag{3.27}$$

In view of (3.25) and (3.26) we have

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)+\delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_3\right),\Lambda_2\right) &= \frac{1}{2}\delta\left(\Lambda_1,\Lambda_2\right)\eta\left(\Lambda_3\right)+\frac{1}{2}\delta\left(\Lambda_1,\Lambda_3\right)\eta\left(\Lambda_2\right)\\ &\quad -\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right),\end{aligned} \tag{3.28}$$

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_2,\Lambda_1\right),\Lambda_3\right)+\delta\left(\mathcal{H}\left(\Lambda_2,\Lambda_3\right),\Lambda_1\right) &= \frac{1}{2}\delta\left(\Lambda_2,\Lambda_1\right)\eta\left(\Lambda_3\right)+\frac{1}{2}\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right)\\ &\quad -\delta\left(\Lambda_1,\Lambda_3\right)\eta\left(\Lambda_2\right),\end{aligned} \tag{3.29}$$

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_3,\Lambda_1\right),\Lambda_2\right)+\delta\left(\mathcal{H}\left(\Lambda_3,\Lambda_2\right),\Lambda_1\right) &= \frac{1}{2}\delta\left(\Lambda_3,\Lambda_1\right)\eta\left(\Lambda_2\right).+\frac{1}{2}\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right)\\ &\quad -\delta\left(\Lambda_1,\Lambda_2\right)\eta\left(\Lambda_3\right).\end{aligned} \tag{3.30}$$

In view of (3.27), (3.28), (3.29) and (3.30), we have

$$\begin{aligned} & \delta\left(\overline{T}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)+\delta\left(\overline{T}\left(\Lambda_3,\Lambda_1\right),\Lambda_2\right)+\delta\left(\overline{T}\left(\Lambda_3,\Lambda_2\right),\Lambda_1\right) \\ = & \ \delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)-\delta\left(\mathcal{H}\left(\Lambda_2,\Lambda_1\right),\Lambda_3\right)+\delta\left(\mathcal{H}\left(\Lambda_3,\Lambda_1\right),\Lambda_2\right) \\ & -\delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_3\right),\Lambda_2\right)+\delta\left(\mathcal{H}\left(\Lambda_3,\Lambda_2\right),\Lambda_1\right)-\delta\left(\mathcal{H}\left(\Lambda_2,\Lambda_3\right),\Lambda_1\right) \\ = & \ 2\delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)-2\delta\left(\Lambda_1,\Lambda_2\right)\eta\left(\Lambda_3\right) \\ & +\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right)-\delta\left(\Lambda_1,\Lambda_3\right)\eta\left(\Lambda_2\right). \end{aligned} \tag{3.31}$$

Setting

$$\delta\left(\overline{T}\left(\Lambda_3,\Lambda_1\right),\Lambda_2\right) = \delta\left(T^*\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right), \tag{3.32}$$

$$\delta\left(\overline{T}\left(\Lambda_3,\Lambda_2\right),\Lambda_1\right) = \delta\left(T^*\left(\Lambda_2,\Lambda_1\right),\Lambda_3\right), \tag{3.33}$$

in (3.31), we get

$$\begin{aligned} & \delta\left(\overline{T}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)+\delta\left(T^*\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)+\delta\left(T^*\left(\Lambda_2,\Lambda_1\right),\Lambda_3\right) \\ = & \ 2\delta\left(\mathcal{H}\left(\Lambda_1,\Lambda_2\right),\Lambda_3\right)-2\delta\left(\Lambda_1,\Lambda_2\right)\eta\left(\Lambda_3\right) \\ & +\delta\left(\Lambda_2,\Lambda_3\right)\eta\left(\Lambda_1\right)-\delta\left(\Lambda_1,\Lambda_3\right)\eta\left(\Lambda_2\right), \end{aligned} \tag{3.34}$$

which implies that

$$\begin{aligned} 2\mathcal{H}\left(\Lambda_1,\Lambda_2\right) = & \ \frac{1}{2}\left[\overline{T}\left(\Lambda_1,\Lambda_2\right)+T^*\left(\Lambda_1,\Lambda_2\right)+T^*\left(\Lambda_2,\Lambda_1\right)\right] \\ & +\delta\left(\Lambda_1,\Lambda_2\right)\varsigma+\frac{1}{2}\left[\eta\left(\Lambda_2\right)\Lambda_1-\eta\left(\Lambda_1\right)\Lambda_2\right]. \end{aligned} \tag{3.35}$$

From (3.25), (3.32) and (3.33), it follows that

$$T^*\left(\Lambda_1,\Lambda_2\right) = -\delta\left(\Lambda_1,\Lambda_2\right)\varsigma+\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right), \tag{3.36}$$

$$T^*\left(\Lambda_2,\Lambda_1\right) = -\delta\left(\Lambda_1,\Lambda_2\right)\varsigma+\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right). \tag{3.37}$$

Substituting (3.25), (3.36) and (3.37) in (3.35), we obtain

$$\mathcal{H}\left(\Lambda_1,\Lambda_2\right)=\frac{1}{2}\left[\eta\left(\Lambda_2\right)\Lambda_1-\eta\left(\Lambda_1\right)\Lambda_2\right]. \tag{3.38}$$

In reference to (3.26) and (3.38), we can easily bring out the equation (3.23).

Theorem 3.1. *There exists a unique semi-symmetric nonmetric connection* $\overline{\nabla}$ *on a para-Sasakian manifold* M *given by (3.23).*

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\begin{aligned} \left(\overline{\nabla}_{\Lambda_1}\eta\right)\Lambda_2 &= -\frac{1}{2}\delta\left(\phi\Lambda_1,\phi\Lambda_2\right), \qquad (3.39)\\ \overline{\nabla}_{\Lambda_1}\varsigma &= -\phi\Lambda_1+\frac{1}{2}\left[\Lambda_1-\eta\left(\Lambda_1\right)\varsigma\right], \qquad (3.40)\end{aligned}$$

for all $\Lambda_1,\ \Lambda_2 \in \chi(M)$.

Let $\overline{R}$ be the Riemannian curvature tensor with respect to SSNM-connection on a para-Sasakian manifold defined as

$$\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3=\overline{\nabla}_{\Lambda_1}\overline{\nabla}_{\Lambda_2}\Lambda_3-\overline{\nabla}_{\Lambda_2}\overline{\nabla}_{\Lambda_1}\Lambda_3-\overline{\nabla}_{[\Lambda_1,\Lambda_2]}\Lambda_3. \qquad (3.41)$$

In reference of (2.13), (2.14) and (3.23) we have

$$\begin{aligned} \overline{\nabla}_{\Lambda_1}\overline{\nabla}_{\Lambda_2}\Lambda_3 &= \nabla_{\Lambda_1}\nabla_{\Lambda_2}\Lambda_3+\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2+\eta\left(\nabla_{\Lambda_1}\Lambda_3\right)\Lambda_2+\eta\left(\Lambda_3\right)\nabla_{\Lambda_1}\Lambda_2\right]\\ &\quad -\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3+\eta\left(\nabla_{\Lambda_1}\Lambda_2\right)\Lambda_3+\eta\left(\Lambda_2\right)\nabla_{\Lambda_1}\Lambda_3\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_{\Lambda_2}\Lambda_3\right)\Lambda_1-\eta\left(\Lambda_1\right)\nabla_{\Lambda_2}\Lambda_3\right]\\ &\quad +\frac{1}{4}\left[\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right)\Lambda_3-\eta\left(\Lambda_1\right)\eta\left(\Lambda_3\right)\Lambda_2\right], \qquad (3.42)\\ \overline{\nabla}_{[\Lambda_1,\Lambda_2]}\Lambda_3 &= \nabla_{[\Lambda_1,\Lambda_2]}\Lambda_3+\frac{1}{2}\left[\eta\left(\Lambda_3\right)\nabla_{\Lambda_1}\Lambda_2-\eta\left(\Lambda_3\right)\nabla_{\Lambda_2}\Lambda_1\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_{\Lambda_2}\Lambda_1\right)\Lambda_3-\eta\left(\nabla_{\Lambda_1}\Lambda_2\right)\Lambda_3\right]. \qquad (3.43)\end{aligned}$$

Interchanging Λ_1 and Λ_2 in (3.42) and using it along with (3.42) and (3.43) in (3.41) we get

$$\begin{aligned} \overline{R}(\Lambda_1,\Lambda_2)\Lambda_3 &= R(\Lambda_1,\Lambda_2)\Lambda_3+\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2-\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1-2\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3\right]\\ &\quad +\frac{1}{4}\left[\eta\left(\Lambda_2\right)\Lambda_1-\eta\left(\Lambda_1\right)\Lambda_2\right]\eta\left(\Lambda_3\right), \qquad (3.44)\end{aligned}$$

for all $\Lambda_1,\ \Lambda_2,\ \Lambda_3 \in \chi(M)$.

Writing the equation (3.44) by cyclic permutations of Λ_1, Λ_2 and Λ_3 and using first Bianchi identity with respect to Levi-Civita connection we get

$$\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3+\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1+\overline{R}(\Lambda_3,\Lambda_1)\Lambda_2=2\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2-\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1-\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3\right].$$

Proposition 3.1. *The SSNM-connection satisfies first Bianchi identity if and only if*

$$\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2=\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1+\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3,$$

holds for all $\Lambda_1,\ \Lambda_2$ *and* $\Lambda_3 \in \chi(M)$.

Taking inner product of (3.44) with a vector field Λ and contracting over Λ_1 and Λ we get

$$\begin{aligned}\overline{S}(\Lambda_2,\Lambda_3) &= S(\Lambda_2,\Lambda_3)-\frac{1}{2}(n-3)\delta(\Lambda_2,\phi\Lambda_3)\\ &\quad+\frac{1}{4}(n-1)\eta(\Lambda_2)\eta(\Lambda_3), \qquad (3.45)\end{aligned}$$

where $\overline{S}$ denotes Ricci tensor with respect to $\overline{\nabla}$.

Lemma 3.1. *Let M be an n-dimensional para-Sasakian manifold admitting SSNM-connection, then*

$$\begin{aligned}\eta\left(\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3\right) &= \delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2)-\delta(\Lambda_2,\Lambda_3)\eta(\Lambda_1)-\delta(\Lambda_1,\phi\Lambda_2)\eta(\Lambda_3)\\ &\quad\frac{1}{2}\left[\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_2)-\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_1)\right], \qquad (3.46)\\ \overline{R}(\Lambda_1,\Lambda_2)\varsigma &= \frac{3}{4}\left[\eta(\Lambda_1)\Lambda_2-\eta(\Lambda_2)\Lambda_1\right]-\delta(\Lambda_1,\phi\Lambda_2)\varsigma, \qquad (3.47)\end{aligned}$$

$$\begin{aligned}\overline{R}(\varsigma,\Lambda_2)\Lambda_3 &= -\delta(\Lambda_2,\Lambda_3)\varsigma-\frac{1}{2}\delta(\Lambda_2,\phi\Lambda_3)\varsigma\\ &\quad+\frac{3}{4}\eta(\Lambda_3)\Lambda_2+\frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_3)\varsigma, \qquad (3.48)\\ \overline{R}(\Lambda_1,\varsigma)\Lambda_3 &= \delta(\Lambda_1,\Lambda_3)\varsigma+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\varsigma\\ &\quad-\frac{3}{4}\eta(\Lambda_3)\Lambda_1-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\varsigma, \qquad (3.49)\\ \overline{Q}\Lambda_1 &= Q\Lambda_1-\frac{1}{2}(n-3)\phi\Lambda_1+\frac{1}{4}(n-1)\eta(\Lambda_1)\varsigma, \qquad (3.50)\\ \overline{S}(\Lambda_1,\varsigma) &= -\frac{3}{4}(n-1)\eta(\Lambda_1), \qquad (3.51)\\ \overline{Q}\varsigma &= -\frac{3}{4}(n-1)\varsigma, \qquad (3.52)\\ \overline{r} &= r+\frac{1}{4}(n-1)-\frac{1}{2}(n-3)\psi, \qquad (3.53)\end{aligned}$$

for all Λ_1, Λ_2 and $\Lambda_3 \in \chi(M)$, where $\psi = trace(\phi)$ and $\overline{R}$, $\overline{Q}$, $\overline{r}$ denote Riemannian curvature tensor, Ricci operator, scalar curvature with respect to $\overline{\nabla}$, respectively.

Eigen value of Ricci operator with respect to SSNM-connection corresponding to the eigen vector is $-\frac{3}{4}(n-1)$.

4. η-Einstein soliton on para-Sasakian manifold with respect to SSNM-connection

In this section we find the condition of η-Einstein soliton on a para-Sasakian manifold M to be invariant under SSNM-connection. Further, we study η-Einstein soliton on M with respect to SSNM-connection in which the potential vector field being pointwise collinear with the structure vector field of M.

The equation (1.7) with respect to SSNM-connection takes the form

$$0=\left(\overline{L}_{V}\delta\right)(\Lambda_1,\Lambda_2)+2\overline{S}(\Lambda_1,\Lambda_2)+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2), \tag{4.54}$$

for all Λ_1, Λ_2, Λ_3, $V\in\chi(M)$. Expanding $\overline{L}_V$ and using (3.45), (3.53) in (4.54) we get

$$\begin{aligned}0 &= \delta(\overline{\nabla}_{\Lambda_1}V,\Lambda_2)+\delta(\Lambda_1,\overline{\nabla}_{\Lambda_2}V)+2\overline{S}(\Lambda_1,\Lambda_2)\\ &\quad+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2)\\ &= (L_V\delta)(\Lambda_1,\Lambda_2)+2S(\Lambda_1,\Lambda_2)+(2\lambda-r)\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2)\\ &\quad+\left[\eta(V)-\frac{1}{4}(n-1)+\frac{1}{2}(n-3)\psi\right]\delta(\Lambda_1,\Lambda_2)-\frac{1}{2}\delta(V,\Lambda_2)\eta(\Lambda_1)\\ &\quad-\frac{1}{2}\delta(V,\Lambda_1)\eta(\Lambda_2)-(n-3)\delta(\Lambda_1,\phi\Lambda_2)+\frac{1}{2}(n-1)\eta(\Lambda_1)\eta(\Lambda_2).\end{aligned} \tag{4.55}$$

Theorem 4.1. *An η-Einstein soliton (δ,V,λ,β) on a para-Sasakian manifold M to be invariant under SSNM-connection if and only if*

$$\begin{aligned}0 &= \left[\eta(V)-\frac{1}{4}(n-1)+\frac{1}{2}(n-3)\psi\right]\delta(\Lambda_1,\Lambda_2)-\frac{1}{2}\delta(V,\Lambda_2)\eta(\Lambda_1)\\ &\quad-\frac{1}{2}\delta(V,\Lambda_1)\eta(\Lambda_2)-(n-3)\delta(\Lambda_1,\phi\Lambda_2)+\frac{1}{2}(n-1)\eta(\Lambda_1)\eta(\Lambda_2),\end{aligned}$$

holds for Λ_1, Λ_2, Λ_3, $V\in\chi(M)$.

Consider the distribution D on M as $D=\ker\eta$. If $V\in D$, then

$$\eta(V)=0.$$

Taking covariant derivative with respect to ς and using $(\nabla_\varsigma\eta)V=0$, we get

$$\eta(\nabla_\varsigma V)=0. \tag{4.56}$$

In view of (3.23) and (4.56) we have

$$\eta\left(\nabla^*_\varsigma V\right)=0. \tag{4.57}$$

After expanding the Lie derivative in (4.54) we get

$$\begin{aligned}0 &= \delta(\overline{\nabla}_{\Lambda_1}V,\Lambda_2)+\delta(\Lambda_1,\overline{\nabla}_{\Lambda_2}V)+2\overline{S}(\Lambda_1,\Lambda_2)\\ &\quad+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2).\end{aligned} \tag{4.58}$$

Setting $\Lambda_1 = \Lambda_2 = \varsigma$ and using (3.45), (3.53) and (4.57) in (4.58) we obtain

$$r = 2\left(\lambda + \beta\right) - \frac{7}{4}(n-1) + \frac{1}{2}(n-3)\psi, \tag{4.59}$$

where $trace(\phi) = \psi$.

Theorem 4.2. *Let M be a para-Sasakian manifold admitting η-Einstein soliton $(\delta, V, \lambda, \beta)$ with respect to SSNM-connection such that $V \in D$, then scalar curvature of M is given by (4.59).*

Setting $V = \varsigma$ in (4.54) we get

$$\begin{aligned} 0 &= \delta(\overline{\nabla}_{\Lambda_1}\varsigma, \Lambda_2) + \delta(\Lambda_1, \overline{\nabla}_{\Lambda_2}\varsigma) + 2\overline{S}(\Lambda_1, \Lambda_2) \\ &\quad + (2\lambda - \overline{r})\delta(\Lambda_1, \Lambda_2) + 2\beta\eta(\Lambda_1)\eta(\Lambda_2). \end{aligned} \tag{4.60}$$

Using (3.40) and (4.60) we obtain

$$\overline{S}(\Lambda_1, \Lambda_2) = -\frac{1}{2}(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_2) - \frac{1}{2}\left(2\beta - 1\right)\eta(\Lambda_1)\eta(\Lambda_2). \tag{4.61}$$

Using (3.45) and (3.53) in (4.61) we get

$$S(\Lambda_1, \Lambda_2) = k\delta(\Lambda_1, \Lambda_2) + l\eta(\Lambda_1)\eta(\Lambda_2) + m\delta(\Lambda_1, \phi\Lambda_2), \tag{4.62}$$

where

$$\begin{aligned} k &= -\frac{1}{2}\left[2\lambda - r - \frac{1}{4}(n-5) + \frac{1}{2}(n-3)\psi\right], \\ l &= -\frac{1}{4}\left[4\beta + n - 3)\right], \\ m &= -\frac{1}{2}(n-3). \end{aligned}$$

Corollary 4.1. *If a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection, then M is generalized η-Einstein.*

Corollary 4.2. *If a para-Sasakian manifold M contains an η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection such that the structure vector field ς be parallel i.e., $\nabla_{\Lambda_1}\varsigma = 0$, then M is generalized η-Einstein manifold.*

Setting $\Lambda_2 = \varsigma$ and using (3.51) and (3.53) in (4.61) we have

$$r = 2(\lambda + \beta) - \frac{7}{4}(n-1) + \frac{1}{2}(n-3)\psi, \tag{4.63}$$

where $trace(\phi) = \psi$.

Corollary 4.3. *If a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection, then the scalar curvature of M is given by (4.63).*

Putting $\beta = 0$ and $\psi = 0$ in (4.63) we get

$$\lambda = \frac{1}{2}r + \frac{7}{8}(n-1).$$

Corollary 4.4. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta, \varsigma, \lambda)$ with respect to SSNM-connection, then the soliton is shrinking, steady or expanding if*

$$r < -\frac{7}{4}(n-1), r = -\frac{7}{4}(n-1), r > -\frac{7}{4}(n-1),$$

respectively, provided $trace(\phi) = 0$.

5. η-Einstein soliton on para-Sasakian satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$

The condition that must be satisfied by $\overline{S}$ is

$$\overline{S}(\overline{R}(\varsigma, \Lambda_1)\Lambda_2, \Lambda_3) + \overline{S}(\Lambda_2, \overline{R}(\varsigma, \Lambda_1)\Lambda_3) = 0, \tag{5.64}$$

for all Λ_1, Λ_2, $\Lambda_3 \in \chi(M)$.

Using (3.48) and replacing the expression of $\overline{S}$ from (4.61) in (5.64) we get

$$\begin{aligned} 0 &= \frac{1}{2}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right] \\ &+ \frac{1}{4}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \phi\Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \phi\Lambda_3)\eta(\Lambda_2)\right] \\ &- \frac{3}{8}\left[2\lambda - \overline{r} + 1\right]\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right] \\ &- \frac{1}{4}\left[2\lambda + 8\beta - \overline{r} - 3\right]\eta(\Lambda_1)\eta(\Lambda_2)\eta(\Lambda_3). \end{aligned} \tag{5.65}$$

Setting $\Lambda_3 = \varsigma$ in (5.65) we get

$$\begin{aligned} 0 &= \frac{1}{2}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2)\right] \\ &+ \frac{1}{4}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \phi\Lambda_2)\right] \\ &- \frac{3}{8}\left[2\lambda - \overline{r} + 1\right]\left[\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2)\right] \\ &- \frac{1}{4}\left[2\lambda + 8\beta - \overline{r} - 3\right]\eta(\Lambda_1)\eta(\Lambda_2). \end{aligned} \tag{5.66}$$

Contracting (5.66) over Λ_1 and Λ_2 we get

$$\begin{aligned} 0 &= \frac{1}{4}(n-1+2\psi)\lambda + \frac{1}{2}\left[2(n-1)+\psi\right]\beta \\ &\quad -\frac{1}{8}\left[n-1+2\psi\right]\left[r+\frac{1}{4}(n-1)-\frac{1}{2}(n-3)\psi\right] \\ &\quad -\frac{3}{8}(n-1), \end{aligned} \tag{5.67}$$

where $trace(\phi) = \psi$.

Theorem 5.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$, then the soliton constants are given by the equation (5.67).*

Setting $\beta = \psi = 0$ in (5.67) we obtain

$$2\lambda = r + \frac{1}{4}(n+11).$$

Corollary 5.1. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta, \varsigma, \lambda)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$, then the soliton is shrinking, steady or expanding if*

$$r < -\frac{1}{4}(n+11), r = -\frac{1}{4}(n+11), r > -\frac{1}{4}(n+11),$$

respectively, provided $trace(\phi) = 0$.

6. η-Einstein soliton on para-Sasakian satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$

The condition that must be satisfied by $\overline{S}$ is

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\varsigma - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\Lambda_1 \\ &\quad + S(\Lambda_1, \Lambda_2)\overline{R}(\varsigma, \Lambda_3)\Lambda_4 - \overline{S}(\varsigma, \Lambda_2)\overline{R}(\Lambda_1, \Lambda_3)\Lambda_4 \\ &\quad + \overline{S}(\Lambda_1, \Lambda_3)\overline{R}(\Lambda_2, \varsigma)\Lambda_4 - \overline{S}(\varsigma, \Lambda_3)\overline{R}(\Lambda_2, \Lambda_1)\Lambda_4 \\ &\quad + \overline{S}(\Lambda_1, \Lambda_4)\overline{R}(\Lambda_2, \Lambda_3)\varsigma - \overline{S}(\varsigma, \Lambda_4)\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1, \end{aligned} \tag{6.68}$$

for all Λ_1, Λ_2, Λ_3, $\Lambda_4 \in \chi(M)$. Taking inner product with ς the relation (6.68) becomes

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4) - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\eta(\Lambda_1) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_2)\eta(\overline{R}(\varsigma, \Lambda_3)\Lambda_4) - \overline{S}(\varsigma, \Lambda_2)\eta(\overline{R}(\Lambda_1, \Lambda_3)\Lambda_4) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_3)\eta(\overline{R}(\Lambda_2, \varsigma)\Lambda_4) - \overline{S}(\varsigma, \Lambda_3)\eta(\overline{R}(\Lambda_2, \Lambda_1)\Lambda_4) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_4)\eta(\overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \Lambda_4)\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1). \end{aligned} \tag{6.69}$$

Setting $\Lambda_4 = \varsigma$ in (6.69) we obtain

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\varsigma)\eta(\Lambda_1) \\ &+ \overline{S}(\Lambda_1, \Lambda_2)\eta(\overline{R}(\varsigma, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \Lambda_2)\eta(\overline{R}(\Lambda_1, \Lambda_3)\varsigma) \\ &+ \overline{S}(\Lambda_1, \Lambda_3)\eta(\overline{R}(\Lambda_2, \varsigma)\varsigma) - \overline{S}(\varsigma, \Lambda_3)\eta(\overline{R}(\Lambda_2, \Lambda_1)\varsigma) \\ &+ \overline{S}(\Lambda_1, \varsigma)\eta(\overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \varsigma)\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1). \end{aligned} \tag{6.70}$$

Using (3.46), (3.47), (3.49), (4.61) in (6.70) we get

$$\begin{aligned} 0 &= \frac{3}{8}\left[(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_2) + (2\beta - 1)\eta(\Lambda_1)\eta(\Lambda_2)\right]\eta(\Lambda_3) \\ &- \frac{3}{8}\left[(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_3) - (2\beta - 1)\eta(\Lambda_1)\eta(\Lambda_3)\right]\eta(\Lambda_2) \\ &+ \left(\lambda + \beta - \frac{\overline{r}}{2}\right)\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) - \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right]. \end{aligned} \tag{6.71}$$

Setting $\Lambda_1 = \varsigma$ in (6.71) we get

$$\beta = \frac{1}{2}. \tag{6.72}$$

In view of (4.63) and (6.72) we get

$$\lambda = r + \frac{1}{8}(7n - 11) - \frac{1}{4}(n - 3)\psi, \tag{6.73}$$

where $trace\,(\phi) = \psi$.

Theorem 6.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$, then the soliton constants are given by equations (6.72) and (6.73).*

Corollary 6.1. *There exists no Einstein soliton with respect to SSNM-connection on M satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$.*

7. η-Einstein soliton on para-Sasakian satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{R} = 0$.

The condition must be satisfied by R is

$$\begin{aligned} 0 &= \overline{R}(\varsigma, \Lambda_1)\overline{R}(\Lambda_2, \Lambda_3)\Lambda_4 - \overline{R}(\overline{R}(\varsigma, \Lambda_1)\Lambda_2, \Lambda_3)\Lambda_4 \\ &- \overline{R}(\Lambda_2, \overline{R}(\varsigma, \Lambda_1)\Lambda_3)\Lambda_4 - \overline{R}(\Lambda_2, \Lambda_3)\overline{R}(\varsigma, \Lambda_1)\Lambda_4. \end{aligned} \tag{7.74}$$

Using (3.44), (3.46), (3.47) and (3.48) in (7.74) we get

$$
\begin{aligned}
0 &= -\delta(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\varsigma - \frac{1}{2}\delta(\Lambda_1, \phi\overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\varsigma + \frac{3}{4}\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\Lambda_1 \\
&+\frac{1}{4}\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\eta(\Lambda_1)\varsigma - \frac{3}{4}\eta(\Lambda_1)\overline{R}(\Lambda_1, \Lambda_3)\Lambda_4 - \frac{3}{4}\eta(\Lambda_4)\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1 \\
&-\delta(\Lambda_1, \Lambda_2)\left[\delta(\Lambda_3, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
&-\frac{1}{2}\delta(\Lambda_1, \phi\Lambda_2)\left[\delta(\Lambda_3, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
&+\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\left[\delta(\Lambda_3, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
&-\frac{3}{4}\left[\eta(\Lambda_3)\Lambda_2 - \eta(\Lambda_2)\Lambda_3\right]\left[\delta(\Lambda_1, \Lambda_4) + \frac{1}{2}\delta(\Lambda_1, \phi\Lambda_4) - \frac{1}{4}\eta(\Lambda_4)\eta(\Lambda_1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\delta(\Lambda_2, \phi\Lambda_3)\left[\delta(\Lambda_1, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_1, \phi\Lambda_4)\varsigma - \frac{1}{4}\eta(\Lambda_4)\eta(\Lambda_1)\varsigma\right] - \frac{3}{4}\eta(\Lambda_3)\overline{R}(\Lambda_2, \Lambda_1)\Lambda_4 \\
&+\delta(\Lambda_1, \Lambda_3)\left[\delta(\Lambda_2, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right] \\
&+\frac{1}{2}\delta(\Lambda_1, \phi\Lambda_3)\left[\delta(\Lambda_2, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right] \\
&-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\left[\delta(\Lambda_2, \Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2, \phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right]. \qquad (7.75)
\end{aligned}
$$

Setting $V = \varsigma$ in (7.75) we get

$$
\begin{aligned}
0 &= -\frac{3}{4}\delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\varsigma + \frac{3}{4}\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3)\varsigma + \frac{3}{4}\delta(\Lambda_2, \phi\Lambda_3)\eta(\Lambda_1)\varsigma \\
&-\frac{3}{8}\delta(\Lambda_1, \phi\Lambda_3)\eta(\Lambda_2)\varsigma + \frac{3}{8}\delta(\Lambda_1, \phi\Lambda_2)\eta(\Lambda_3)\varsigma - \frac{3}{4}\delta(\Lambda_2, \phi\Lambda_3)\Lambda_1 \\
&+\frac{3}{4}\left[\delta(\Lambda_1, \Lambda_2) + \frac{1}{2}\delta(\Lambda_1, \phi\Lambda_2) - \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\right]\left[-\eta(\Lambda_3)\varsigma + \Lambda_3\right] \\
&-\frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_1)\Lambda_3 - \eta(\Lambda_3)\Lambda_1\right\} - \delta(\Lambda_1, \phi\Lambda_3)\varsigma\right]\eta(\Lambda_2) \\
&+\frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_2)\Lambda_3 - \eta(\Lambda_3)\Lambda_2\right\} - \delta(\Lambda_2, \phi\Lambda_3)\varsigma\right]\eta(\Lambda_1) - \frac{3}{4}\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1 \\
&+\frac{3}{4}\left[\delta(\Lambda_1, \Lambda_3) + \frac{1}{2}\delta(\Lambda_1, \phi\Lambda_3) - \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\right]\left[-\eta(\Lambda_2)\varsigma + \Lambda_2\right] \\
&-\frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_1)\Lambda_2\right\} - \delta(\Lambda_2, \phi\Lambda_1)\varsigma\right]\eta(\Lambda_3). \qquad (7.76)
\end{aligned}
$$

Taking inner product of (7.76) with a vector field Λ_5 we get

$$\begin{aligned}0 &= -\frac{3}{4}\delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2)\eta(\Lambda_5)+\frac{3}{4}\delta(\Lambda_1,\Lambda_2)\eta(\Lambda_3)\eta(\Lambda_5)+\frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_1)\eta(\Lambda_5)\\ &-\frac{3}{8}\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_2)\eta(\Lambda_5)+\frac{3}{8}\delta(\Lambda_1,\phi\Lambda_2)\eta(\Lambda_3)\eta(\Lambda_5)-\frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\delta(\Lambda_1,\Lambda_5)\\ &+\frac{3}{4}\left[\delta(\Lambda_1,\Lambda_2)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_2)-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\right][-\eta(\Lambda_3)\eta(\Lambda_5)+\delta(\Lambda_3,\Lambda_5)]\\ &-\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_1)\delta(\Lambda_3,\Lambda_5)-\eta(\Lambda_3)\delta(\Lambda_1,\Lambda_5)\}-\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_5)\right]\eta(\Lambda_2)\\ &+\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_2)\delta(\Lambda_3,\Lambda_5)-\eta(\Lambda_3)\delta(\Lambda_2,\Lambda_5)\}-\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_5)\right]\eta(\Lambda_1)\\ &+\frac{3}{4}\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\right][-\eta(\Lambda_2)\eta(\Lambda_5)+\delta(\Lambda_2,\Lambda_5)]\\ &-\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_2)\delta(\Lambda_1,\Lambda_5)-\eta(\Lambda_1)\delta(\Lambda_2,\Lambda_5)\}-\delta(\Lambda_2,\phi\Lambda_1)\eta(\Lambda_5)\right]\eta(\Lambda_3)\\ &-\frac{3}{4}\delta(\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1,\Lambda_5). \qquad (7.77)\end{aligned}$$

Contracting (7.77) over Λ_2 and Λ_5 we obtain

$$\overline{S}(\Lambda_1,\Lambda_3)=-(n-1)\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\eta(\Lambda_1)\eta(\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\right]. \qquad (7.78)$$

Using (4.61) in (7.78) we get

$$\begin{aligned}0 &= \frac{1}{2}(2\lambda-\overline{r}+1)\delta(\Lambda_1,\Lambda_2)+\frac{1}{2}(2\beta-1)\eta(\Lambda_1)\eta(\Lambda_2)\\ &-(n-1)\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\eta(\Lambda_1)\eta(\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\right]. \qquad (7.79)\end{aligned}$$

Setting $\Lambda_2=\varsigma$ in (7.79) we have

$$2(\lambda+\beta)=r+\frac{13}{4}(n-1)-\frac{1}{2}(n-3)\psi, \qquad (7.80)$$

where $trace\,(\phi)=\psi$.

Theorem 7.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma,\Lambda_1).\overline{R}=0$, then the relation between the soliton constants are given by equation (7.80).*

Setting $\beta=0$ in (7.80) we get

$$\lambda=\frac{1}{2}r+\frac{13}{8}(n-1)-\frac{1}{4}(n-3)\psi.$$

Corollary 7.1. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta,\varsigma,\lambda)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma,\Lambda_1).\overline{R}=0$, then the soliton is shrinking, steady and expanding if*

$$r<-\frac{13}{4}(n-1), r=-\frac{13}{4}(n-1), r>-\frac{13}{4}(n-1),$$

respectively, provided $trace(\phi)=0$.

8. Example of para-Sasakian manifold admitting SSNM-connection

Let us consider 3-dimensional manifold

$$M^3=\left\{(x,y,z)\in R^3\right\},$$

where (x,y,z) are the standard co-ordinates in R^3. We choose the linearly independent vector fields

$$E_1=e^x\frac{\partial}{\partial y}, E_2=e^x\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right), E_3=-\frac{\partial}{\partial x}.$$

Let g be the pseudo Riemannian metric defined by $g\left(E_i,E_j\right)=0,\ if\ i\neq j\ for\ i,j=1,2,3,$ and $g\left(E_1,E_1\right)=-1, g\left(E_2,E_2\right)=-1, g\left(E_3,E_3\right)=1$

Let η be the 1-form defined by $\eta\left(X\right)=g\left(X,E_3\right)$ for any $X\in\chi\left(M^3\right)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi E_1 = E_1, \phi E_2=E_2, \phi E_3=0. \tag{8.81}$$

$$trace(\phi) = \sum_{i=1}^{3} g(E_i,\phi E_i)=-2 \tag{8.82}$$

Let X, Y, $Z\in\chi\left(M^3\right)$ be given by

$$X = x_1E_1+x_2E_2+x_3E_3,$$

$$Y = y_1E_1+y_2E_2+y_3E_3,$$

$$Z = z_1E_1+z_2E_2+z_3E_3.$$

Then, we have

$$g\left(X,Y\right) = x_1y_1+x_2y_2+x_3y_3,$$

$$\eta\left(X\right) = x_3,$$

$$g\left(\phi X,\phi Y\right) = x_1y_1+x_2y_2.$$

Using the linearity of g and ϕ, $\eta(E_3) = 1, \phi^2 X = X - \eta(X) E_3$ and $g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$. We have

$$\begin{aligned} [E_1, E_2] &= 0, [E_1, E_3] = -E_1, [E_2, E_3] = E_2, \\ [E_2, E_1] &= 0, [E_3, E_1] = E_1, [E_3, E_2] = -E_2. \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above results we see that the structure (ϕ, ξ, η, g) satisfies

$$(\nabla_X \phi) Y = -g(X,Y)\xi + \eta(Y) X,$$

for all $X, Y \in \chi(M^3)$, where $\eta(\xi) = \eta(E_3) = 1$. Hence $M^3(\phi, \xi, \eta, g)$ is a para-Sasakian manifold.

The components of Riemannian curvature tensor of M^3 are given by

$$\begin{pmatrix} R(E_1,E_2)E_2 & R(E_1,E_3)E_3 & R(E_1,E_2)E_3 \\ R(E_2,E_1)E_1 & R(E_2,E_3)E_3 & R(E_2,E_3)E_1 \\ R(E_3,E_1)E_1 & R(E_3,E_2)E_2 & R(E_3,E_1)E_2 \end{pmatrix} = \begin{pmatrix} -E_1 & -E_1 & 0 \\ E_2 & E_2 & 0 \\ E_3 & E_3 & 0 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 are given by

$$S(E_1,E_1) = S(E_3,E_3) = 0, S(E_2,E_2) = 2. \tag{8.83}$$

Therefore the scalar curvature of M^3 is

$$r = \sum_{i=1}^{3} S(E_i, E_i) = 2. \tag{8.84}$$

Using (3.23) we have the following values of $\overline{\nabla}$:

$$\begin{pmatrix} \overline{\nabla}_{E_1} E_1 & \overline{\nabla}_{E_1} E_2 & \overline{\nabla}_{E_1} E_3 \\ \overline{\nabla}_{E_2} E_1 & \overline{\nabla}_{E_2} E_2 & \overline{\nabla}_{E_2} E_3 \\ \overline{\nabla}_{E_3} E_1 & \overline{\nabla}_{E_3} E_2 & \overline{\nabla}_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -\frac{1}{2}E_1 \\ 0 & E_3 & -\frac{1}{2}E_2 \\ \frac{1}{2}E_1 & \frac{1}{2}E_2 & 0 \end{pmatrix}.$$

By the help of (3.41) and above matrix we get the components of Riemannian curvature tensor of M^3 with respect to SSNM-connection as follows

$$\begin{pmatrix} \overline{R}(E_1,E_2)E_1 & \overline{R}(E_1,E_3)E_1 & \overline{R}(E_2,E_3)E_1 \\ \overline{R}(E_1,E_2)E_2 & \overline{R}(E_1,E_3)E_2 & \overline{R}(E_2,E_3)E_2 \\ \overline{R}(E_1,E_2)E_3 & \overline{R}(E_1,E_3)E_3 & \overline{R}(E_2,E_3)E_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}E_2 & -\frac{3}{2}E_3 & 0 \\ -\frac{1}{2}E_1 & 0 & -\frac{1}{2}E_3 \\ 0 & -\frac{1}{4}E_1 & \frac{1}{4}E_2 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 with respect to SSNM-connection are given by

$$\overline{S}(E_1, E_1) = \overline{S}(E_2, E_2) = 1, \overline{S}(E_3, E_3) = \frac{1}{2}. \tag{8.85}$$

Therefore the scalar curvature of M^3 with respect to SSNM-connection is

$$\overline{r} = \sum_{i=1}^{3} S(E_i, E_i) = \frac{5}{2}. \tag{8.86}$$

In view of (8.82), (8.84) and (8.86) we have

$$\begin{aligned} \overline{r} &= \frac{5}{2} \\ &= 2 + \frac{1}{4}(3-1) - \frac{1}{2}(3-3).(-2) \\ &= r + \frac{1}{4}(n-1) - \frac{1}{2}(n-3)\psi, \end{aligned}$$

which verifies the relation (3.53). Similarly, we can verify all the results obtained.

9. Conclusion

From the results obtained in this paper we can conclude that if a para-Sasakian manifold $M(\phi, \varsigma, \eta, \delta)$ admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to semi-symmetric nonmetric connection, then M is generalized η-Einstein manifold. We also conclude that if a para-Sasakian manifold M admitting η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to semi-symmetric nonmetric connection satisfies $\overline{R}.\overline{S} = 0, \overline{S}.\overline{R} = 0$ and $\overline{R}.\overline{R} = 0$, then the soliton constants depend on scalar curvature of M and trace of the function ϕ on M

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