International Journal of Maps in Mathematics
Volume 8, Issue 2, 2025, Pages:425-435
E-ISSN: 2636-7467
www.simadp.com/journalmim

$\mathcal{N}(\kappa)$-QUASI-EINSTEIN MANIFOLDS ADMITTING SCHOUTEN TENSOR

MOHD VASIULLA, MOHABBAT ALI, AND ABDUL HASEEB *

Abstract. In this note, we study $\mathcal{N}(\kappa)$-quasi-Einstein (in short, $\mathcal{N}(\kappa)$-QE) manifolds admitting the Schouten tensor satisfying certain curvature conditions. At last, the existence of an $\mathcal{N}(\kappa)$-QE manifold is verified by an example.

Keywords: k-nullity distribution, $\mathcal{N}(\kappa)$-(QE) manifolds, Schouten tensor.

2020 Mathematics Subject Classification: 53C18, 53C21, 53C25.

1. Introduction

A Riemannian manifold $(\mathcal{M}^n, g)$ is named an Einstein manifold [1] if its $(0,2)$ type Ricci tensor $Ric(\neq 0)$ satisfies: $Ric = \frac{scal}{n} g$, here $scal$ denotes the scalar curvature of $\mathcal{M}^n$. Einstein manifolds play a key role in mathematical physics, Riemannian geometry as well as in general theory of relativity . Due to its significant physical applications in broad perspectives, these manifolds have been explored by many geometers.

An $(\mathcal{M}^n, g)$ is said to be a quasi-Einstein (QE) [3] if its $Ric(\neq 0)$ fulfills

$$Ric(\Upsilon_1, \Upsilon_2) = \vartheta g(\Upsilon_1, \Upsilon_2) + \Phi \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2), \tag{1.1}$$

for some smooth functions $\vartheta, \Phi(\neq 0)$, and 1-form $\mathcal{A}(\neq 0)$ such that

$$g(\Upsilon_1, \ell) = \mathcal{A}(\Upsilon_1), \quad g(\ell, \ell) = \mathcal{A}(\ell) = 1, \tag{1.2}$$

Received: 2024.11.13 *Revised: 2025.02.01* *Accepted: 2025.02.14*

* Corresponding author

Mohd Vasiulla ⋄ mohd.vasiulla@miet.ac.in ⋄ https://orcid.org/0000-0002-4214-8148

Mohabbat Ali ⋄ ali.math509@gmail.com ⋄ https://orcid.org/0000-0001-8518-4942

Abdul Haseeb ⋄ malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa ⋄ https://orcid.org/0000-0002-1175-6423.

for any vector field Υ_1 and a unit vector field ℓ named the generator of QE manifold. $\mathcal{A}$ is also called the associated 1-form. It is clear from (1.1) that for $\Phi = 0$, a QE manifold reduce to an Einstein manifold.

From (1.1) and (1.2), we have

$$scal = n\vartheta + \Phi, \quad R(\Upsilon_1) = \vartheta\Upsilon_1 + \Phi\mathcal{A}(\Upsilon_1)\ell, \tag{1.3}$$

where R is the Ricci operator defined by

$$g(R(\Upsilon_1), \Upsilon_2) = Ric(\Upsilon_1, \Upsilon_2), \tag{1.4}$$

for $\Upsilon_1, \Upsilon_2 \in \Gamma(TM)$.

The concept of QE manifolds came into existence during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces as well as during the study of exact solutions to Einstein's field equations. For example, the Robertson-Walker spacetimes are QE manifolds. Also, QE spacetime can be used as a model of the perfect fluid spacetime in general relativity [13].

The QE manifolds have also been studied by many authors such as Bilal et. al. [2], Chaki [4], De and Ghosh [8], Vasiulla et al. [20] and many others.

Let $\mathcal{K}$ denotes the Riemannian curvature tensor and κ-nullity distribution $\mathcal{N}(\kappa)$ of an $(\mathcal{M}^n, g)$ [19] is defined by

$$\mathcal{N}(\kappa) : p \to \mathcal{N}_p(\kappa) = \{\Upsilon_3 \in T_pM : \mathcal{K}(\Upsilon_1, \Upsilon_2)\Upsilon_3 = \kappa[g(\Upsilon_2, \Upsilon_3)\Upsilon_1 - g(\Upsilon_1, \Upsilon_3)\Upsilon_2]\}, \tag{1.5}$$

for all vector fields Υ_1, Υ_2, $\Upsilon_3 \in \Gamma(\mathcal{TM})$ ($\Gamma(\mathcal{TM})$: the set of all smooth vector fields on $\mathcal{M}^n$) and κ being some smooth function. Also see [12, 14].

From (1.2) and (1.5), we have

$$Ric(\Upsilon_1, \ell) = \kappa(n-1). \tag{1.6}$$

Similarly, κ-nullity distribution $\mathcal{N}(\kappa)$ of a Lorentzian manifold can also be defined. In a QE manifold, if ℓ belongs to some κ-nullity distribution $\mathcal{N}(\kappa)$, then $\mathcal{M}$ is named $\mathcal{N}(\kappa)$-QE manifold [15]. For more detailed study of $\mathcal{N}(\kappa)$-QE manifolds, we refer the papers [5, 6, 18, 22]. Indeed, κ is not arbitrary as the following lemma:

Lemma 1.1. [15] *In an n-dimensional $\mathcal{N}(\kappa)$-QE manifold it follows that*

$$\kappa = \frac{\vartheta + \Phi}{n-1}. \tag{1.7}$$

It is to be noted that in an n-dimensional $\mathcal{N}(\kappa)$-QE manifold [15]

$$\mathcal{K}(\Upsilon_1,\Upsilon_2,\ell)=\frac{\vartheta+\Phi}{n-1}[\mathcal{A}(\Upsilon_2)\Upsilon_1-\mathcal{A}(\Upsilon_1)\Upsilon_2], \tag{1.8}$$

which is equivalent to

$$\mathcal{K}(\ell,\Upsilon_1,\Upsilon_2)=\frac{\vartheta+\Phi}{n-1}[g(\Upsilon_1,\Upsilon_2)\ell-\mathcal{A}(\Upsilon_2)\Upsilon_1]=-\mathcal{K}(\Upsilon_1,\ell,\Upsilon_2). \tag{1.9}$$

Taking $\Upsilon_2=\ell$ in (1.9), we have

$$\mathcal{K}(\ell,\Upsilon_1,\ell)=\frac{\vartheta+\Phi}{n-1}[\mathcal{A}(\Upsilon_1)\ell-\Upsilon_1]. \tag{1.10}$$

A conformally flat QE manifold of dimension n is an $\mathcal{N}(\frac{\vartheta+\Phi}{n-1})$-QE manifold and hence, a QE manifold of dimension 3 is an $\mathcal{N}(\frac{\vartheta+\Phi}{2})$-QE manifold, as demonstrated in [15]. The conformally flat QE manifolds are certain $\mathcal{N}(\kappa)$-QE manifolds [17]. In 2021, Hazra and Sarkar [11] studied certain curvature conditions on $\mathcal{N}(k)$-QE manifolds. The derivation conditions $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{K}=0$ and $\mathcal{K}(\ell,\Upsilon_1)\cdot Ric=0$ have also been studied in [17]. In [15], the authors studied the derivation conditions $N(\ell,\Upsilon_1)\cdot N=0$ and $N(\ell,\Upsilon_1)\cdot\mathcal{K}=0$ on $\mathcal{N}(\kappa)$-QE manifolds, where N denotes the concircular curvature tensor.

The Weyl conformal curvature tensor $\mathcal{C}$ [7, 9] of an $(\mathcal{M}^n,g)$ is defined by

$$\begin{aligned}\mathcal{C}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=\mathcal{K}(\Upsilon_1,\Upsilon_2,\Upsilon_3)-\frac{1}{n-2}[&Ric(\Upsilon_2,\Upsilon_3)\Upsilon_1-Ric(\Upsilon_1,\Upsilon_3)\Upsilon_2\\&+g(\Upsilon_2,\Upsilon_3)R(\Upsilon_1)-g(\Upsilon_1,\Upsilon_3)R(\Upsilon_2)]\\&+\frac{scal}{(n-1)(n-2)}[g(\Upsilon_2,\Upsilon_3)\Upsilon_1-g(\Upsilon_1,\Upsilon_3)\Upsilon_2],\end{aligned} \tag{1.11}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$. Also, in n-dimensional $\mathcal{N}(\kappa)$-QE manifolds, $\mathcal{C}$ satisfies:

$$\mathcal{C}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\Upsilon_1-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\Upsilon_2], \tag{1.12}$$

$$\mathcal{C}(\Upsilon_1,\Upsilon_2,\ell)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\Upsilon_1-\mathcal{A}(\Upsilon_1)\Upsilon_2], \tag{1.13}$$

$$\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)=0,\quad \mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\ell)=0, \tag{1.14}$$

$$\mathcal{C}(\ell,\Upsilon_2,\Upsilon_3)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\ell-\mathcal{A}(\Upsilon_3)\Upsilon_2], \tag{1.15}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$.

The projective curvature tensor $\mathcal{P}$ is defined by [10, 21]

$$\mathcal{P}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=\mathcal{K}(\Upsilon_1,\Upsilon_2,\Upsilon_3)-\frac{1}{n-1}[Ric(\Upsilon_2,\Upsilon_3)\Upsilon_1-Ric(\Upsilon_1,\Upsilon_3)\Upsilon_2], \tag{1.16}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$. Also, in n-dimensional $\mathcal{N}(\kappa)$-QE manifolds, $\mathcal{P}$ satisfies:

$$\mathcal{P}(\Upsilon_1,\Upsilon_2,\ell)=0, \tag{1.17}$$

$$\mathcal{P}(\ell,\Upsilon_1,\Upsilon_2)=\frac{\Phi}{n-1}[g(\Upsilon_1,\Upsilon_2)\ell-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\ell], \tag{1.18}$$

$$\mathcal{A}(\mathcal{P}(\Upsilon_1,\Upsilon_2,\Upsilon_4)=\frac{\Phi}{n-1}[g(\Upsilon_2,\Upsilon_4)\mathcal{A}(\Upsilon_1)-g(\Upsilon_1,\Upsilon_4)\mathcal{A}(\Upsilon_2)], \tag{1.19}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_4\in\Gamma(\mathcal{TM})$.

In an $(\mathcal{M}^n,g)$, the Schouten tensor $\mathcal{S}$ is given by [16]

$$S(\Upsilon_1,\Upsilon_2)=\frac{1}{n-2}\Big[Ric(\Upsilon_1,\Upsilon_2)-\frac{scal}{2(n-1)}g(\Upsilon_1,\Upsilon_2)\Big], \tag{1.20}$$

for $\Upsilon_1,\Upsilon_2\in\Gamma(\mathcal{TM})$.

In an $\mathcal{N}(\kappa)$-QE manifold, the Schouten tensor takes the form

$$S(\Upsilon_1,\Upsilon_2)=\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}g(\Upsilon_1,\Upsilon_2)+\Phi\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\Big]. \tag{1.21}$$

By contracting (1.21) over Υ_1 and Υ_2, we find

$$scal=\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}n+\Phi\Big]. \tag{1.22}$$

Taking $\Upsilon_1=\ell$ in (1.21), we have

$$S(\ell,\Upsilon_2)=\frac{\mathcal{A}(\Upsilon_2)}{n-2}\Big[\vartheta+\Phi-\frac{scal}{2(n-1)}\Big]. \tag{1.23}$$

2. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S}=0$

Let us consider an $\mathcal{N}(\kappa)$-QE manifold that satisfies $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S}=0$. Then

$$(\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S})(\Upsilon_2,\ell)=-\mathcal{S}(\mathcal{K}(\ell,\Upsilon_1)\Upsilon_2,\ell)-\mathcal{S}(\Upsilon_2,\mathcal{K}(\ell,\Upsilon_1)\ell)=0. \tag{2.24}$$

From (1.9) and (1.21), we find

$$\mathcal{S}(\mathcal{K}(\ell,\Upsilon_1)\Upsilon_2,\ell)=\frac{1}{n-2}\Big[\Big(\vartheta+\Phi-\frac{scal}{2(n-1)}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)\Big], \tag{2.25}$$

where $g(\mathcal{K}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)=\overline{\mathcal{K}}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)$, $\overline{\mathcal{K}}$ is the $(0,4)$ type curvature tensor.

Also, from (1.10) and (1.21), we find

$$\mathcal{S}(\Upsilon_2,\mathcal{K}(\ell,\Upsilon_1)\ell)=-\frac{1}{n-2}\Big[\Big(\vartheta-\frac{scal}{2(n-1)}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)\Big], \tag{2.26}$$

where $g(\mathcal{K}(\ell,\Upsilon_1)\ell,\Upsilon_2)=-\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)$ and $g(\mathcal{K}(\ell,\Upsilon_1)\ell,\ell)=0$ being used.

By virtue of (2.25) and (2.26), the relation (2.24) yields,

$$\Big(\frac{\Phi}{n-2}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)=0. \tag{2.27}$$

From (1.9) and (2.27), we have

$$\Big(\frac{\Phi k}{n-2}\Big)(g(\Upsilon_1,\Upsilon_2)-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2))=0, \tag{2.28}$$

Since $\Phi \neq 0$ and $g(\Upsilon_1, \Upsilon_2) \neq \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)$, then we have $\kappa = 0$, i.e., $\vartheta + \Phi = 0$. Conversely, if $\kappa = 0$, then in view of (1.9) and (1.10) $\mathcal{M}$ satisfies $\mathcal{K}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$. Thus, we can state:

Theorem 2.1. *An $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{K}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$ if and only if $\vartheta + \Phi = 0$.*

3. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$

Let an $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$. Then

$$(\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S})(\Upsilon_2, \Upsilon_3) = -\mathcal{S}(P(\ell, \Upsilon_1)\Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, \mathcal{P}(\ell, \Upsilon_1)\Upsilon_3) = 0, \tag{3.29}$$

which in view of (1.18) takes the form

$$\begin{aligned} &\frac{\Phi}{n-1}\Big[g(\Upsilon_1, \Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) \\ &+g(\Upsilon_1, \Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\mathcal{S}(\Upsilon_2, \ell)\Big] = 0. \end{aligned} \tag{3.30}$$

Since $\Phi (\neq 0)$, therefore, we have

$$\begin{aligned} &g(\Upsilon_1, \Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) \\ &+g(\Upsilon_1, \Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) = 0. \end{aligned} \tag{3.31}$$

In view of (1.23), (3.31) gives

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big)\Big(g(\Upsilon_1, \Upsilon_2)\mathcal{A}(\Upsilon_3) + g(\Upsilon_1, \Upsilon_3)\mathcal{A}(\Upsilon_2) - 2\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\Big) = 0,$$

which by contracting over Υ_1 and Υ_2 gives

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big)\mathcal{A}(\Upsilon_3) = 0. \tag{3.32}$$

This gives $\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) = 0$, as $\mathcal{A}(\Upsilon_3) \neq 0$. Thus, we can state:

Theorem 3.1. *An $\mathcal{N}(\kappa)$-QE manifold $\mathcal{M}$ $(n \geq 3)$ satisfies $\mathcal{P}(\ell, \Upsilon_1).\mathcal{S} = 0$ if and only if $\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) = 0$.*

4. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K} = 0$

Let an $\mathcal{N}(\kappa)$-QE manifold satisfies the condition $(\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4 = 0$. We know that

$$(\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4 = ((\Upsilon_1 \wedge_{\mathcal{S}} \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4, \tag{4.33}$$

where the endomorphism $(\Upsilon_1 \wedge_{\mathcal{S}} \Upsilon_2)\Upsilon_3$ is given by

$$(\Upsilon_1 \wedge_{\mathcal{S}} \Upsilon_2)\Upsilon_3 = \mathcal{S}(\Upsilon_2, \Upsilon_3)\Upsilon_1 - \mathcal{S}(\Upsilon_1, \Upsilon_3)\Upsilon_2. \tag{4.34}$$

Rewriting (4.33) as

$$\begin{aligned}(\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K})(\Upsilon_2,\Upsilon_3)\Upsilon_4 &= ((\Upsilon_1\wedge_{\mathcal{S}}\ell)\mathcal{K})(\Upsilon_2,\Upsilon_3)\Upsilon_4-\mathcal{K}((\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_2,\Upsilon_3)\Upsilon_4\\ &\quad -\mathcal{K}(\Upsilon_2,(\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_3)\Upsilon_4-\mathcal{K}(\Upsilon_2,\Upsilon_3)(\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_4,\end{aligned}$$

which by using $\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K}=0$ and (4.34) turns to

$$\begin{aligned}&\mathcal{S}(\ell,\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_4)\Upsilon_1-\mathcal{S}(\Upsilon_1,\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_4)\ell-\mathcal{S}(\ell,\Upsilon_2)\mathcal{K}(\Upsilon_1,\Upsilon_3)\Upsilon_4\\ &+\mathcal{S}(\Upsilon_1,\Upsilon_2)\mathcal{K}(\ell,\Upsilon_3),\Upsilon_4-\mathcal{S}(\ell,\Upsilon_3)\mathcal{K}(\Upsilon_2,\Upsilon_1)\Upsilon_4+\mathcal{S}(\Upsilon_1,\Upsilon_3)\mathcal{K}(\Upsilon_2,\ell)\Upsilon_4 \qquad (4.35)\\ &-\mathcal{S}(\ell,W)\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_1+\mathcal{S}(\Upsilon_1,\Upsilon_4)\mathcal{K}(\Upsilon_2,\Upsilon_3)\ell=0.\end{aligned}$$

By using (1.8), (1.9) and (1.21) in (4.35), and taking the inner product with ℓ, we have

$$\frac{\Phi\kappa}{n-2}[g(\Upsilon_1,\Upsilon_2)\mathcal{A}(\Upsilon_3)\mathcal{A}(\Upsilon_4)-g(\Upsilon_1,\Upsilon_3)\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_4)]=0. \qquad (4.36)$$

Putting $\Upsilon_3=\ell$ in (4.36), we have

$$\Phi\kappa\mathcal{A}(\Upsilon_4)[g(\Upsilon_1,\Upsilon_2)-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)]=0. \qquad (4.37)$$

Since $\Phi(\neq 0)$, $\mathcal{A}(\neq 0)$ and $g(\Upsilon_1,\Upsilon_2)\neq\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)$, then $\kappa=0$. If $\kappa=0$, then the converse is trivial. Thus, we have:

Theorem 4.1. *An $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K}=0$ if and only if $\vartheta+\Phi=0$.*

5. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{C}\cdot\mathcal{S}=0$

Let an $\mathcal{N}(\kappa)$-(QE) manifold holds $\mathcal{C}\cdot\mathcal{S}=0$. We know that

$$(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4)=-\mathcal{S}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)-\mathcal{S}(\Upsilon_3,\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4). \qquad (5.38)$$

Making use of (1.21) in (5.38), we have

$$\begin{aligned}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4) &= -\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}\Big(g(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)\\ &\quad +g(\Upsilon_3,\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big)+\Phi\Big(\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)\mathcal{A}(\Upsilon_4) \qquad (5.39)\\ &\quad +\mathcal{A}(\Upsilon_3)\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big)\Big],\end{aligned}$$

which by using the symmetric property of the metric tensor, and the skew-symmetric property of $\overline{\mathcal{K}}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)$ reduces to

$$(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4)=-\frac{\Phi}{n-2}\Big(\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)\mathcal{A}(\Upsilon_4)+\mathcal{A}(\Upsilon_3)\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big).$$

In view of (1.14), the foregoing equation turns to

$$\mathcal{C}((\Upsilon_1, \Upsilon_2) \cdot \mathcal{S})(\Upsilon_3, \Upsilon_4) = 0. \tag{5.40}$$

Thus, we have:

Theorem 5.1. *In an $\mathcal{N}(\kappa)$-QE manifold, the relation $\mathcal{C} \cdot \mathcal{S} = 0$ holds for all $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$.*

6. Schouten-recurrent $\mathcal{N}(\kappa)$-QE manifolds

In 1952, Patterson [16] proposed the idea of Ricci recurrent manifolds . According to him, an $(\mathcal{M}^n, g)$ is said to be Ricci recurrent if

$$(D_{\Upsilon_1} Ric)(\Upsilon_2, \Upsilon_3) = \mathcal{A}(\Upsilon_1) Ric(\Upsilon_2, \Upsilon_3), \tag{6.41}$$

for some 1-form $\mathcal{A}(\neq 0)$.

An $(\mathcal{M}^n, g)$ is named a Schouten recurrent manifold if its Schouten tensor satisfies

$$(D_{\Upsilon_1} \mathcal{S})(\Upsilon_2, \Upsilon_3) = \mathcal{A}(\Upsilon_1) \mathcal{S}(\Upsilon_2, \Upsilon_3). \tag{6.42}$$

We write

$$(D_{\Upsilon_1} \mathcal{S})(\Upsilon_2, \Upsilon_3) = \Upsilon_1 \mathcal{S}(\Upsilon_2, \Upsilon_3) - \mathcal{S}(D_{\Upsilon_1} \Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, D_{\Upsilon_1} \Upsilon_3). \tag{6.43}$$

From (6.42) and (6.43), we have

$$\Upsilon_1 \mathcal{S}(\Upsilon_2, \Upsilon_3) - \mathcal{S}(D_{\Upsilon_1} \Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, D_{\Upsilon_1} \Upsilon_3) = \mathcal{A}(\Upsilon_1) \mathcal{S}(\Upsilon_2, \Upsilon_3). \tag{6.44}$$

Putting $\Upsilon_2 = \Upsilon_3 = \ell$ in (6.44) and using (1.21), we find

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) \mathcal{A}(\Upsilon_1) = \Upsilon_1 \Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big). \tag{6.45}$$

Thus, we have the following result:

Theorem 6.1. *If $(\mathcal{M}^n, g)$ is a Schouten recurrent $\mathcal{N}(\kappa)$-QE manifold, then*

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) \mathcal{A}(\Upsilon_1) = \Upsilon_1 \Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big),$$

for all $\Upsilon_1 \in \Gamma(\mathcal{TM})$.

A Schouten recurrent manifold is Schouten symmetric iff $\mathcal{A} = 0$. Thus, we have:

Corollary 6.1. *If $(\mathcal{M}^n, g)$ is a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold, then $\vartheta + \Phi - \frac{scal}{2(n-1)}$ is constant.*

Corollary 6.2. *If $(\mathcal{M}^n, g)$ is a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold and if $\vartheta+\Phi-\frac{scal}{2(n-1)}$ is constant, then either $\vartheta+\Phi-\frac{scal}{2(n-1)}=0$ or $(\mathcal{M}^n, g)$ reduces to a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold.*

7. Example

Define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by

$$ds^2 = g_{ij}d\varkappa^i d\varkappa^j = (1+2p)[(d\varkappa^4)^2+(d\varkappa^3)^2+(d\varkappa^2)^2+(d\varkappa^1)^2], \tag{7.46}$$

where $\varkappa^1, \varkappa^2, \varkappa^3, \varkappa^4$ are non-zero finite and $p = e^{\varkappa^1}k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{ij} = (1+2p),\ for\ i=j=1,2,3,4, \quad g_{ij}=0,\ otherwise \tag{7.47}$$

and

$$g^{ij} = \frac{1}{1+2p},\ for\ i=j=1,2,3,4, \quad g^{ij}=0,\ otherwise, \tag{7.48}$$

respectively. The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned}\begin{Bmatrix}1\\11\end{Bmatrix}=\begin{Bmatrix}2\\12\end{Bmatrix}=\begin{Bmatrix}3\\13\end{Bmatrix}=\begin{Bmatrix}4\\14\end{Bmatrix}&=\frac{p}{1+2p},\\ \begin{Bmatrix}1\\22\end{Bmatrix}=\begin{Bmatrix}1\\33\end{Bmatrix}=\begin{Bmatrix}1\\44\end{Bmatrix}&=\frac{-p}{1+2p}.\end{aligned} \tag{7.49}$$

The non-zero derivatives of (7.49) are

$$\begin{aligned}\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}4\\14\end{Bmatrix}=\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}3\\13\end{Bmatrix}=\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}2\\12\end{Bmatrix}=\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}1\\11\end{Bmatrix}&=\frac{p}{(1+2p)^2},\\ \frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}1\\44\end{Bmatrix}=\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}1\\33\end{Bmatrix}=\frac{\partial}{\partial\varkappa^1}\begin{Bmatrix}1\\22\end{Bmatrix}&=\frac{-p}{(1+2p)^2}.\end{aligned} \tag{7.50}$$

For the Riemannian curvature tensor,

$$\mathcal{K}^l_{ijk} = \underbrace{\begin{vmatrix}\frac{\partial}{\partial\varkappa^j} & \frac{\partial}{\partial\varkappa^k}\\ \begin{Bmatrix}l\\ij\end{Bmatrix} & \begin{Bmatrix}l\\ik\end{Bmatrix}\end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix}\begin{Bmatrix}m\\ik\end{Bmatrix} & \begin{Bmatrix}m\\ij\end{Bmatrix}\\ \begin{Bmatrix}l\\mk\end{Bmatrix} & \begin{Bmatrix}l\\mj\end{Bmatrix}\end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$\mathcal{K}^1_{221} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^1_{331} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^1_{441} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{p}{(1+2p)^2}$$

and the non-zero components of (II) are:

$$\mathcal{K}^2_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$\mathcal{K}^2_{442} = \begin{Bmatrix} m \\ 42 \end{Bmatrix}\begin{Bmatrix} 2 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$\mathcal{K}^3_{443} = \begin{Bmatrix} m \\ 43 \end{Bmatrix}\begin{Bmatrix} 3 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ m3 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p^2}{(1+2p)^2}.$$

Adding the corresponding components of (I) and (II), we have

$$\mathcal{K}^1_{221} = \mathcal{K}^1_{331} = \mathcal{K}^1_{441} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^2_{332} = \mathcal{K}^2_{442} = \mathcal{K}^3_{443} = \frac{p^2}{(1+2p)^2}.$$

Thus, the non-zero components of the curvature tensor, up to symmetry are

$$\overline{K}_{1221} = \overline{K}_{1331} = \overline{K}_{1441} = \frac{p}{1+2p},$$

$$\overline{K}_{2332} = \overline{K}_{2442} = \overline{K}_{3443} = \frac{p^2}{1+2p}$$

and the non-zero components of the Ricci tensor are

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} + g^{44}\overline{K}_{1414} = \frac{3p}{(1+2p)^2},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} + g^{44}\overline{K}_{2424} = \frac{p}{(1+2p)},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} + g^{44}\overline{K}_{3434} = \frac{p}{(1+2p)},$$

$$Ric_{44} = g^{jh}\overline{K}_{4j4h} = g^{11}\overline{K}_{4141} + g^{22}\overline{K}_{4242} + g^{33}\overline{K}_{4343} = \frac{p}{(1+2p)}.$$

The scalar curvature *scal* is

$$scal = \frac{6p(1+p)}{(1+2p)^2}.$$

Let us consider the associated scalars ϑ, Φ are defined by

$$\vartheta = \frac{p}{(1+2p)^2}, \quad \Phi = \frac{2p(1-p)}{(1+2p)^3}$$

and the 1-form

$$\mathcal{A}_i(x) = \begin{cases} \sqrt{1+2p}, & \text{if } i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where generators are unit vector fields, then from (1.11), we have

$$Ric_{11} = \vartheta g_{11} + \Phi \mathcal{A}_1 \mathcal{A}_1, \tag{7.51}$$

$$Ric_{22} = \vartheta g_{22} + \Phi \mathcal{A}_2 \mathcal{A}_2, \tag{7.52}$$

$$Ric_{33} = \vartheta g_{33} + \Phi \mathcal{A}_3 \mathcal{A}_3, \tag{7.53}$$

$$Ric_{44} = \vartheta g_{44} + \Phi \mathcal{A}_4 \mathcal{A}_4. \tag{7.54}$$

$$\begin{aligned} R.H.S. \quad of \quad (7.51) &= \vartheta g_{11} + \Phi \mathcal{A}_1 \mathcal{A}_1 \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S. \quad of \quad (7.51) \end{aligned}$$

By similar way it can be shown that (7.52) to (7.54) are also true. Hence $(\mathbb{R}^4, g)$ is an $\mathcal{N}\left(\frac{p}{(1+2p)^3}\right)$-QE manifold.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

References

[1] Besse, A. L. (1987). Einstein manifolds. Ergeb. Math. Grenzgeb., 3. Folge, Bd. 10. Springer-Verlag, Berlin.

[2] Bilal, M., Vasiulla, M., Haseeb, A., Ahmadini, A. A. H., & Ali, M. (2023). A study of mixed generalized quasi-Einstein spacetimes with applications in general relativity. AIMS Mathematics, 8(10), 24726-24739.

[3] Chaki, M. C., & Maity, R. K. (2000). On quasi-Einstein manifolds. Publ. Math. Debrecen, 57, 297-306.

[4] Chaki, M. C. (2001). On generalized quasi-Einstein manifolds. Publ. Math. Debrecen, 58, 683-691.

[5] Crasmareanu, M. (2012). Parallel tensors and Ricci solitons in $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Indian Journal of Pure and Applied Mathematics, 43(4), 359-369.

[6] Chaubey, S. K. (2017). Existence of $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Facta Universitatis (NIS), Ser. Math. Inform., 32(3), 369-385.

[7] De, U. C., & Shaikh, A. A. (2007). Differential Geometry of Manifolds. Narosa Publishing House (Pvt. Ltd.), New Delhi.

[8] De, U. C. & Ghosh, G. C. (2004). On quasi-Einstein manifolds. Period. Math. Hungar., 48, 223-231.

[9] Haseeb, A. (2015). Some new results on para-Sasakian manifold with a quarter-symmetric metric connection. Facta Universitatis (NIS), Ser. Math. Inform., 30(5), 765-776.

[10] Haseeb, A. (2017). Some results on projective curvature tensor in an ϵ-Kenmotsu manifold. Palestine Journal of Mathematics, 6(special issue: II), 196-203.

[11] Hazra, D., & Sarkar, A. (2021). Quasi-conformal curvature tensor on $\mathcal{N}(k)$-quasi Einstein manifolds. Korean J. Math., 29(4), 801-810.

[12] Mandal P., Shahid M. H., & Yadav S. K. (2024). Conformal Ricci soliton on paracontact metric (κ, μ)-manifolds with Schouten van Kampen connection. Commun. Korean Math. Soc, 39(1), 161-173.

[13] Mantica, C. A., Suh, Y. J., & De, U. C. (2016). A note on generalized Robertson-Walker space-times. Int. J. Geom. Meth. Mod. Phys., 13, 1650079, (9 pages.).

[14] Montano B .C., Erken I. K., & Murathan C. (2012). Nullity conditions in paracontact geometry. Differential Geom. Appl., 30(6), 665-693.

[15] Özgür, C., & Tripathi, M. M. (2007). On the concircular curvature tensor of an $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Math. Pann., 18(1), 95-100.

[16] Patterson, E. M. (1952). Some theorem on Ricci-recurrent space. J. London Math. Soc., 27, 287-295.

[17] Tripathi, M. M., & Kim, J. S. (2007). On $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Commun. Korean Math. Soc., 22(3), 411-417.

[18] Taleshian, A., & Hosseinzadeh, A. A. (2011). Investigation of some conditions on $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Bull. Malays. Math. Sci. Soc., 34(3), 455-464.

[19] Tanno, S. (1988). Ricci curvatures of contact Riemannian manifolds. Tohoku Math. J., 40, 441-448.

[20] Vasiulla, M., Haseeb A., Mofarreh, F. & Ali, M. (2022). Application of mixed generalized quasi-Einstein spacetimes in general relativity. Mathematics, 10, 3749.

[21] Yano, K., & Kon, M. (1984). Structures on Manifolds. Vol. 3, World Scientific Publishing Co., Singapore.

[22] Yang, Y., & Xu, S. (2012), Some conditions of $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Int. J. Dig. Cont.Tech. Appl., 6(8), 144-150.

Department of Applied Sciences, Meerut Institute of Engineering & Technology, Meerut-250005, India

School of Basic and Applied Sciences, K. R. Mangalam University, Sohna Road, Gurugram – 122103, India

Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan-45142, Kingdom of Saudi Arabia