# ON KANNAN-GERAGHTY MAPS AS AN EXTENSION OF KANNAN MAPS

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ABSTRACT. Extending the concept of weakly Kannan maps on metric spaces, we study the maps as $f : X \to X$ on a metric space $(X, d)$ satisfying condition $d(f(x), f(y)) \leq (1/2)\beta(d(x, y))[d(x, f(x)) + d(y, f(y))]$ for every $x, y \in X$ and a function $\beta : [0, \infty) \to [0, 1)$ where for every sequence $t = \{t_n\}$ of non-negative real numbers satisfying $\beta(t_n) \to 1$, while $t_n \to 0$. Such a map is named the Kannan-Geraghty map because of its relation to weakly Kannan map and Geraghty contraction. Firstly, we show that our new condition is different from weakly Kannan condition. Having proven the fixed point theorem, we present two useful results on Kannan-Geraghty maps. Also, we illustrate some examples of Kannan-Graghty map having interesting properties.

## 1. INTRODUCTION

In 1968, R. Kannan started the study of fixed point theory on some contractive maps. A map $f : X \to X$ on a metric space $(X, d)$ is said to be *contractive* if it satisfies the condition $d(f(x), f(y)) \leq qd(x, y)$ for any $x, y \in X$ and a fixed real number $q \in [0, 1)$. If the coefficient $q$ (instead of a constant number) be a function as $q : X \times X \to [0, 1)$ satisfying the condition $sup\{q(x, y)|x, y \in X, a \leq d(x, y) \leq b\} < 1$ for every positive real numbers $a$ and $b$ (with $a \leq b$), then $f$ is said to be *weakly contractive*.

The well known Kannan Theorem was a variant version of the Banach contraction principle ([5]). The Banach contraction principle says that every contractive map on a complete metric space has a unique fixed point. In [1] Ruis and Melando extended Kannan theorem to the class of weakly Kannan maps, and then they gave a continuation method for this class. Recently, the single and set-valued $\alpha$-$\eta$-$\psi$-contractive mappings have been studied in [4].

In this paper, based on the articles [1] and [8], we present the concept of Kannan-Geraghty maps, and we prove that Kannan-Graghty self mapping has a unique fixed point, and also Kannan-Geraghty non-self mapping has a best proximity point. Then we show two theorems; in the first theorem, we show the relation between weakly Kannan and Kannan Geraghty and in the second, relation between Kannan-Geraghty and weakly Kannan mappings.

## 2. Preliminaries

In this section, we recall some basic notations, definitions and theorems from references [7, 2, 5, 1, 8]. We discuss on the class $\Gamma$ consisting of all of functions $\beta : [0, \infty) \to [0, 1)$ such that for every convergent sequence $t = \{t_n\}$ of non-negative real numbers satisfying $\beta(t_n) \to 1$ while $t_n \to 0$.

**Definition 2.1.** ([2]) Let $(X, d)$ be a metric space. A mapping $f : X \to X$ is said to be a Geraghty-contraction if it satisfies the condition $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ for a continuous function $\beta \in \Gamma$.

**Theorem 2.1.** ([2]) Let $d(X, d)$ be a complete metric space and $f : X \to X$ be a Geraghty-contraction. Then, $f$ has a unique fixed point.

Following the Geraghty notations, for any two disjoint sequences $\bar{x} = \{x_n\}$ and $\bar{y} = \{y_n\}$ of points in a metric space $(X, d)$ (s.t. $x_n \neq y_n$ for $n = 1, 2, 3, \cdots$), we use two sequences of non-negative real numbers $\bar{\delta}(\bar{x}, \bar{y}) := \{\delta_n(\bar{x}, \bar{y})\}$ and $\bar{\Delta}(\bar{x}, \bar{y}) := \{\Delta_n(\bar{x}, \bar{y})\}$ defined by

$$\delta_n(\bar{x}, \bar{y}) = d(x_n, y_n) \ and \ \Delta_n(\bar{x}, \bar{y}) = \frac{d(f(x_n), f(y_n))}{d(x_n, y_n)}.$$

**Theorem 2.2.** ([5]) Let $f : X \to X$ be a contractive mapping on a complete metric space $(X, d)$ and take $x_0 \in X$ and $x_n = f(x_{n-1})$ for $n = 1, 2, 3, \cdots$. Then, $x_n \to x_\infty$ in $X$, where $x_\infty$ is the unique fixed point of $f$, if and only if for any two subsequences $\hat{x} := \{x_{h_n}\}$ and $\tilde{x} := \{x_{k_n}\}$ ( where $x_{h_n} \neq x_{k_n}$ for $n = 1, 2, 3, \cdots$) we have

$$\Delta_n(\hat{x}, \tilde{x}) \to 1 \Rightarrow d_n(\hat{x}, \tilde{x}) \to 0.$$

**Definition 2.2.** ([9]) *Let $(X, d)$ be a metric space. A mapping $f : X \to X$ is said to be weakly Kannan if it satisfies the condition*

$$d(f(x), f(y)) \leq \frac{\overline{\alpha}(x, y)}{2}[d(x, f(x)) + d(y, f(y))]$$

*for every points $x, y \in X$, where $\overline{\alpha} : X \times X \to [0, 1)$ is a real-valued function satisfying the condition*

$$\Theta(a, b) := \sup\{\overline{\alpha}(x, y) : x, y \in X \ \text{and} \ a \leq d(x, y) \leq b\} < 1$$

*for every positive real numbers $a \leq b$.*

**Theorem 2.3.** ([1]) *Let $(X, d)$ be a complete metric space. If $f : X \to X$ is a weakly Kannan mapping, then $f$ has a unique fixed point $x^*$ and the Picard sequence of iterates $\{f^n(x)\}_{n \in N}$ converges to $x^*$ for every $x \in X$.*

Now, let $A, B$ be two nonempty subsets of a metric space $(X, d)$. The subsets $A_0$ and $B_0$ of $A$ and $B$ (respectively) are defined as follow:

$$A_0 := \{x \in A : d(x, y) = d(A, B), \ \text{for some} \ y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B), \ \text{for some} \ x \in A\},$$

where $d(A, B) := \inf\{d(x, y) : x \in A \ \text{and} \ y \in B\}$.

**Definition 2.3.** ([9]) *Let $A, B \subset X$ be two nonempty subsets of a metric space $(X, d)$ and $f : A \to B$ be an arbitrary mapping. An element $x \in A$ is said to be a best proximity point of the mapping $f$ if it satisfies the equality $d(x, f(x)) = d(A, B)$.*

**Definition 2.4.** ([9]) *Assume that $A, B \subset X$ be two nonempty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$.*

*(i) The pair $(A, B)$ is said to have P-property if for any points $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, we have:*

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

*(ii) The pair $(A, B)$ is said to have weak P-property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, we have:*

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2)$$

Here, we remember the straightforward generalization of the concept of weakly Kannan map and Geraghty contraction to the non-self-mapping case.

**Definition 2.5.** *([9]) Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $(X, d)$. A map $f : A \to B$ is said to be weakly Kannan if it satisfies the inequality*

$$d(f(x), f(y)) \leq \frac{\overline{\alpha}(x, y)}{2}[d(x, f(x)) + d(y + f(y)) - 2d(A, B)]$$

*for every $x, y \in X$ and a real-valued function $\overline{\alpha} : X \times X \to [0, 1)$ such that $\Theta(a, b) := \sup\{\overline{\alpha}(x, y) : a \leq d(x, y) \leq b\} \leq 1$ for every real numbers $0 < a \leq b$.*

**Definition 2.6.** *([2]) Let $A, B$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : A \to B$ is said to be a Geraghty contraction if there exists $\beta \in \Gamma$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

*for every $x, y \in A$.*

Notice that since $\beta : [0, \infty) \to [0, 1)$, we have

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) < d(x, y),$$

Therefore, every Geraghty-contraction is a contractive mapping.

Finally, we introduce two new versions of contractive mappings, namely *Kannan-Geraghty maps* separately in selfmapping and non-selfmapping cases, on which we will prove fixed point theorem in the next section.

**Definition 2.7.** *Let $(X, d)$ be a metric space. A mapping $f : X \to X$, is said to be a selfmapping Kannan Geraghty map if there exists a real valued function $\beta \in \Gamma$ such that, for all $x, y \in X$ we have*

$$d(f(x), f(y)) \leq \frac{\beta(d(x, y))}{2}[d(x, f(x)) + d(y, f(y))].$$

**Definition 2.8.** *Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space. A mapping $f : A \to B$ is said to be a non-selfmapping Kannan Geraghty map if there exists real valued function $\beta \in \Gamma$ where we have*

$$d(f(x), f(y)) \leq \frac{\beta(d(x, y))}{2}[d(x, f(x)) + d(y, f(y))] \tag{2.1}$$

*for every $x, y \in A$.*

## 3. Main Results

In this section, we prove some theorems among them Theorem 3.1, and theorem 3.2 is our main result. Also we give some examples.

**Theorem 3.1.** *Let $f : X \to X$ be a Kannan-Geraghty map on a complete metric space $(X, d)$. Then, $f$ has a unique fixed point $u \in X$ and for any $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ converges to $u$.*

**Proof.** Since $f : X \to X$ is Kannan-Geraghty mapping, there exists a function $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Consider any $x_0 \in X$ and define $x_n = f(x_{n-1}), n = 1, 2, \dots$. We assume that $d(x_0, x_1) > 0$, otherwise there is nothing to prove. We prove that $d(x_n, x_{n+1}) \to 0$, and then, $\{x_n\}$ converges to a point $u$ which is the unique fixed point of $f$.

From the following inequality

$$d(f(x_n), f(x_{n-1})) \le \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, f(x_n)) + d(x_{n-1}, f(x_{n-1}))]$$

we have

$$d(x_{n+1}, x_n) \le \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)],$$

which gives

$$d(x_{n+1}, x_n) \le \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)],$$
$$\le [\frac{1}{2}d(x_n, x_{n+1}) + \frac{\beta(d(x_n, x_{n-1}))}{2}d(x_{n-1}, x_n)],$$

and hence, we have

$$d(x_{n+1}, x_n) \le \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}),$$

then

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \tag{3.2}$$

So, by (3.2), $\{d(x_n, x_{n-1})\}$ is a decreasing sequence of non-negative real numbers, and hence there exists $r \ge 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r.$$

In the sequel, we prove that $r = 0$.

Assume $r > 0$, then from (3.2) we have

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \le \beta(d(x_{n-1}, x_n)) < 1, \tag{3.3}$$

for any $n \in \mathbb{N}$. By the Sandwich theorem, from the inequality 3.3, we get $\lim_{n \to \infty} \beta(d(x_{n-1}, x_n)) = 1$, which contradicts with the continuity of $\beta \in \Gamma$. Hence we obtain $r = 0$. Therefore, we have $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$, which means $\lim_{n \to \infty} d(x_n, f(x_n)) = 0$. Now, from the inequality

$$d(f(x_n), f(x_m)) \le \frac{1}{2}\beta(d(x_n, x_m))[d(x_n, f(x_n)) + d(x_m, f(x_m))]$$

for all $m, n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{m+1}) \le \frac{1}{2}\beta(d(x_n, x_m))[dx_n, x_{n+1}) + d(x_m, x_{m+1})],$$

which implies that both sequences $\{x_n\}$ or $\{f(x_n)\}$ are Cauchy sequences. Since $(X, d)$ is complete, the sequence $\{f(x_n)\}$ is convergent to a point $u$, and also, $x_n \to u$. Indeed, $u$ is the fixed point of $f$, because we have:

$$d(u, f(u)) = \lim_{n \to \infty} d(f(x_n), f(u))$$
$$\le \lim_{n \to \infty} \tfrac{1}{2}\beta(d(x_n, u))[d(u, f(u) + d(x_n, fx_n)]$$
$$= \tfrac{1}{2}(\lim_{n \to \infty} \beta(d(x_n, u)))[d(u, f(u)) + 0]$$
$$\le \tfrac{1}{2}d(u, f(u)),$$

which gives $d(u, f(u)) = 0$, and then $f(u) = u$. Finally, we show that $f$ cannot to have another fixed point. Assuming a point $z$ to be a fixed point of $f$, we have

$$d(u, z) = d(f(u), f(z)) \le \beta(d(u, z))[d(u, f(u)) + d(z, f(z))] = 0,$$

hence $z = u$.

**Theorem 3.2.** *Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty. Let $T : A \to B$ be a Kannan-Geraghty mapping defined as Definition 2.8. Suppose that $T(A_0) \subseteq B_0$ and the pair $(A, B)$ has the weak P-property. Then $T$ has a unique best proximity point $x^\star$ in $A$ such that $d(x^\star, Tx^\star) = d(A, B)$.*

**Proof.**          We first prove that $B_0$ is closed. Let $\{y_0\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak $P-$property that

$$d(y_n, y_m) \to 0 \Rightarrow d(x_n, x_m) \to 0$$

as $n, m \to \infty$, where $x_n, x_m \in A_0$ and $d(x_n, y_n) = d(A, B), d(x_m, y_m) = d(A, B)$. Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric d we have $d(p, q) = d(A, B)$, that is, $q \in B_0$ and hence $B_0$ is closed.

Let $\overline{A_0}$ be the closure of $A_0$. We claim that $T(\overline{A_0}) \subseteq B_0$. In fact, if $x \in \overline{A_0}$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of $T$ and the closedness of $B_0$ we have $Tx = \lim_{n \to \infty} Tx \in B_0$. That is $T(\overline{A_0}) \subseteq B_0$.

Define an operator $PA_0 : T(\overline{A_0}) \to A_0$, by $PA_0 = \{x \in A_0 : d(x, y) = d(A, B)\}$. Since the pair $(A, B)$ has the weak $P-$property

$$d(PA_0Tx_1, PA_0Tx_2) \leq d(Tx_1, Tx_2)$$
$$\leq \frac{\beta(d(x, y))}{2}[d(x_1Tx_1) + d(x_2, Tx_2) - d(A, B)$$
$$\leq \frac{\beta(d(x, y))}{2}[d(x_1, PA_0Tx_1) + d(x_1, PA_0Tx_1)$$
$$+ d(x_2, PA_0Tx_2) + d(x_2, PA_0Tx_2) - 2d(A, B)$$
$$\leq \frac{\beta(d(x, y))}{2}[d(x_1, PA_0Tx_1) + d(x_2, PA_0Tx_2)].$$

For any $x_1, x_2 \in \overline{A_0}$. This shows that $PA_0T : \overline{A_0} \to \overline{A_0}$ is a Kannan Graghty mapping from complete metric subspace $\overline{A_0}$ into itself. Using Theorem 3.1, we can see that $PA_0T$ a unique fixed point $x^*$. That is, $PA_0Tx^* = x^* \in A_0$, which implies that

$$d(x^*, Tx^*) = d(A, B).$$

Therefore, $x^*$ is the unique one in $A_0$ such that $d(x^*, Tx^*) = d(A, B)$. It is easy to see that $x^*$ is also the unique one in A such that $d(x^*, Tx^*) = d(A, B)$. The Picard iteration sequence

$$x_{n+1} = PA_0Tx_n, n = 0, 1, 2, ...$$

converges, for every $x_0 \in A_0$, to $x^*$. Since the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by

$$x_{2k+1} = Tx_{2k}, \ d(x_{2k+1}, x_{2k+2}) = d(A, B), \ k = 0, 1, 2...,$$

is exactly the subsequence of $\{x_n\}$, so it converges, for every $x_0 \in A_0$, to $x^*$. This completes the proof.

Now we proof the following theorem which shows the relation between Kannan-Geraghty and weakly Geraghty.

**Theorem 3.3.** *Let $(X, d)$ be a complete metric space. If $f : X \to X$ satisfies the following conditions such that*

(1) *Let $f : X \to X$ is weakly Kannan mapping, then $f$ has a unique fixed point $x^*$;*

(2) *Let $f : X \to X$ is Kannan-Geraghty mapping, then $f$ has a unique fixed point $x^*$;*

*We have if 1 then 2.*

**Proof.**        $1 \to 2$

A mapping $f : X \to X$ is said to be weakly Kannan provided that

$$d(f(x), (y)) \leq \frac{\overline{\alpha}}{2}(x, y)[d(x, fx) + f(y, fy)]$$

for all $x, y \in X$, where the function $\overline{\alpha} : X \times X \to [0, 1)$, for every $0 < a \leq b$, satisfy

$$\Theta(a, b) = \sup\{\overline{\alpha}(x, y) : a \leq d(x, y) \leq b\} < 1.$$

Put $\overline{\alpha}(x, y) = \beta(d(x, y))$, suppose $L = \sup\{\alpha(x, y) \mid a \leq d(x, y) \leq b\} < 1$, for $0 < a < b$. Let $\sup\{L\} \to 1$ then, $\alpha(x, y) \to 1$ or $\beta(d(x_n, y_n)) \to 1$ therefore $d(x_n, y_n) \to 0$, this is a contradiction . Because $0 < a < d(x_n, y_n)$. Hence $a = b = 0$, then $f$ is Kannan Graghty and

$$d(f(x), (y)) \leq \beta(d(x, y))[d(x, fx) + f(y, fy)].$$

By using theorem 2.1 $f$ has a unique fixed point.

**Example 3.1.** *Let $X = \{(1, x) : 0 \leq x \leq \frac{1}{10}\}$, and define $f : X \to X$ as follows:*

$$f(1, y) = (1, \frac{y^2}{y + 1}).$$

*We have*

$$|\frac{y_1^2}{y_1 + 1} - \frac{y_2^2}{y_2 + 1}| \leq |y_2^2 - y_2^2|$$

$$\leq |y_1 - y_2| \, |y_1 + y_2|$$

$$\leq \frac{1}{5}|y_1 - y_2|$$

$$\leq \frac{1}{5}|y_1 - \frac{y_1^2}{y_1 + 1} + \frac{y_1^2}{y_1 + 1} - (y_2 - \frac{y_2^2}{y_2 + 1} + \frac{y_2^2}{y_2 + 1})|$$

$$\leq \frac{1}{5}|y_1 - \frac{y_1^2}{y_1 + 1}| + |\frac{y_1^2}{y_1 + 1} - \frac{y_2^2}{y_2 + 1}| + |y_2 - \frac{y_2^2}{y_2 + 1}|$$

*Then*

$$\frac{4}{5}\left|\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}\right| \leq \frac{1}{5}\left(\left|y_1 - \frac{y_1^2}{y_1+1}\right| + \left|y_2 - \frac{y_2^2}{y_2+1}\right|\right),$$

*now we get result*

$$\left|\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}\right| \leq \frac{1}{4}\left(\left|y_1 - \frac{y_1^2}{y_1+1}\right| + \left|y_2 - \frac{y_2^2}{y_2+1}\right|\right)$$

$$\leq \frac{1}{2}\cdot\frac{1}{1+d(y_1-y_2)}\left(\left|y_1 - \frac{y_1^2}{y_1+1}\right| + \left|\frac{y_2^2}{y_2+1}\right|\right)$$

*then $f$ is weakly Kannan. with*

$$\beta(d(y_1,y_2)) = \frac{1}{1 + d(y_1 - y_2)}.$$

*Then $f$ is weakly Graghty. Put $\dfrac{1}{1 + d(y_1 - y_2)} = \overline{\beta}(x,y)$. So for every $0 < a \leq b$. By using theorem 3.3, $f$ is Kannan Graghty and has a unique fixed point.*

**Theorem 3.4.** *Let $X$ be a compact metric space and $f : X \to X$ be a Kannan mapping, and for arbitrary $x_0 \in X$, the Picard iteration process defined by $x_n = f(x_{n-1})$ for $n > 0$. Then $f$ has a unique fixed point $x_\infty$ in $X$, $x_n \to x_\infty$ in $X$, iff there exists a subsequence $x_{h_n}$ and $x_{k_n}$ ($x_{h_n} \neq x_{k_n}$) such that*

$$\Delta_n \to 1 \ \text{only if} \ d_n \to 0.$$

**Proof.** There exists a subsequence $x_{h_n}$ and $x_{k_n}$, such that $x_n \to x_\infty$. Then clearly $d_n = d(x_{h_n}, x_{k_n}) \to 0$, and the condition holds.

Next, for given initial point $x_0$ in $X$, we assume that the condition is satisfied. then $d_n = d(x_n, x_{n+1})$ is non-increasing, for $d(x_{h_n}, x_{k_n}) \leq 1$, and then it is convergent to the real number $d$, such that $d \to \varepsilon(0 \leq \varepsilon)$. Assume $\epsilon > 0$, and $h_n = n$ and $k_n = n+1$, so we have $d_n \to \epsilon > 0$, While $\Delta \to 1$, which is a contradiction. Thus $d(x_n, x_{n+1}) \to 0$.

Suppose, for a contradiction, that the sequence of iterates $\{x_n\}$ is not Cauchy, the real number

$$D_N = \sup_{m,n \geq N} d(x_n, x_m) > \varepsilon.$$

is called the diameter of the sequence $\{x_n\}_{n \geq N}$, so there exists $\varepsilon > 0$ such that $D_N > \varepsilon$. For any $n > 0$, We choose $N_n$ sufficiently large number, such that $d(x_m, x_{m+1}) < \dfrac{1}{n}$ for all $m > N_m$, Let $h_n$ is the smallest integer such that $h_n \geq N_n$. For $k_n > h_n$, we have $d(x_{h_n}, x_{k_n}) > \varepsilon$. Such pairs exist by the above diameter condition.

Again we consider the sequence $k_n$, and put $k_n - 1 = h_n$ or else $d(x_{h_n}, x_{k_n-1}) \leq \varepsilon$. In either case we have $\varepsilon \leq d_n = d(x_{h_n}, x_{k_n}) < \varepsilon + 1$.

Moreover, by using the triangular inequality, for all $x_{h_n}, x_{k_n} \in X$, we have

$$d(x_{h_n+1}, x_{h_n+2}) = d(f(x_{h_n}), f(x_{h_n+1}))$$

$$= d(f(x_{h_n}), f(x_{k_n}))$$

$$\leq \frac{\beta}{2}(d(x_{h_n}, x_{h_n+1}) + d(x_{h_n+1}, x_{h_n+2}))$$

$$= d(f(x_{h_n}), f(x_{h_n+1})).$$

Where $\beta \in [0, 1)$. So

$$d(x_{h_n+1}, x_{h_n+2}) \leq \beta d(x_{h_n}, x_{h_n+1}).$$

Without loss of generality, we may assume that $\dfrac{d(x_{h_n+1}, x_{h_n+2})}{d(x_{h_n}, x_{h_n+1})} > 1 - \dfrac{1}{n}$, so

$$1 \geq \Delta_n = \frac{d(x_{h_n+1}, x_{h_n+2})}{d(x_{h_n}, x_{h_n+1})} > 1 - \frac{1}{n}.$$

So $d_n \to \varepsilon > 0$ while $\Delta_n \to 1$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in $X$ and $X$ is complete, we have $x_n \to x_\infty$ for some $x_\infty$ in $X$, then $x_\infty$ is a unique fixed point of $f$ and the proof is complete.

we present a theorem and after that we bring S−Kannan theorem which shows the relation between Kannan and contractive mapping. The proof of our main theorem is inspired by this theorem.

**Theorem 3.5.** *[S−Kannan] Let $X$ be a compact metric space and $f : X \to X$ be a Kannan mapping, and let for arbitrary $x_0 \in X$ the Picard iteration process defined by $x_n = f(x_{n-1})$ for $n > 0$. Then $f$ has a unique fixed point $x_\infty$ in $X$, $x_n \to x_\infty$ in $X$, iff*
*(i) there exists $\beta : X \times X \to [0, 1)$, such that for every $0 < a \leq b$ and for all $n, m$ and $x_n, x_m \in X$*

$$\beta(x_n, x_m) = \sup \ \{ \ \alpha(x_n, x_m) : a \leq d(x_n, x_m) \leq b\} < 1$$

*and*

$$d(f(x_n), f(x_m)) \leq \frac{\beta(x_n, x_m)}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))).$$

*(ii) $\beta(x_n, x_m) \in S$*

**Proof.**            It suffices to prove that $\beta$ in $S$ satisfies the condition of Theorem 2.2. Let $\beta = \gamma(d(x_n, y_n))$, where the class $\gamma$ denotes function $\gamma : [0, \infty) \to [0, 1)$ satisfying the following condition

$$\gamma(t_n) \to 1 \Rightarrow t_n \to 0$$

for every $0 < a < b$, let $L = \sup\{\alpha(x_n, y_n)|a \leq d(x_n, y_n) \leq b\}$, and let $\limsup\{L\} = 1$ implies $\beta(x_n, y_n) \to 1$ or $\gamma(d(x_n, y_n)) \to 1$. Hence $d(x_n, y_n) \to 0$. it is a contradiction, because $0 < a < d(x_n, y_n)$, so $a = b = 0$. So $\gamma$ holds in delta, the conclusion is exactly the same as what we had in theorem 3.4.

Define $\beta : \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta = \sup\{\frac{2d(f(x_n), f(x_m))}{d(x_n, f(x_n)) + d(x_m, f(x_m))} = d(x_n, x_m) \geq t\} = \alpha(d(x_n, x_m)) = \alpha(t_n).$$

Since $f$ is a Kannan , all quotients are below 1 and so $\beta$ is defined for all $t > 0$ and $\beta \leq 1$. It is clear that $\beta$ satisfies in (i).

Before presenting an important result, we first present a preliminary result:

let $x_0$ be an arbitrary point in $X$. We define the iterative sequence $\{x_n\}$ by $x_n = fx_{n_1}, n = 1, 2, ...$, we have

$$d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq \frac{\beta}{2}\left(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\right),$$

so

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{2}\frac{\beta}{1 - \beta} d(x_n, x_{n+1}),$$

by the same argument,

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{2}\left(\frac{\beta}{1 - \beta}\right)^n d(x_1, x_0). \tag{3.4}$$

By (3.4), for every $m, n \in \mathbb{N}$ such that $n > m$ we have

$$d(f(x_n), f(x_m)) \leq \frac{\beta}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))),$$

or

$$d(x_{n+1}, x_{m+1}) \leq \frac{\beta}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))).$$

So

$$d(x_{n+1}, x_{m+1}) \leq \frac{\beta}{2}(d(x_n, x_{n+1}) + d(x_m, x_{m+1}))$$

$$\leq \frac{\beta}{2}(d(x_n, x_m) + d(x_{m+1}, x_{n+1})) + \beta d(x_m, x_{m+1})$$

$$\leq \frac{\beta}{2}(d(x_n, x_m) + d(x_{m+1}, x_{n+1})) + (\frac{\beta}{1 - \beta})^m d(x_0, x_1).$$

So

$$d(x_{n+1}, x_{m+1}) \leq \beta \ d(x_n, x_m),$$

or

$$d(f(x_n), f(x_m)) \leq \beta \ d(x_n, x_m).$$

Now, let $\beta(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that

$$1 - \frac{1}{n} < \frac{d(f(x_n), f(x_m))}{d(x_n, x_m)} \leq \beta = \alpha(t_n) \leq 1.$$

We must show that $t_n \to 0$. Since $\alpha(t_n)$ is an upper bound. So for each $n > 0$, there are $x_{h_n}$ and $x_{k_m}$ in $\{x_n\}$, such that

$$d(x_{h_n}, x_{h_m}) \geq t_n,$$

and

$$1 - \frac{1}{n} < \frac{d(fx_{k_n}), f(x_{k_m})}{d(x_{k_n}, x_{k_m})} \leq \beta = \alpha(t_n) \leq 1.$$

So $\Delta_n \to 1$. Hence from theorem 3.4, we have $d(x_{h_n}, x_{k_n}) \to 0$. So $t_n \to 0$. This completes the proof.

## REFERENCES

[1] D. Ariza-Ruiz, A. Jimenez-Melado, *A continuation method for weakly Kannan mappings.* Fixed Point Theory and Applications, **2010:321594** (2010), https://doi.org/10.1155/2010/321594.

[2] J. Caballero, J. Harjani and K. Sadarangani, *A best proximity point theorem for Graphty-contractions.* Fixed Point Theory and Applications, **2012:231** (2012), https://doi.org/10.1186/1687-1812-2012-231.

[3] J. Dugundji, A. Granas, *Weakly contractive mappings and elementary domain invariance theorem.* Bulletin of Greek Mathematical Society. **19**, 141-151 (1978).

[4] N. Hussain, P. Salimi and A. Latif, *Fixed point results for single and set-valued $\alpha$-$\eta$-$\psi$-contractive mappings.* Fixed Point Theory and Application, **2013:212** (2013), http://doi.org/10.1186/1687-1812-2013-212.

[5] R. Kannan, *Some results on fixed points.* Bulletin of the Calcutta Mathematical Society, **60**, 71-76 (1968).

[6] N. Shioji, T. Suzuki, W. Takahashi, Contractive mappings, Kannan mappings and metric completeness. Fixed Point Theory, **126**, 3117-3124 (1998).

[7] W.A. Kirk, S. Reich, P. Veeramani, *Proximinal retracts and best proximity pair theorems.* Numerical Functional Analysis and Optimizations, **24**, 851-862 (2003).

[8] M. Geraghty, *On contractive mappings.* Proceedings of the American Mathematical Society, **40:2**, 604-608 (1973).

[9] Y. Sun, Y. Su, J. Zhang, *A new method for the research of best proximity point theorems of nonlinear mappings.* Fixed Point Theory and Applications, **2014:116** (2014).

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