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A NEW TYPE OF IRRESOLUTE FUNCTION VIA δgp-OPEN SET

JAGADEESH B. TORANAGATTI * AND SAEID JAFARI

Abstract. In this article, a new class of complete continuity called complete δgp-irresolute is introduced. Properties and characterizations of completely δgp-irresolute functions are investigated.

Keywords: Completely δgp-irresolute, Strongly δgp-normal, Countable δgp-compact, δgp-Lindelöf.

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1. Introduction

Many researchers have examined and analyzed various forms of continuity in academic literature. In general topology, continuity remains a vital and foundational concept in mathematics. In 1972, Crossley and Hildebrand [7] introduced the concept of irresoluteness. In 1999, Arokiarani et al. [3] studied gp-irresolute functions, followed by Balasubramanian and Sarada [5] in 2012, who explored the properties of gpr-irresolute functions. Over time, several variants of irresolute functions have been introduced. Recently, J. B. Toranagatti proposed and investigated δgp-continuity [23] as a broader interpretation of continuity. This research aims to introduce and investigate a completely δgp-irresolute function, which serves as a more robust variant of the existing gpr-irresolute function.

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* Corresponding author

Jagadeesh B. Toranagatti ⋄ jagadeeshbt20000@gmail.com ⋄ https://orcid.org/0000-0002-3271-8629

Saeid Jafari ⋄ jafaripersia@gmail.com ⋄ https://orcid.org/0000-0001-5744-7354.

2. Preliminaries

Throughout this paper ($\mathcal{M}$, τ), ($\mathcal{N}$, γ) and ($\mathcal{P}$, η) (or simply $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise stated. For a subset K of $\mathcal{M}$, $\acute{c}$(K) and $\acute{i}$(K) denote the closure of K and the interior of K, respectively.

Definition 2.1. *A set $J \subseteq \mathcal{M}$ is called:*

(i) regular closed [22] *if $\acute{c}(\acute{i}(J))=J$,*

(ii) pre-closed [18] *$\acute{c}(\acute{i}(J)) \subseteq J$.*

Definition 2.2. *A set $J \subseteq \mathcal{M}$ is called δ-closed* [28] *if $J = \acute{c}_\delta(J)$ where $\acute{c}_\delta(J) = \{b \in \mathcal{M}$: $\acute{i}(\acute{c}(U)) \cap J = \emptyset$, $U \in \Im$ and $b \in U\}$.*

Definition 2.3. *A set $J \subseteq \mathcal{M}$ is called δgp-closed* [6] *(resp., gp-closed* [17] *and gpr-closed* [10]*) if $p\acute{c}(J) \subseteq H$ whenever $J \subseteq H$ and H is δ-open (resp., open, regular open) in $\mathcal{M}$.*

Their complements are the open sets that are related to the previously listed closed sets. $\delta\mathbb{O}(\mathcal{M})$ is the collection of all δ-open sets in ($\mathcal{M}$, τ). The families of open sets, pre-open sets, regular open sets, gp-open sets, gpr-open sets, and δgp-open sets are denoted as $\mathbb{O}(\mathcal{M})$, $\mathbb{PO}(\mathcal{M})$, $\mathbb{RO}(\mathcal{M})$, $\mathbb{GPO}(\mathcal{M})$, $\mathbb{GPRO}(\mathcal{M})$ and $\delta\mathbb{GPO}(\mathcal{M})$ correspondingly.

Definition 2.4. *A function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is called:*

(i) R-maps [12] *if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \mathbb{RO}(\mathcal{N})$;*

(ii) completely continuous [4] *if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \gamma$;*

(iii) completely preirresolute [14] *(resp., completely gp-irresolute* [14] *and completely gpr-irresolute) if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \mathbb{PO}(\mathcal{N})$ (resp., $K \in \mathbb{GPO}(\mathcal{N})$ and $K \in \mathbb{GPRO}(\mathcal{N})$);*

(iv) δgp-irresolute [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \delta\mathbb{GPO}(\mathcal{N})$;*

(v) δgp-continuous [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \gamma$;*

(vi) pre δgp-continuous [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \mathbb{PO}(\mathcal{N})$;*

(vii) gpr-irresolute [5] *if $\ell^{-1}(K) \in \delta\mathbb{GPR}(\mathcal{M})$ for every $K \in \delta\mathbb{GPR}(\mathcal{N})$.*

Definition 2.5. *A space $(\mathcal{M}, \Im)$ is called:*

(i) δgp-additive [24] *if $\delta\mathbb{GPC}(\mathcal{M})$ is closed under arbitrary intersections;*

(ii) $T_{\delta gp}$-space [6] *if $\delta\mathbb{GPC}(\mathcal{M}) = \mathbb{C}(\mathcal{M})$;*

(iii) preregular $T_{1/2}$-space [10] *if $\mathbb{GPRC}(\mathcal{M}) = \mathbb{PC}(\mathcal{M})$;*

(iv) locally indiscrete [13] *if $\Im = \mathbb{RO}(\mathcal{M})$.*

3. Completely δgp-irresolute Functions

Definition 3.1. *A function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is called as completely δgp-irresolute (briefly, c.δgp-i.) if for every point b in $\mathcal{M}$ and for any δgp-open set H that includes $\ell(b)$, there exists a δ-open set G around b such that $\ell(G) \subseteq H$.*

Theorem 3.1. *The following conditions are equivalent for a function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*
(i) ℓ is c.δgp-i.;
(ii) For each $q \in \mathcal{M}$ and each $D \in \delta\mathbb{GPC}(\mathcal{N}, \ell(q))$, there exists a $C \in \mathbb{RO}(\mathcal{M}, q)$ such that $\ell(C) \subseteq D$.

Proof. (i) $\to$ (ii): Let q $\in \mathcal{M}$ and D $\in \delta\mathbb{GPC}(\mathcal{N}, \ell(\mathrm{q}))$.
$\overset{\text{(i)}}{\Longrightarrow}(\exists\ \mathrm{J} \in \delta\mathbb{O}(\mathcal{M}, \mathrm{q}))(\ell(\mathrm{J}) \subset \mathrm{D})$.
Now, J $\in \delta\mathbb{O}(\mathcal{M}, \mathrm{q}) \implies (\exists\ \mathrm{C} \in \mathbb{RO}(\mathcal{M}, \mathrm{q})(\mathrm{C} \subset \mathrm{J}))$.
Therefore, $(\ \exists\ \mathrm{C} \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))\ (\ell(\mathrm{C}) \subset \ell(\mathrm{J}) \subset \mathrm{D})$.
(ii) $\to$ (i): Obvious. □

Theorem 3.2. *The following conditions are identical for a function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*
(i) ℓ is c.δgp-i.;
(ii) For each $q \in \mathcal{M}$ and each $G \in \delta\mathbb{GPC}(\mathcal{N})$ where $\ell(q) \notin G$, there exists an $H \in \delta\mathbb{C}(\mathcal{M})$ such that $q \notin H$ and $\ell^{-1}(G) \subseteq H$;
(iii) For each $q \in \mathcal{M}$ and each $G \in \delta\mathbb{GPC}(\mathcal{N})$ where $\ell(q) \notin G$, there exists an $H \in \mathbb{RC}(\mathcal{M})$ such that $q \notin H$ and $\ell\ ^{-1}(G) \subseteq H$;
(iv) For every $q \in \mathcal{M}$ and each $N \in \delta\mathbb{GPC}(\mathcal{N},\ell(q))$, there exists a $G \in \mathbb{O}(\mathcal{M}, q)$ such that $\ell(\acute{i}(\acute{c}(G)) \subseteq N$;
(v) For every $q \in \mathcal{M}$ and each $H \in \delta\mathbb{GPC}(\mathcal{N},\ell(q))$, there exists a $G \in \mathbb{O}(\mathcal{M}, q)$ such that $\ell(sc(G)) \subseteq H$.

Proof. Obvious. □

Remark 3.1. *We can generate the following diagram for the function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ by using Definitions 2.4 and 3.1.*

c.gpr.i. $\to$ *c.δgp.i.* $\to$ *c.gp.i.* $\to$ *c.p.i.* $\to$ *c.c.* $\to$ *R-m.*

$\downarrow$ $\downarrow$

gpr.i. $\to$ *δgp.i.* $\to$ *p.δgp.c.* $\to$ *δgp.c.*

Notations:

c.gpr.i.: completely gpr-irresolute, **c.δgp.i.:** completely δgp-irresolute, **c.gp.i.:** completely gp-irresolute, **c.p.i.:** completely pre-irresolute, **c.c.:** completely continuous, **R-m.:** R-maps, **gpr.i.:** gpr-irresolute, **δgp.i.:** δgp-irresolute, **p.δgp.c.:** pre δgp-continuous, **δgp.c.:** δgp-continuous.

None of the implications in above diagram is reversible as shown in the following examples.

Example 3.1. *Let* $\mathcal{M} = \eta = \{u_1, u_2, u_3, u_4\}$, $\tau = \{\mathcal{M}, \emptyset, \{u_1\},\{u_2\},\{u_1, u_2\},\{u_1, u_2, u_3\}\}$ *and* $\gamma = \{\eta, \emptyset, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$. *Then:*

(i) The identity function ℓ*:* $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ *is* δ*gp-irresolute, but it is not completely* δ*gp-irresolute.*

(ii) Let us define ℓ*:* $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ *by* $\ell(u_1) = u_1$, $\ell(u_2) = u_3 = \ell(u_3)$ *and* $\ell(u_4) = u_4$*, In this case,* ℓ *is* δ*gp-i. but not gpr-i., since* $\{u_1, u_2\} \in \mathbb{GPRC}(\mathcal{M}, \gamma)$ *implies that* $\ell^{-1}(\{u_1, u_2\})=\{u_1\} \notin \mathbb{GPRC}(\mathcal{M}, \Im)$*.*

Example 3.2. *Consider* $\mathcal{M} = \{u_1, u_2, u_3, u_4\}$ *with the topologies*
$\Im = \{\emptyset, \mathcal{M}, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$ *and*
$\gamma = \{\emptyset, \mathcal{M}, \{u_1\},\{u_2\},\{u_1, u_2\},\{u_1, u_2, u_3\}\}$.
Let the function ℓ*:* $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ *be defined by* $\ell(u_1) = u_2 = \ell(u_3)$*, and* $\ell(u_2) = u_4$ *with* $\ell(u_4)= u_4$*. In this scenario,* ℓ *is c.gp-i. but not c.*δ*gp-i.. This is evident as* $\{u_4\} \in \delta\mathbb{GPC}((M, \gamma)$ *leads to the conclusion that* $\ell^{-1}(\{u_4\})) = \{u_2, u_4\} \notin \mathbb{RO}(\mathcal{M}, \Im)$*.*

Example 3.3. *Consider* $(\mathcal{M}, \Im)$ *as in Example 3.2. We define the function* ℓ*:* $(\mathcal{M}, \Im) \to (M, \Im)$ *by specifying* $\ell(u_1)= u_2$, $\ell(u_3)= u_2$*, and* $\ell(u_2)= u_4$ *with* $\ell(u_4) = u_4$*. In this context,* ℓ *is c.*δ*gp-i., but it is not c.gpr-i.. This is because* $\{u_1, u_4\} \in \mathbb{GPRO}(\mathcal{M}, \sigma)$ *leads to the conclusion that* $\ell^{-1}(\{u_1, u_4\}) = \{u_2, u_4\} \notin \mathbb{RO}(\mathcal{M}, \Im)$*.*

Theorem 3.3. *For any* $J \subseteq \mathcal{M}$*, the following are the same where* $(\mathcal{M}, \Im)$ *is locally indiscrete space* [25].

(i) J *is gp-closed;*

(ii) J *is* δ*gp-closed;*

(iii) J *is gpr-closed.*

As a consequence of Theorem 3.3, we can state the following theorem.

Theorem 3.4. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is locally indiscrete space:*

(i) ℓ is c.gp-i.;

(ii) ℓ is c.δgp-i.;

(iii) ℓ is c.gpr-i..

Theorem 3.5. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is $T_{\delta gp}$-space:*

(i) ℓ is c.c.;

(ii) ℓ is c.p-i.;

(iii) ℓ is c.gp-i.;

(iv) ℓ is c.δgp-i..

Theorem 3.6. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is preregular $T_{1/2}$-space:*

(i) ℓ is c.p-i.;

(ii) ℓ is c.gp-i.;

(iii) ℓ is c.δgp-i.;

(iv) ℓ is c.gpr-i..

Definition 3.2. [11] *Every gpr-closed set for a space $(\mathcal{M}, \Im)$ is closed if and only if $\tau_g^*= \tau$ where $\tau_g^*=\{L \subseteq \mathcal{M}$: gprcl($\mathcal{M}$-L)=($\mathcal{M}$-L)$\}$.*

Theorem 3.7. *If $\gamma_g^*= \gamma$ in $(\mathcal{N}, \gamma)$. Then, the assertions that follow are the same:*

(i) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gpr-i.;

(ii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i.;

(iii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gp-i.;

(iv) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.p-i.;

(v) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.c..

Theorem 3.8. *If $\gamma_g^*= \gamma$ in $(\mathcal{N}, \gamma)$ and $(\mathcal{N}, \gamma)$ is locally indiscrete. Then, the assertions that follow are the same:*

(i) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gpr-i.;

(ii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i.;

(iii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gp-i.;

(iv) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.p-i.;

(v) ℓ*:* $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ *is c.c.;*

(vi) ℓ*:* $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ *is R-map.*

Theorem 3.9. *Let* ℓ*:* $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ *be such that the space* $(\mathcal{N}, \gamma)$ *is* δ*gp-additive. The statement that follow are interchangeable:*

(i) ℓ *is c.*δ*gp-*$\acute{i}$*.;*

(ii) ℓ^{-1} $(\delta gp\text{-}\acute{i}(R)) \subseteq \acute{i}_\delta(\ell^{-1}(R))$ *for each* $R \subseteq \mathcal{N}$*;*

(iii) ℓ $(\acute{c}_\delta(S)) \subseteq \delta gp\text{-}\acute{c}(\ell(S))$ *for each* $S \subseteq \mathcal{M}$*;*

(iv) $\acute{c}_\delta(\ \ell^{-1}(R)) \subseteq \ell^{-1}(\delta gp\text{-}\acute{c}(R))$ *for each* $R \subseteq \mathcal{N}$*;*

(v) $\ell^{-1}(B) \in \delta\mathbb{C}(\mathcal{M})$ *for each* $B \in \delta\mathbb{GPC}(\mathcal{N})$*;*

(vi) $\ell^{-1}(A) \in \delta\mathbb{O}(\mathcal{M})$ *for each* $A \in \delta\mathbb{GPO}(\mathcal{N})$*;*

(vii) $\ell^{-1}(A) \in \mathbb{RO}(\mathcal{M})$ *for each* $A \in \delta\mathbb{GPO}(\mathcal{N})$*;*

(viii) $\ell^{-1}(B) \in \mathbb{RC}(\mathcal{M})$ *for each* $B \in \delta\mathbb{GPC}(\mathcal{N})$*.*

Proof. (i) $\Longrightarrow$ (ii): Let R $\subseteq \mathcal{N}$ and x $\in \ell^{-1}(\delta\text{gp-}\acute{i}(\text{R}))$.

$$\text{b} \in \ell^{-1}\ (\delta\text{gp-}\acute{i}(\text{R})) \implies \delta\text{gp-}\acute{i}(\text{R}) \in \delta\mathbb{GPO}(\mathcal{N}, \ell(\text{b}))$$
$$\overset{(i)}{\Longrightarrow} (\exists\ \text{S} \in \mathbb{RO}(\mathcal{M}, \text{q})\ (\ell(\text{S}) \subseteq \delta\text{gp-}\acute{i}(\text{R}) \subset \text{R})$$
$$\implies (\exists\ \text{S} \in \mathbb{RO}(\mathcal{M}, \text{q}))(\text{S} \subseteq \ell^{-1}(\text{R})) \implies \text{q} \in \acute{i}_\delta(\ell^{-1}(\text{R})).$$

(ii) $\Longrightarrow$ (iii) : Let S $\subseteq \mathcal{M}$.

$$\text{S} \subseteq \mathcal{M} \implies \ell(\text{S}) \subseteq \mathcal{N} \implies \mathcal{N} \setminus \ell(\text{S}) \subseteq \mathcal{N} \overset{(ii)}{\Longrightarrow} \ell^{-1}\ [\delta\text{gp-}\acute{i}(\mathcal{N}\setminus\text{S})] \subseteq \acute{i}_\delta(\ell^{-1}(\mathcal{N}\setminus\ell(\text{S})))$$
$$\implies \mathcal{M} \setminus \ell^{-1}\ (\delta\text{gp-}\acute{c}(\ell(\text{S})) \subseteq \mathcal{M} \setminus \acute{c}_\delta(\ell^{-1}(\ell(\text{S}))))$$
$$\implies \acute{c}_\delta\ (\text{S}) \subset \acute{c}_\delta\ ((\ell^{-1}(\ell(\text{S}))) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\ell(\text{S})))$$
$$\implies \ell(\acute{c}_\delta(\text{S})) \subseteq \delta\text{gp-}\acute{c}(\ell(\text{S})).$$

(iii) $\Longrightarrow$ (iv): Let R $\subseteq \mathcal{N}$.

$$\text{R} \subseteq \mathcal{N} \implies \ell^{-1}(\text{R}) \subseteq \mathcal{M} \overset{(iii)}{\Longrightarrow} \ell(\acute{c}_\delta\ (\ell^{-1}(\ell(\text{R}))) \subseteq \delta\text{gp-}\acute{c}(\ell^{-1}(\text{R})) \subseteq \delta\text{gp-}\acute{c}(\text{R})$$
$$\implies \acute{c}_\delta\ ((\ell^{-1}(\text{R})) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\text{R})).$$

(iv) $\Longrightarrow$ (v): Let H $\in \delta\mathbb{GPC}(\mathcal{N})$.

$$\text{H} \in \delta\mathbb{GPC}(\mathcal{N}) \implies \text{H} = \delta\text{gp-}\acute{c}(\text{H})$$
$$\overset{(iv)}{\Longrightarrow} \acute{c}_\delta((\ell^{-1}(\text{H})) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\text{H})) = \ell^{-1}(\text{H})$$
$$\implies \ell^{-1}(\text{H}) = \acute{c}_\delta\ (\ell^{-1}(\text{H})) \implies \ell^{-1}(\text{H}) \in \delta\mathbb{C}(\text{M}).$$

(i) $\Longrightarrow$ (vi): Obvious.

(viii) $\Longleftrightarrow$ (vii) $\Longrightarrow$ (vi) : Obvious.

(vi) $\Longrightarrow$ (i): Let K $\in \delta\mathbb{GPO}(\mathcal{N})$ and q $\in \ell^{-1}(\text{K})$.

$(\mathrm{K} \in \delta\mathbb{GPO}(\mathcal{N})\ (\mathrm{q} \in \ell^{-1}(\mathrm{K})) \implies \mathrm{K} \in \delta\mathbb{GPO}(\mathcal{N},\, \ell(\mathrm{q}))$

$\overset{\text{(vi)}}{\Longrightarrow} (\mathrm{L}\!:= \ell^{-1}(\mathrm{K}) \in \delta\mathbb{O}(\mathcal{M},\, \mathrm{q}))\ (\ell(\mathrm{L}) \subseteq \mathrm{K}).$ □

Theorem 3.10. *The following assertions are identical for a bijection ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*

(i) ℓ is c.δgp-i.;

(ii) δgp-$\acute{i}(\ell(H)) \subseteq \ell(\acute{i}_\delta(H))$ for each $H \subseteq \mathcal{M}$.

Proof. (i) $\implies$ (ii) Let $\mathrm{H} \subseteq \mathcal{M}$.

$\mathrm{H} \subseteq \mathcal{M} \implies \mathcal{M} \setminus \mathrm{H} \subseteq \mathcal{M}$

$$\left.\begin{array}{r} \overset{\text{(i)}}{\Longrightarrow} \ell[(\mathcal{M}\backslash \acute{i}_\delta(\mathrm{H}))] = \ell[\acute{c}_\delta(\mathcal{M} \setminus\mathrm{H})] \subseteq \delta\text{gp-}\acute{c}(\ell[\mathcal{M}/\mathrm{H}]) \\ \ell \text{ is bijection} \end{array}\right\} \implies$$

$$\implies \mathcal{N} \setminus\ell[\acute{i}_\delta(\mathrm{H})] \subseteq \mathcal{N} \setminus \delta\text{gp-}\acute{i}(\ell[\mathrm{H}])$$

$\implies \delta\text{gp-}\acute{i}(\ell[\mathrm{H}]) \subseteq \ell(\acute{i}_\delta[\mathrm{H}])$.

(ii) $\implies$ (i) : Let $\mathrm{K} \subseteq \mathcal{M}$.

$$\left.\begin{array}{r} \mathrm{K} \subseteq \mathcal{M} \implies \mathcal{M}\backslash\mathrm{K}\subseteq\mathcal{M} \overset{\text{(ii)}}{\Longrightarrow} \delta\text{gp-}\acute{i}(\ell[\mathcal{M}\backslash\mathrm{K}]) \subseteq \ell[\acute{i}_\delta(\mathcal{M}\backslash\mathrm{K})] \\ \ell \text{ is bijection} \end{array}\right\} \implies$$

$$\implies \mathcal{N}\setminus \delta\text{gp-}\acute{c}(\ell[\mathrm{K}]) \subseteq \mathcal{N} \setminus \ell[\acute{c}_\delta(\mathrm{K})]$$

$$\implies \ell(\acute{c}_\delta(\mathrm{K})) \subseteq \delta\text{gp-}\acute{c}(\ell(\mathrm{K})).$$ □

Lemma 3.1. *Let $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{N} \in \mathbb{O}$ $(\mathcal{M})$. The following hold* [15].

(i) $K \in \mathbb{RO}(\mathcal{M}) \implies Y \cap K \in \mathbb{RO}(\mathcal{N},\, \tau_{\mathcal{N}})$.

(ii) $H \in \mathbb{RO}(\mathcal{N},\, \tau_{\mathcal{N}}) \implies (\exists$ a $K \in \mathbb{RO}(\mathcal{M})$ such that $H = \mathcal{N} \cap K$.

where $\tau_{\mathcal{N}}$=$\{\mathcal{N} \cap G \mid G \in \mathbb{O}(\mathcal{M})\}$.

Theorem 3.11. *If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. and $K \in \Im$, then the restriction $\ell/_K : K \to \mathcal{N}$ is c.δgp-i..*

Proof. Let $\mathrm{J} \in \delta\mathbb{GPO}(\mathcal{N})$.

$$\left.\begin{array}{r} \mathrm{J} \in \delta\mathbb{GPO}(\mathcal{N}) \xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}} \ell^{-1}(\mathrm{J}) \in \mathbb{RO}(\mathcal{M}) \\ \mathrm{K} \in \Im \end{array}\right\} \implies$$

$\xrightarrow{\text{lemma 3.1}} (\ell/_K)^{-1}(\mathrm{J}) = \ell^{-1}(\mathrm{J}) \cap \mathrm{K} \in \mathbb{RO}(\mathrm{K}).$ □

Lemma 3.2. *Let $\mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{N} \in \mathbb{PO}(\mathcal{M})$. Then, $\mathcal{N} \cap K \in \mathbb{RO}(\mathcal{N})$ for each $K \in \mathbb{RO}(\mathcal{M})$* [2].

Theorem 3.12. *If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \Re)$ is a c.δgp-i. and $K \in \mathbb{PO}(\mathcal{M})$, then $\ell/_k : K \rightarrow \mathcal{N}$ is c. δgp-i..*

Proof. This can be inferred from Lemma 3.2. □

Theorem 3.13.

(i) If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ is δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \rightarrow (\mathcal{P}, \sigma)$ is δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \rightarrow (P, \sigma)$ is also c.δgp-i.

(ii) If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ is c.δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \rightarrow (\mathcal{P}, \sigma)$ is c.δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \rightarrow (P, \sigma)$ remains c.δgp-i.

(iii) If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ is an R-map and $\acute{k}$: $(\mathcal{N}, \gamma) \rightarrow (\mathcal{P}, \sigma)$ is c.δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \rightarrow (P, \sigma)$ is c.δgp-i.

(iv) If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ is c.δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \rightarrow (\mathcal{P}, \sigma)$ is δgp-c., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \rightarrow (\mathcal{P}, \sigma)$ is also c.c..

Proof. Straightforward. □

Definition 3.3. *If J, H $\in \mathbb{RO}(\mathcal{M})$ (resp., $\delta\mathbb{GPO}(\mathcal{M})$) cannot be found such that J $\cap$ H $= \emptyset$ and J $\cup$ H $= \mathcal{M}$, then a space $(\mathcal{M}, \Im)$ is referred to as almost connected* [8] *(resp., δgp-connected* [24]*).*

Theorem 3.14. *If ℓ: $(\mathcal{M}, \Im) \rightarrow (\mathcal{N}, \gamma)$ is a surjective function that is c.δgp-i. and $(\mathcal{M}, \Im)$ is almost connected, then $(\mathcal{N}, \gamma)$ is δgp-connected.*

Proof. Let us consider that $(\mathcal{N}, \gamma)$ is not δgp-connected.

$(\mathcal{N}, \gamma)$ is not δgp-connected $\Longrightarrow$

$$\left.\begin{array}{r}(\exists \text{ C, D} \in \delta\mathbb{GPO}(\mathcal{N})\backslash\{\emptyset\})(\text{C}\cap\text{D} = \emptyset)(\text{C}\cup\text{D} = \mathcal{N}) \\ \ell \text{ is c.}\delta\text{gp-i. surjection}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\ell^{-1}(C), \ell^{-1}(D) \in \mathbb{RO}(\text{M}) \setminus\{\emptyset\})(\ell^{-1}(\text{C}\cap\text{D}) = \ell^{-1}(\emptyset))(\ell^{-1}(\text{C}\cup\text{D}) = \ell^{-1}(\mathcal{N}))$

$\Longrightarrow$ $(\ell^{-1}(\text{C}), \ell^{-1}(\text{D}) \in \mathbb{RO}(\text{M})\setminus\{\emptyset\})(\ell^{-1}(\text{C}) \cap \ell^{-1}(\text{D}) = \emptyset)(\ell^{-1}(\text{C}) \cup \ell^{-1}(\text{D}) = \mathcal{M})$.

This $(\mathcal{M}, \Im)$ is not almost connected. □

Definition 3.4.

(i) If each regular open cover of a space $(\mathcal{M}, \Im)$ has a finite subcover, then the space is said

to as nearly compact (briefly, n.c.) [20]*;*

(ii) Every countable cover of a space $(\mathcal{M}, \Im)$ *by regular open sets that has a finite subcover is called nearly countably compact (briefly, n.c.c.)* [9].

(iii) If there is a countable subcover for each cover of $\mathcal{M}$ *by regular open sets, then the space* $(\mathcal{M}, \Im)$ *is referred to as nearly Lindelöf(briefly, n.L.)* [8].

(iv) If each δgp*-open cover of a space* $(\mathcal{M}, \Im)$ *has a finite subcover, then the space is said to as* δgp*-compact (briefly,* $\delta gp.c.$*)* [26]*;*

(v) Every countable cover of a space $(\mathcal{M}, \Im)$ *by* δgp*-open sets that has a finite subcover is called countably* δgp*-compact (briefly,* $c.\delta gp$*-c.);*

(vi) If there is a countable subcover for each cover of $\mathcal{M}$ *by* δgp*-open sets, then the space* $(\mathcal{M}, \Im)$ *is referred to as* δgp*-Lindelöf(briefly,* $\delta gp.L.$*)*

Theorem 3.15. *Let* ℓ*:* $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *be a c.* δgp*-i. surjection, then the following hold:*
(i) If $(\mathcal{M}, \Im)$ *is n.c., then* $(\mathcal{N}, \gamma)$ *is* δgp*-c.;*
(ii) If $(\mathcal{M}, \Im)$ *is n.L., then* $(\mathcal{N}, \gamma)$ *is* δgp*-L.;*
(iii)If $(\mathcal{M}, \Im)$ *is n.c.c., then* $(\mathcal{N}, \gamma)$ *is* $c.\delta gp$*-c..*

Proof. (i) Let $\mathcal{M}$ be n.c. and $\mathcal{A}$ be an δgp-open cover of $\mathcal{N}$.
$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))\ (\mathcal{N} = \cup\ \mathcal{A}) \xRightarrow{\ell \text{ is c.}\delta \text{ gp.i}}$

$$\left.\begin{array}{r}(\mathrm{K}:=\{\ell^{-1}(\mathrm{J})|\ \mathrm{J} \in \mathcal{A}\} \subset \mathbb{RO}(\mathcal{M}))(\mathcal{M} = \cup\ \mathrm{K}) \\ \mathcal{M} \text{ is nearly compact}\end{array}\right\} \Longrightarrow (\exists K^* \subset K)(|\ K^* \ | < \aleph_0(\mathcal{M} = \cup K^*))$$

$\xRightarrow{\ell \text{ is surjective}}$ (K: $= (\ell(\mathrm{K}^*){\subset}\ell(\mathrm{K}){=}\mathcal{A})(|\ell(\mathrm{B}^*)| < \aleph_0)\ (\mathrm{K} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{B}^*) = \cup_{N \in N^*}\ \ell(\mathrm{K}))$.

(ii) Let $\mathcal{M}$ be n.L. and $\mathcal{A}$ be an δgp-open cover of $\mathcal{N}$.
$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))(|\mathcal{A}| \leq \aleph_0(\mathcal{N} = \cup\ \mathcal{A})) \xRightarrow{\ell \text{ is c.}\delta \text{ gp.i.}}$

$$\Longrightarrow \left.\begin{array}{r}\left(K := \{\ell^{-1}(J) \mid J \in \mathcal{A}\} \subseteq \mathrm{RO}(\mathcal{M})\right)(\mathcal{M} = \cup K) \\ \mathcal{M} \text{ is n.c.}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{r}(\exists K^* \subseteq K \quad |\ K \ | < \aleph_0 \quad \wedge \quad \mathcal{M} = \cup K^*) \\ \ell \text{ is surjective}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow\ (\ell(\mathrm{K}^*) \subset \ell(\mathrm{K}) = \mathcal{A})(|\ell(\mathrm{K}^*)\ | < \aleph_0)(\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{K}^*) = \mathrm{U}_{\mathcal{N} \in \mathcal{N}^*}\ \ell(\mathrm{K})).$$

(iii) Let $\mathcal{M}$ be n.c.c. and $\mathcal{A}$ be an δgp-open countable cover of $\mathcal{N}$.

$$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))(|\mathcal{A}| \leq \aleph_0)(\mathcal{N} = \cup\, \mathcal{A}) \xrightarrow{\ell \text{ is c.}\delta \text{ gp.i.}}$$

$$\left.\begin{array}{r} \Longrightarrow \ \Big(K := \{\ell^{-1}(J) \mid J \in \mathcal{A}\} \subseteq \mathrm{RO}(\mathcal{M})\Big)\ (\mathcal{M} = \cup K) \\ \mathcal{M} \text{ is n.c.c.} \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{r} \Longrightarrow \ \big(\exists K^* \subseteq K \ \text{ with } |K^*| < \aleph_0 \ \wedge \ \mathcal{M} = \cup K^*\big) \\ \ell \text{ is surjective} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\ell\ (\mathrm{K}^*) \subset \ell(\mathrm{K}) = \mathcal{A})(|\ \ell(\mathrm{K}^*)| < \aleph_0)(\mathrm{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{K}^*)) = \cup_{K \in K^*}\ \ell(\mathrm{K}). \qquad \square$$

Definition 3.5.

(i) If there is a finite subcover for each regular closed (resp., δgp-closed) cover of a space ($\mathcal{M}$, $\Im$) then the space is said to be S-closed [27] *(resp., δgp-closed compact).*

(ii) If every countable cover of $\mathcal{M}$ by regular closed (resp., δgp-closed) sets has a finite subcover, then the space ($\mathcal{M}$, $\Im$) is called countably S-closed compact [1]*(resp., countably δgp-closed compact).*

(iii) If any cover of $\mathcal{M}$ by regular closed (resp., δgp-closed) sets admits a countable subcover, then the space ($\mathcal{M}$, $\Im$) is called S-Lindelöf [16] *(resp., δgp-closed Lindelöf).*

Theorem 3.16. *Let ℓ: ($\mathcal{M}$, $\Im$) $\to$ ($\mathcal{N}$, γ) be a c.δgp-i. surjection. The following is true:*

(i) If ($\mathcal{M}$, $\Im$) is S-closed, then ($\mathcal{N}$, γ) is δgp-closed compact.

(ii) If ($\mathcal{M}$, $\Im$) is S-Lindelöf, then ($\mathcal{N}$, γ) is δgp-closed Lindelöf.

(iii) If ($\mathcal{M}$, $\Im$) is countably S-closed compact, then ($\mathcal{N}$, γ) is countably δgp-closed compact.

Proof. (i) Let ($\mathcal{M}$, $\Im$) be S-closed and compact and $\mathcal{A}$ be an δgp-closed cover of ($\mathcal{N}$, γ).

$$\left.\begin{array}{r} (\mathcal{A} \subset \delta\mathbb{GPC}(\mathcal{N}))(\mathcal{N} = \cup\ \mathcal{A}) \xrightarrow{\ell \text{ is c.}\delta \text{ gp.i.}} (\mathrm{H}{:} = \{\ell^{-1}(\mathrm{K})|\mathrm{K} \in \mathcal{A}\} \subset \mathbb{RC}(\mathcal{M}))(\mathcal{M} = \cup\ \mathrm{H}) \\ \mathcal{M} \text{ is S-closed} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\exists\ \mathcal{N}^* \subset \mathrm{H})\ (|\mathrm{H}^*| < \aleph_0)\ (\mathcal{M} = \cup\ \mathrm{H}) \xrightarrow{\ell \text{ is surjective}}$$

$$\Longrightarrow (\mathrm{H} : = (\ell(\mathrm{H}^*) \subset \ell(\mathrm{H}){=}\mathcal{A})(|\ell(\mathrm{H}^*)| < \aleph_0)(\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{H}^*) = \cup_{\ H \in H^*}\ \ell(\mathrm{H})).$$

(ii) Let ($\mathcal{M}$, $\Im$) be S - Lindelöf and $\mathcal{A}$ be an δgp-closed countable cover of $\mathcal{N}$.

$$(\mathcal{A} \subset \delta\mathbb{GPC}(\mathrm{Q}))\ (|\ \mathcal{A}\ | \leq \aleph_0)(\mathcal{N} = \cup\ \mathcal{A}) \xrightarrow{\ell \text{ is c.}\delta \text{ gp.i.}}$$

$$\left.\begin{array}{r} \Longrightarrow \ \Big(B := \{\ell^{-1}[A] \mid A \in \mathcal{A}\} \subseteq \delta RC(\mathcal{M})\Big)\ (\mathcal{M} = \cup B) \\ \mathcal{M} \text{ is Lindelöf closed} \end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{B}^{*} \subset \mathrm{B})(|\mathrm{B}^{*}| < \aleph_0)(\mathcal{M} = \cup\ \mathrm{B}^{*})$

$\xrightarrow{\ell \text{ is surjective}}$ $(\ell[\mathrm{B}^{*}] \subset \ell[\mathrm{B}] = \mathcal{A})\ (|\ell[\mathrm{B}^{*}]| \leq \aleph_0)$

$(\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{B}^{*})) = \cup_{B \in B^{*}}\ \ell(\mathrm{B})$.

(iii) Let $(\mathcal{M}, \Im)$ be countable S-closed compact and $\mathcal{A}$ be an δgp-closed countable cover of $\mathcal{N}$.

$(\mathcal{A} \subset \delta\mathbb{GPC}(\mathcal{N}))\ (|\mathcal{A}| \leq \aleph_0)(\mathcal{N} = \cup\ \mathcal{A})$ $\xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$

$$\left.\begin{array}{r}(\mathrm{J}{:=}\{\ell^{-1}\ (\mathrm{A}) \mid \mathrm{A} \in \mathcal{A}\} \subset \mathbb{RC}(\mathcal{M}))(|\mathrm{J}| \leq \aleph_0)(\mathcal{M} = \cup\ \mathrm{J}) \\ \mathcal{M} \text{ is countable S-closed compact})\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{r}\left(\exists J^{*} \subseteq J \text{ with } |J^{*}| < \aleph_0 \quad \wedge \quad M = \cup J^{*}\right) \\ \ell \text{ is surjective}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\ell(\mathrm{J}^{*}) \subset \ell(\mathrm{J}) = \mathcal{A})\ (|\ \ell(\mathrm{J}^{*})\ | < \aleph_0)(\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{J}^{*}) = \cup_{J \in J^{*}}\ \ell(\mathrm{J})$. □

Definition 3.6. *A space $(\mathcal{M}, \Im)$ is defined as almost regular* [19] *(or strongly δgp-regular) if for any $L \in \mathbb{RC}(\mathcal{M})$ (or $\delta\mathbb{GPC}(\mathcal{M})$ and any point $q \in \mathcal{M}\backslash L$, there exist C, $D \in \Im$ (or $\delta\mathbb{GPO}(\mathcal{M})$) such that $q \in C$, $L \subseteq D$ and $C \cap D = \emptyset$).*

Example 3.4. *Consider $\mathcal{M} = \{u_1, u_2, u_3, u_4, u_5\}$ with the topology $\Im = \{\emptyset, \mathcal{M}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_2, u_3, u_4\}\}$. Then, $(\mathcal{M}, \Im)$ is strongly δgp-regular*

Theorem 3.17. *If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \Re)$ c.δgp-i. δgp-open bijection.*
If $(\mathcal{M}, \Im)$ is an almost regular, then $(\mathcal{N}, \gamma)$ is strongly δgp-regular.

Proof. Let $\mathrm{F} \in \delta\mathbb{GPC}(\mathcal{N})$ and $\ell(\mathrm{r}) = \mathrm{s} \notin \mathrm{F}$.

$$\left.\begin{array}{r}\ell(\mathrm{r}){=}\mathrm{s} \notin \mathrm{F} \in \delta\mathbb{GPC}(\mathcal{N}) \xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}} \mathrm{r} \notin \ell^{-1}\ (\mathrm{F}) \in \mathbb{RC}(\mathcal{M}) \\ \mathcal{M} \text{ is almost regular}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{U}, \mathrm{V} \in \delta\mathbb{GPO}(\mathcal{M}))\ (\mathrm{r} \in \mathrm{U})\ (\ell^{-1}(\mathrm{F}) \subset \mathrm{V})(\mathrm{U} \cap \mathrm{V} = \emptyset)$

$\xrightarrow{\ell \text{ is } \delta\text{gp-open bijection}}$ $(\ell(\mathrm{U}), \ell(\mathrm{V}) \in \delta\mathbb{GPO}(\mathcal{N}))(\mathrm{s} = \ell(\mathrm{r}) \in \ell(\mathrm{U}))(\mathrm{F} \subset \ell(\mathrm{V}))(\ell(\mathrm{U}) \cap \ell(\mathrm{V}) = \emptyset)$.

□

Definition 3.7. *A space $(\mathcal{M}, \Im)$ is defined as follows:*

(a) Almost normal: [21] *For each $G \in C(\mathcal{M})$ and each $H \in \mathbb{RC}(M)$ such that $G \cap H = \emptyset$, there exist J, $K \in \Im$ such that $J \cap K = \emptyset$, $G \subseteq J$ and $H \subseteq K$.*

(b) Strongly δgp-normal: For any pair G, $H \in \delta\mathbb{GPC}(\mathcal{M})$ such that $G \cap H = \emptyset$, there exist J, $K \in \delta\mathbb{GPO}(\mathcal{M})$ such that $J \cap K = \emptyset$, $G \subseteq J$ and $H \subseteq K$.

Example 3.5. *Consider $\mathcal{M}$ =$\{u_1, u_2, u_3, u_4\}$ with the topology $\Im$ =$\{\emptyset, \mathcal{M}, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$. Then, $(\mathcal{M}, \Im)$ is strongly δgp-normal.*

Theorem 3.18. *If $(\mathcal{M}, \Im)$ is an almost normal space then $(\mathcal{N}, \gamma)$ is strongly δgp-normal whenever ℓ:$(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c. δgp-i. and δgp-open bijection.*

Proof. Let C, D $\in \delta\mathbb{GPC}(\mathcal{N})$ and C $\cap$ D $= \emptyset$.

$$\left.\begin{array}{r}(\mathrm{C}, \mathrm{D} \in \delta\mathbb{GPC}(\mathcal{N}))\ (\mathrm{C} \cap \mathrm{D})= \emptyset \\ \ell \text{ is c.}\delta\text{gp.i.}\end{array}\right\} \implies (\ell^{-1}(C), \ell^{-1}(D) \in \mathbb{RC}(\mathcal{M}))(\ell^{-1}(C \cap D) = \ell^{-1}(\emptyset))$$

$$\implies \left.\begin{array}{r}\Big(\ell^{-1}(C),\ \ell^{-1}(D) \in \mathrm{RC}(\mathcal{M}) \ \ \wedge\ \ \ell^{-1}(C) \cap \ell^{-1}(D) = \emptyset\Big) \\ \mathrm{RC}(\mathcal{M}) \subseteq C(\mathcal{M})\end{array}\right\} \implies$$

$$\implies (\ell^{-1}(\mathrm{C}) \in \mathrm{C}(\mathcal{M}))\ (\ \ell^{-1}(\mathrm{D}) \in \mathbb{RC}(\mathcal{M}))\ (\ell^{-1}(\mathrm{C}) \cap \ell^{-1}(\mathrm{D}) = \emptyset)$$

$$\xRightarrow{(\mathcal{M}, \Im) \text{ is almost normal}}$$

$$\implies \left.\begin{array}{r}\Big(\exists U, V \in \delta\mathrm{GPO}(\mathcal{M})\ :\ \ell^{-1}(C) \subseteq U,\ \ell^{-1}(D) \subseteq V,\ U \cap V = \emptyset\Big) \\ \ell \text{ is a } \delta\text{-gp-open bijection}\end{array}\right\} \implies$$

$$\implies (\ell(\mathrm{U}), \ell(\mathrm{V}) \in \delta\mathbb{GPO}(\mathcal{N}))(\mathrm{C} \subseteq \ell(\mathrm{U}))(\mathrm{D} \subseteq \ell(\mathrm{V}))\ (\ell(\mathrm{U}) \cap \ell(\mathrm{V}) = \emptyset.$$ □

Definition 3.8. *A space $(\mathcal{M}, \Im)$ is said to be δgp-T_1 [25] (resp., r-T_1 [8]) if for each r, s (r $\neq$ s) $\in \mathcal{M}$, there exist K_1 and $K_2 \in \delta\mathbb{GPO}(\mathcal{M})$ (resp., $\mathbb{O}(\mathcal{M})$) $r \in K_1$, $s \in K_2$, $r \notin K_2$ and $s \notin K_1$.*

Theorem 3.19. *If ℓ:$(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. injection and $(\mathcal{N}, \gamma)$ is δgp-T_1, then $(\mathcal{M}, \Im)$ is r-T_1.*

Proof. Let r, s $\in \mathcal{M}$ and r $\neq$ s.

$$\left.\begin{array}{r}((\mathrm{r}, \mathrm{s}) \in \mathcal{M})(\mathrm{r} \neq \mathrm{s}) \xRightarrow{\ell \text{ is injective}} \ell(\mathrm{r}) \neq \ell(\mathrm{s}) \\ (\mathcal{N}, \gamma) \text{ is } \delta\text{gp - T}_1\end{array}\right\} \implies$$

$$\implies (\exists\ \mathrm{U} \in \delta\mathbb{GPO}(\mathcal{N}, \ell(\mathrm{r}) \text{ and } \mathrm{V} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{s})))\ (\ell(\mathrm{r}) \notin \mathrm{V})\ (\ell(\mathrm{s}) \notin \mathrm{U})$$

$$\xRightarrow{\ell \text{ is c.}\delta\text{gp.i.}} (\ell^{-1}(\mathrm{U}) \in \mathbb{RO}(\mathcal{M}, \mathrm{r}))\ (\ell^{-1}(\mathrm{V}) \in \mathbb{RO}(\mathcal{M}, \mathrm{s}))(\mathrm{r} \notin \ell^{-1}(\mathrm{V}))\ (\mathrm{s} \notin \ell^{-1}(\mathrm{U})).$$ □

Definition 3.9. *A space* $(\mathcal{M}, \Im)$ *is said to be* δ*gp-Hausdorff* [24] *(resp.* r-T_2 [8]*) for each* p, q $(p \neq q) \in \mathcal{M}$, *there exist* $\in J$, K $\delta\mathbb{GPO}(\mathcal{M})$ *(resp.,* $\mathbb{RO}(\mathcal{M})$*) such that* $p \in J$, $q \in K$ *and* $J \cap K = \emptyset$.

Theorem 3.20. *If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N},\gamma)$ *is c.*δ*gp-i. injection and* $(\mathcal{N}, \gamma)$ *is* δ*gp-Hausdorff, then* $(\mathcal{M}, \Im)$ *is* r-T_2.

Proof. Let r, s $\in \mathcal{M}$ and r $\neq$ s.

$$\left.\begin{aligned} (\mathrm{r, s}) \in \mathcal{M}\times\mathcal{M})(\mathrm{r} \neq \mathrm{s}) \xrightarrow{\ell \text{ is injective}} \ell(\mathrm{r}) \neq \ell(\mathrm{s}) \\ (\mathcal{N}, \gamma) \text{ is } \delta\text{gp-Hausdorff} \end{aligned}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{A} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{r}))\ (\exists\ \mathrm{B} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{s}))\ (\ \mathrm{A} \cap \mathrm{B}= \emptyset)$

$\xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$ $(\ell^{-1}(\mathrm{A}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{r}))\ (\ell^{-1}(\mathrm{B}) \in \mathbb{RO}(\mathcal{N}, \mathrm{s}))(\ell^{-1}(\mathrm{A}) \cap \ell^{-1}(\mathrm{B}) = \emptyset)$. □

Theorem 3.21. *Let* $(\mathcal{N}, \gamma)$ *be* δ*gp-Hausdorff space. If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N},\gamma)$ *and* $\acute{k}$: $(\mathcal{M}, \Im) \to (\mathcal{N},\gamma)$ *are c.*δ*gp-i.e, then* $L = \{q \mid \ell(q) = \acute{k}(q)\}$ *is* δ*-closed in* $\mathcal{M}$.

Proof. Suppose that q $\notin$ L.

$$\left.\begin{aligned} \mathrm{q} \notin \mathrm{L} \implies \ell(\mathrm{q}) \neq \acute{\mathrm{k}}(\mathrm{q}) \\ (\mathcal{N}, \gamma) \text{ is } \delta\text{gp-Hausdorff} \end{aligned}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{G} \in \delta\mathbb{GPO}(\mathrm{N}, \ell(\mathrm{q})))(\exists\ \mathrm{H} \in \delta\mathbb{GPO}(\mathcal{N}, \acute{\mathrm{k}}(\mathrm{q})))(\mathrm{G} \cap \mathrm{H} = \emptyset)$

$\xrightarrow{\ell \text{ and } \acute{\mathrm{k}} \text{ are c.}\delta\text{gp.i.}}$

$(\ell^{-1}(\mathrm{G}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))(\acute{\mathrm{k}}^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))(\ell^{-1}(\mathrm{G} \cap \mathrm{H}) = \emptyset)(\acute{\mathrm{k}}^{-1}(\mathrm{G} \cap \mathrm{H}) = \emptyset)$

$\Longrightarrow$ $(\mathrm{U} : = \ell^{-1}(\mathrm{G}) \cap \acute{\mathrm{k}}^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))\ (\mathrm{U} \cap \mathrm{L} = \emptyset) \implies \mathrm{q} \notin \acute{c}_\delta(\mathrm{L})$.

Then, L is δ-closed in $\mathcal{M}$. □

Theorem 3.22. *Let* $(\mathcal{N}, \gamma)$ *be* δ*gp-Hausdorff space. If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N},\gamma)$ *is c.*δ*gp-i., then* $K = \{(p, q) \mid \ell(p) = \ell(q)\}$ *is* δ*-closed in* $\mathcal{M}\times\mathcal{M}$.

Proof. Let (p, q) $\notin$ K.

$$\left.\begin{aligned} (\mathrm{p, q}) \notin \mathrm{K} \implies \ell(\mathrm{p}) \neq \ell(\mathrm{q}) \\ (\mathcal{N}, \gamma) \text{ is } \delta\text{gp-Hausdorff} \end{aligned}\right\} \Longrightarrow$$

$(\exists\ \mathrm{G} \in \delta\mathbb{GPO}(\mathcal{N},\ell(\mathrm{p})))(\exists\ \mathrm{H} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{q})))(\mathrm{G} \cap \mathrm{H} = \emptyset)$

$\xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$ $(\ell^{-1}(\mathrm{G}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{p}))\ (\ell^{-1}(\mathrm{H}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{q}))(\ell^{-1}(\mathrm{G}) \cap \ell^{-1}(\mathrm{H}) = \emptyset)$

$\Longrightarrow$ $(\mathrm{U}: = \ell^{-1}(\mathrm{G}) \times \ell^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}\times\mathcal{M}, (\mathrm{p, q}))(\mathrm{U} \cap \mathrm{K} = \emptyset)$

$\Longrightarrow$ $(\mathrm{p, q}) \notin \acute{c}_\delta(\mathrm{K})$

Then, $\mathcal{M}$ is δ-closed in $\mathcal{M}\times\mathcal{M}$. □

4. Conclusion

In this research paper, we have defined completely δgp-irresolute functions, strongly δgp-regular space, and strongly δgp-normal space in topological spaces with an example and give the proof of the theorems based on their properties. We are interested in extending our research work to convergence in bitopological spaces and nano topological spaces. In addition, we plan to find some interesting concepts in bitopological spaces.

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Department of Mathematics, Karnatak University's Karnatak College, Dharwad - 580001,India

Dr.rer.nat in Mathematics(Graz University of Technology-Graz, Austria) College of Vestsjaelland South Herrestarede 11 and Mathematical and Physical Science Foundation 4200 Slagelse, Denmark.