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ON f-BIHARMONIC HYPERSURFACES

SELCEN YÜKSEL PERKTAŞ, FEYZA ESRA ERDOĞAN*, AND BİLAL EFTAL ACET

ABSTRACT. In the present paper, we study spacelike and timelike f-biharmonic hypersurfaces in Lorentzian para-Sasakian manifolds. We investigate f-biharmonic equations for spacelike hypersurfaces of an Lorentzian para-Sasakian manifold with a constant and harmonic mean curvature. Also we give some results for f-biharmonic timelike hypersurfaces in η-Einstein Lorentzian para-Sasakian manifolds.

1. Introduction

Harmonic maps have an important area of study as a generalization of important ideas like geodesics and minimal submanifolds. A significant literature has been created in the last decade including relationships between harmonic maps and the other disciplines namely theoretical physics.

A map $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds is called harmonic if it is a critical point of energy functional given by

$$E(\Psi) = \frac{1}{2}\int_{\Omega} \mid d\Psi \mid^2 \vartheta_g.$$

Therefore harmonic maps are the solutions of the corresponding Euler-Lagrange equation,

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* Corresponding author

which is characterized by the vanishing of the tension field

$$\tau(\Psi) = trace\nabla d\Psi.$$

As introduced by J. Eells and J. H. Sampson in [1], bienergy of a map Ψ is defined by

$$E_2(\Psi) = \frac{1}{2}\int_\Omega \mid \tau(\Psi) \mid^2 \vartheta_g,$$

and Ψ is said to be biharmonic if it is a critical point of the bienergy.

In [2], the first and second variation formula for the bienergy were derived by G. Y. Jiang, showing that the Euler-Lagrange equation associated to E_2 is

$$\tau_2(\Psi) = -\triangle\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where $\Delta = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator of N. The equation $\tau_2(\Psi) = 0$ is called biharmonic equation and it is clear that any harmonic map is biharmonic.

With another aspect, B. Y. Chen [5] defined biharmonic submanifolds of the Euclidean space by $\Delta H = 0$, that is any submanifold in Euclidean space whose mean curvature vector field H is harmonic is called a biharmonic submanifold, where Δ is the Laplacian of the submanifold acting on functions. Also B. Y. Chen [5] made a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal. If one use the definition of biharmonic maps to Riemannian immersions into Euclidean space, it is easy to see that Chen's definition of biharmonic submanifold coincides with the definition given by using bienergy functional.

In the literature there are many results on the non-existence of biharmonic submanifolds in manifolds with non-positive sectional curvature. These non-existence consequences (see [3, 6],) and as well as Generalized Chen's conjecture: any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal, which was proposed by R. Caddeo, S. Montaldo and C. Oniciuc [7], led the studies to spheres and other non-negatively curved spaces. But in recent years the authors of [8] proved that generalized Chen conjecture is not true by constructing examples of proper biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature.

In the last fifteen years there is a growing interest in biharmonic maps theory and its applications to the other areas. For some recent geometric studies of general biharmonic maps and biharmonic submanifolds see [9, 7, 10, 8, 11, 15, 12, 13, 14] and the references therein.

f-harmonic maps between Riemannian manifolds were first introduced and studied by A. Lichnerowicz in 1970 (see also [16]). A smooth map $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds is called f-harmonic if it is a critical point of f-energy functional defined by

$$E_f(\Psi) = \frac{1}{2} \int_\Omega f \mid d\Psi \mid^2 \vartheta_g,$$

for any compact domain domain Ω. The Euler-Lagrange equation is given by

$$\tau_f(\Psi) = f\tau(\Psi) + d\Psi(gradf) = 0,$$

where $\tau(\Psi)$ is the tension field of Ψ. If f is a constant function then it is obvious that f-harmonic maps are harmonic. So f-harmonic maps, where f is a non-constant function are more interesting to study. We call such maps as proper f-harmonic maps.

The concept of f-biharmonic maps have been introduced by W.-J. Lu [17] as a generalization of biharmonic maps. A differentiable map between Riemannian manifolds is said to be f-biharmonic if it is a critical point of the f-bienergy functional defined by integral of f times the square-norm of the tension field, where f is a smooth positive function on the domain. If $f = 1$ then f-biharmonic maps are biharmonic. To avoid the confussion with the types of map called by the same name in [18] and defined as a critical point of the square-norm of the f-tension field, some authors (see [17], [19]) called the map defined in [18] as bi-f-harmonic map.

In the present paper, our aim is to study f-biharmonic equations for hypersurfaces in Lorentzian para-Sasakian manifolds. Since the characteristic vector field of a Lorentzian para-Sasakian manifold is timelike then we consider it in the tangent space and as the normal vector of the hypersurface, respectively. We investigate f-biharmonic equations for spacelike hypersurfaces of an LP-Sasakian manifold with a constant and harmonic mean curvature. Also we give some results for f-biharmonic timelike hypersurfaces in η-Einstein LP-Sasakian manifolds.

2. Preliminaries

2.1. **Harmonic maps.** A map $\Psi \in C^\infty(M, N)$ is called *harmonic* if it is a critical point of the *energy* functional

$$E(\Psi) = \frac{1}{2} \int_\Omega |d\Psi|^2 v_g, \tag{2.1}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the harmonic map equation [1]

$$\tau(\Psi) \equiv trace\nabla d\Psi = 0, \tag{2.2}$$

where $\tau(\Psi) \equiv trace\nabla d\Psi$ is called the tension field of Ψ, ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^Ψ.

2.2. **Biharmonic maps.** As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J. H. Sampson in [1]. *Biharmonic maps* between Riemannian manifolds $\Psi : (M,g) \to (N,h)$ are the critical points of the *bienergy functional*

$$E_2(\Psi) = \frac{1}{2}\int_\Omega |\tau(\Psi)|^2 v_g. \tag{2.3}$$

The first variation formula for the bienergy which is derived in [2, 3] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0, \tag{2.4}$$

where $\Delta = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator on N. From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

2.3. ***f*-Harmonic maps.** f-Harmonic maps are critical points of the f-energy functional for maps $\Psi : (M,g) \to (N,h)$ between Riemannian manifolds:

$$E_f(\Psi) = \frac{1}{2}\int_\Omega f\,|d\Psi|^2 v_g, \tag{2.5}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the f-harmonic map equation ([20], [18]):

$$\tau_f(\Psi) \equiv f\,\tau(\Psi) + d\Psi(gradf) = 0, \tag{2.6}$$

where $\tau(\Psi) \equiv trace\nabla d\Psi$ is the tension field of Ψ vanishing of which means Ψ is a harmonic map. $\tau_f(\Psi)$ is called the f-tension field of map Ψ.

2.4. **f-Biharmonic maps.** f-Biharmonic maps are critical points of the f-bienergy functional for maps $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_f(\Psi) = \frac{1}{2}\int_{\Omega} f\ |\tau(\Psi)|^2 v_g, \tag{2.7}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the f-biharmonic map equation [17] :

$$\tau_{2,f}(\Psi) \equiv f\,\tau_2(\Psi) + (\Delta f)\tau(\Psi) + 2\nabla^{\Psi}_{gradf}\ \tau(\Psi) = 0, \tag{2.8}$$

where $\tau(\Psi)$ and $\tau_2(\Psi)$ are the tension and bitension fields of Ψ, respectively. $\tau_{2,f}(\Psi)$ called the f-bitension field of map Ψ.

2.5. **Bi-f-Harmonic maps.** Bi-f-harmonic maps are critical points of the bi-f-energy functional for maps $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E^2_f(\Psi) = \frac{1}{2}\int_{\Omega} |\tau_f(\Psi)|^2 v_g. \tag{2.9}$$

The Euler-Lagrange equation gives the bi-f-harmonic map equation [18] :

$$\tau^2_f(\Psi) \equiv f\,J^{\Psi}(\tau_f(\Psi)) - \nabla^{\Psi}_{gradf}\,\tau_f(\Psi), \tag{2.10}$$

where J^{Ψ} is the Jacobi operator of the map defined by

$$J^{\Psi}(X) = -Tr\,(\nabla^{\Psi}\nabla^{\Psi}X - \nabla^{\Psi}_{\nabla^M}X - R^N(d\Psi, X)d\Psi). \tag{2.11}$$

The following illustrate the relations among these different types of harmonic maps:

Harmonic maps $\subset$ Biharmonic maps $\subset$ f-Biharmonic maps,

Harmonic maps $\subset$ f-Harmonic maps $\subset$ Bi-f-harmonic maps.

3. Lorentzian Para-Sasakian Manifolds

Let $\overline{M}$ be an $(m+1)$-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\overline{M}$ such that [21]

$$\eta(\xi) = -1, \tag{2.2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2.2}$$

where I denotes the identity map of $T_p\overline{M}$ and $\otimes$ is the tensor product. The equations (2.2.1) and (2.2.2) imply that

$$\begin{aligned} \eta\circ\phi &= 0, \\ \phi\xi &= 0, \\ rank(\phi) &= m. \end{aligned} \tag{2.2.3}$$

Then $\overline{M}$ admits a Lorentzian metric $\overline{g}$, i.e., $\overline{g}$ is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p\in\overline{M}$, the tensor $g_p : T_p\overline{M}\times T_p\overline{M}\to R$ is a non-degenerate inner product of index 1, where $T_p\overline{M}$ is the tangent space of $\overline{M}$ at the point p,such that

$$\overline{g}(\phi X,\phi Y)=\overline{g}(X,Y)+\eta(X)\eta(Y), \tag{2.2.4}$$

and $\overline{M}$ is said to admit a Lorentzian almost paracontact structure $(\phi,\xi,\eta,\overline{g})$. Then we get

$$\begin{aligned} \overline{g}(X,\xi) &= \eta(X), \\ \Phi(X,Y) &= \overline{g}(X,\phi Y)=\overline{g}(\phi X,Y)=\Phi(Y,X), \\ (\overline{\nabla}_X\Phi)(Y,Z) &= g(Y,(\overline{\nabla}_X\phi)Z)=(\overline{\nabla}_X\Phi)(Z,Y), \end{aligned} \tag{2.2.5}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to $\overline{g}$. A non-zero vector $X_p\in T_p\overline{M}$ is called spacelike, null or timelike ,if it satisfies $\overline{g}_p(X_p,X_p)\geq 0$, $\overline{g}_p(X_p,X_p)=0$ $(X_p\neq 0)$ or $\overline{g}_p(X_p,X_p)<0$, respectively. It is clear that the Lorentzian metric $\overline{g}$ makes ξ a timelike unit vector field, i.e, $\overline{g}(\xi,\xi)=-1$. The manifold $\overline{M}$ equipped with a Lorentzian almost paracontact structure $(\phi,\xi,\eta,\overline{g})$ is called a Lorentzian almost paracontact manifold (for short LAP-manifold) [21], [22].

In equations (2.2.1) and (2.2.2) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by Satō [23].

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\phi,\xi,\eta,\overline{g})$ is called a Lorentzian para-Sasakian manifold (for short LP-Sasakian) [21] if

$$(\overline{\nabla}_X\phi)Y=\eta(Y)X+\overline{g}(X,Y)\xi+2\eta(X)\eta(Y)\xi. \tag{2.2.6}$$

In an LP-Sasakian manifold the 1-form η is closed and

$$\overline{\nabla}_X\xi=\phi X. \tag{2.2.7}$$

Also, an LP-Sasakian manifold $\overline{M}$ is said to be η-Einstein if its Ricci tensor $\overline{S}$ satisfies

$$\overline{S}(X,Y)=a\overline{g}(X,Y)+b\eta(X)\eta(Y), \tag{2.2.8}$$

for any vector fields X, Y, where a and b are functions on $\overline{M}$. The Ricci tensor of an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold is given by [24]

$$\overline{S}(X,Y) = (\frac{\overline{r}}{m} - 1)\overline{g}(X,Y) + (\frac{\overline{r}}{m} - (m+1))\eta(X)\eta(Y), \tag{2.2.9}$$

where $\overline{r}$ is the scalar curvature of the manifold.

In an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$ with the structure $(\phi, \xi, \eta, \overline{g})$, the following relations hold [21], [25]:

$$\overline{g}(\overline{R}(X,Y)Z, \xi) = \eta(\overline{R}(X,Y)Z) = \overline{g}(Y,Z)\eta(X) - \overline{g}(X,Z)\eta(Y), \tag{2.2.10}$$

$$\overline{R}(\xi, X)Y = \overline{g}(X,Y)\xi - \eta(Y)X \tag{2.2.11}$$

$$\overline{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.2.12}$$

$$\overline{R}(\xi, X)\xi = X + \eta(X)\xi, \tag{2.2.13}$$

$$\overline{S}(X,\xi) = m\eta(X), \tag{2.2.14}$$

$$\overline{S}(\phi X, \phi Y) = \overline{S}(X,Y) + m\eta(X)\eta(Y), \tag{2.2.15}$$

for any vector fields X, Y, Z in $\overline{M}$, where $\overline{R}$ and $\overline{S}$ are the Riemannian curvature and the Ricci tensors of $\overline{M}$, respectively.

A semi-Riemannian hypersurface of a semi-Riemannian manifold is just a semi-Riemannian submanifold of codimension 1. It is well known that a Lorentzian manifold is a semi-Riemannian manifold with a symmetric nondegenerate $(0,2)$ tensor field, namely metric tensor, of index 1. Let M be a hypersurface of a Lorentzian manifold $\overline{M}$. If the normal vector field of M is timelike(respectively, spacelike) then M is called a spacelike (respectively, timelike) hypersurface of $\overline{M}$ (see [26]).

Let M be a hypersurface of an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$. The Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.2.16}$$

$$\overline{\nabla}_X N = -A_N X, \tag{2.2.17}$$

for each X, $Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where ∇ is the Levi-Civita connection on M, B is the second fundamental form of M and A_N is the shape operator with respect to the normal section N. We can write $B(X,Y) = b(X,Y)N$, where b is the function-valued second fundamental form of M. Then the second fundamental form B and the shape operator of the hypersurface with respect to the unit normal vector field N are related by

$$B(X,Y) = \varepsilon\overline{g}(\overline{\nabla}_X Y, N)N = -\varepsilon\overline{g}(Y, \overline{\nabla}_X N)N = \varepsilon\overline{g}(A_N X, Y)N \tag{2.2.18}$$

and

$$\overline{g}(A_N X, Y) = \overline{g}(B(X,Y), N) = \overline{g}(b(X,Y)N, N) = \varepsilon b(X,Y), \tag{2.2.19}$$

where X, $Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$ and $\varepsilon = \overline{g}(N,N)$.

4. f-Biharmonic spacelike hypersurfaces in LP-Sasakian manifolds

In this section we consider that the characteristic vector field of the LP-Sasakian manifold is the unit normal vector field of the hypersurface. Hence, we characterize the spacelike f-biharmonic hypersurfaces in a Lorentzian para-Sasakian (LP-Sasakian) manifold.

Definition 4.1. *A hypersurface in an LP-Sasakian manifold is called an f-biharmonic hypersurface if the isometric immersion defining the hypersurface is an f-biharmonic map.*

Minimal hypersurfaces are well known examples of biharmonic hypersurfaces. Also biharmonic hypersurfaces are f-biharmonic with the function $f = 1$. So we have the following relationship:

Minimal hypersurfaces $\subset$ Biharmonic hypersurfaces $\subset$ f-Biharmonic hypersurfaces.

Neither minimal nor biharmonic hypersurfaces will be called proper f-biharmonic submanifolds.

The f-biharmonic equation for a hypersurface in a Riemannian manifold is given in the following [19].

Theorem 4.1. *Let $\Psi : M^m \to N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then Ψ is an f-biharmonic map if and only if*

$$\begin{cases} \Delta H - H\,|A|^2 + HRic^N(\xi,\xi) + H\frac{\Delta f}{f} + 2(grad\ln f)H = 0, \\ 2A(gradH) + \frac{m}{2}gradH^2 - 2H(Ric^N(\xi))^T + 2HA(grad\ln f) = 0, \end{cases} \tag{3.1}$$

where $Ric^N : T_qN \to T_qN$ denotes the Ricci operator of the ambient space, A is the shape operator of the hypersurface with respect to the unit normal vector ξ, and Δ, $grad$ are the Laplace and the gradient operator of the hypersurface, respectively.

Theorem 4.2. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and $\Psi : M \to \overline{M}$ be an isometric immersion with $\dim M = m$. Assume that the characteristic vector field ξ is the unit normal vector field of the hypersurface M. Then the spacelike hypersurface M is f-biharmonic if and only if*

$$\begin{aligned} \frac{m}{2} f\left(gradH^2\right) - 2A(gradfH) &= 0, &(3.2)\\ -\Delta(fH) + 2H\Delta f + 2mfH &= 0 &(3.3) \end{aligned}$$

where A is the shape operator of the hypersurface with respect to the unit normal vector field ξ and $\mu = H\xi$ is the mean curvature vector.

Proof. Let M be a hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field ξ and $\Psi : M \to \overline{M}$ be an isometric immersion. Assume that $\{e_i\}_{i=1}^{m}$ is a local orthonormal frame of M such that $\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$ is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$. We identify $d\Psi(X)$ by X and $\nabla^{\Psi}_X W$ by $\overline{\nabla}_X W$ for all $X \in \Gamma(TM)$, $W \in \Gamma(\Psi^{-1}TM)$. Note that the tension field of Ψ is $\tau(\Psi) = mH\xi$. Then the bitension field of $\Psi : M \to \overline{M}$ is as follows:

$$\begin{aligned} \tau_2(\psi) &= \sum_{i=1}^{m}\{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}\tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}\tau(\Psi) - \overline{R}(d\Psi(e_i), \tau(\Psi))d\Psi(e_i)\}\\ &= \sum_{i=1}^{m}\{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}(mH\xi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\}\\ &= \sum_{i=1}^{m}\{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(mH\xi) - \overline{\nabla}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\}\\ &= m\sum_{i=1}^{m}\{\overline{\nabla}_{e_i}\left(e_i(H)\xi + H\overline{\nabla}_{e_i}\xi\right) - \left(\nabla_{e_i}e_i\right)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi\\ &\quad -H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\}\\ &= m\sum_{i=1}^{m}\{e_ie_i(H)\xi + 2e_i(H)\overline{\nabla}_{e_i}\xi + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi\\ &\quad -\left(\nabla_{e_i}e_i\right)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi - H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\}\\ &= -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + mH\sum_{i=1}^{m}\overline{R}(\xi, d\Psi(e_i))d\Psi(e_i). \quad (3.4) \end{aligned}$$

Since $\overline{M}$ is a LP-Sasakian manifold then from (2.2.11), we have

$$\sum_{i=1}^{m}\overline{R}(\xi, d\Psi(e_i))d\Psi(e_i) = m\xi. \quad (3.5)$$

By writing (3.5) in (3.4), we get

$$\tau_2(\Psi) = -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + m^2H\xi. \tag{3.6}$$

From (2.8) we can write f-bitension field of Ψ :

$$\begin{aligned} \tau_{2,f}(\Psi) \;\; \equiv \;\; & f\left\{-m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + m^2H\xi\right\} \\ & + m(\Delta f)H\xi + 2m\nabla^{\Psi}_{gradf}\;(H\xi) \end{aligned}$$

Now, to compute the tangential and normal parts of the f-bitenson field, it suffices to find only the normal and tangential parts of $\Delta^{\Psi}\xi$ and $\nabla^{\Psi}_{gradf}\;(H\xi)$:

From (2.2.7) we have

$$\begin{aligned} \overline{g}(\Delta^{\Psi}\xi,\xi) &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi,\xi) \\ &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi,\xi) \\ &= \sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi,\overline{\nabla}_{e_i}\xi), \\ &= \sum_{i=1}^{m}\overline{g}(\phi e_i,\phi e_i). \end{aligned} \tag{3.7}$$

By using (2.2.4) in (3.7), then the normal part of $\Delta^{\Psi}\xi$ is

$$\begin{aligned} \left(\Delta^{\Psi}\xi\right)^{\perp} &= -\overline{g}(\Delta^{\Psi}\xi,\xi)\xi \\ &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi,\overline{\nabla}_{e_i}\xi)\xi \\ &= -m\xi. \end{aligned} \tag{3.8}$$

The tangential part of $\Delta^{\Psi}\xi$ can be calculated by

$$\begin{aligned}
\left(\Delta^{\Psi}\xi\right)^{\top} &= -\sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi, e_k)e_k \\
&= \sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}Ae_i - A(\nabla_{e_i}e_i), e_k)e_k \\
&= \sum_{i,k=1}^{m} \{e_i\overline{g}(Ae_i, e_k) - \overline{g}(Ae_i, \nabla_{e_i}e_k) - \overline{g}(A(\nabla_{e_i}e_i), e_k)\}e_k \\
&= \sum_{i,k=1}^{m} \{-e_i b(e_i, e_k) + b(e_i, \nabla_{e_i}e_k) + b(\nabla_{e_i}e_i, e_k)\}e_k \\
&= -\sum_{i,k=1}^{m} \{\nabla_{e_i} b(e_k, e_i)\}e_k. \qquad (3.9)
\end{aligned}$$

By Codazzi-Mainardi equation, we have

$$\begin{aligned}
\sum_{i=1}^{m}(\nabla_{e_i}b(e_k, e_i) - \nabla_{e_k}b(e_i, e_i)) &= -\sum_{i=1}^{m}\overline{g}\left(\overline{R}(e_i, e_k)e_i, \xi\right) \\
&= \overline{S}(\xi, e_k). \qquad (3.10)
\end{aligned}$$

Since $\overline{S}(\xi, e_k) = 0$, (3.10) implies that

$$\sum_{i=1}^{m}(\nabla_{e_i}b(e_k, e_i) - \nabla_{e_k}b(e_i, e_i)) = 0. \qquad (3.11)$$

If we write (3.11) in (3.9), we get

$$\left(\Delta^{\Psi}\xi\right)^{\top} = -mgradH. \qquad (3.12)$$

On the other hand we have

$$\begin{aligned}
\nabla^{\Psi}_{gradf}\ (H\xi) &= \overline{\nabla}_{gradf}(H\xi) \\
&= gradf\,(H)\xi + H\overline{\nabla}_{gradf}\,\xi \\
&= g(gradf, gradH)\xi - HA(gradf), \qquad (3.13)
\end{aligned}$$

which implies

$$\nabla^{\Psi}_{gradf}\ (H\xi) = div(fgradH)\xi - (f\,\Delta H)\xi - HA(gradf). \qquad (3.14)$$

Then the tangential and normal parts of $\nabla^{\Psi}_{gradf}\ (H\xi)$ are given by

$$\begin{aligned}
\left[\nabla^{\Psi}_{gradf}\ (H\xi)\right]^{\top} &= -HA(gradf), \qquad (3.15) \\
\left[\nabla^{\Psi}_{gradf}\ (H\xi)\right]^{\perp} &= div(fgradH)\xi - (f\,\Delta H)\xi, \qquad (3.16)
\end{aligned}$$

respectively. Finally by considering all these parts, we have the tangential and normal components of the bitension field as follows:

$$(\tau_{2,f}(\Psi))^{\top} = -2mA(gradfH) + \frac{m^2}{2} f\,(gradH^2), \tag{3.17}$$

$$(\tau_{2,f}(\Psi))^{\perp} = -m\Delta(fH) + 2m^2 fH + 2mH\Delta f \tag{3.18}$$

This completes the proof.

Corollary 4.1. *A spacelike hypersurface of an LP-Sasakian manifold with a costant mean curvature is f-biharmonic if and only if either it is minimal or*

$$\begin{cases} A(gradf) = 0, \\ \Delta f = -2mf. \end{cases} \tag{3.19}$$

Proof. Let M be a spacelike hypersurface of an $(m+1)$-dimensional LP-Sasakian manifold with H =constant. From (3.2) and (3.3) we have M is an f-biharmonic hypersurface if and only if

$$\begin{aligned} HA(gradf) &= 0, \\ H\Delta f + 2mfH &= 0, \end{aligned}$$

by virtue of Weitzenböck formula. This completes the proof.

Corollary 4.2. *A spacelike hypersurface of an LP-Sasakian manifold with a harmonic mean curvature is f-biharmonic if and only if*

$$\begin{cases} 2A(grad(\ln fH) = mgradH, \\ g(grad\ln f, grad\ln H) = \frac{\Delta f}{2f} + m. \end{cases} \tag{3.20}$$

From Corollary 4.1 it is obvious that f-biharmonic spacelike hypersurfaces of LP-Sasakian manifolds with a constant mean curvature are either minimal ones or satisfy (3.19).

Theorem 4.3. *Let M be a totally umbilic f-biharmonic spacelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with dimension $(m+1)$. Then we have*

$$\begin{cases} \left(\frac{m}{2}+1\right) grad\lambda = A(grad\ln f), \\ 2g(gradf, gradH) = f\,\Delta\lambda - \lambda\,(\Delta f + 2mf)\,. \end{cases} \tag{3.21}$$

Proof. Assume that $\{e_i\}_{i=1}^m$ is a local orthonormal frame of M such that

$$\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$$

is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$ where $\Psi : M \to \overline{M}$ is an isometric immersion. By identifying $d\Psi(X)$ by X, for all X in TM, we have an orthonormal basis$\{e_1, e_2, ..., e_m, \xi\}$ for the ambient manifold $\overline{M}$ such that $Ae_i = \lambda_i e_i$, where A is the shape operator of M and λ_i, $(1 \leq i \leq m)$, is the principal curvatures in the direction of e_i. Since M is totally umbilical then all the principal curvatures at any point p of M are equal to the same number $\lambda(p)$. Then by taking ξ instead of N in (2.2.18) we have

$$\begin{aligned} H &= -\frac{1}{m}\sum_{i=1}^{m} g(B(e_i, e_i), \xi) \\ &= -\frac{1}{m}\sum_{i=1}^{m} g(Ae_i, e_i) \\ &= -\frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i, e_i) \\ &= -\lambda. \end{aligned} \tag{3.22}$$

On the other hand by using (3.22), we get

$$A(gradH) = -\frac{1}{2}grad\lambda^2. \tag{3.23}$$

Since M is an f-biharmonic spacelike hypersurface of $\overline{M}$, from (3.2), (3.3), (3.22), (3.23) and Weitzenböck formula we complete the proof.

Corollary 4.3. *Let M be a totally umbilic f-biharmonic spacelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with dimension $(m+1)$. If $gradf \perp gradH$ then we have*

$$\begin{cases} \left(\frac{m}{2}+1\right) f\, gradH = -A(grad\, f), \\ \frac{\Delta H}{H} = \frac{\Delta f}{f} + 2m. \end{cases} \tag{3.24}$$

5. f-Biharmonic timelike hypersurfaces in LP-Sasakian manifolds

Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be a hypersurface of $\overline{M}$. Assume that the characteristic vector field of $\overline{M}$ belongs to the tangent hyperplane of the hypersurface M and N is the unit normal vector field of the manifold. Since N is spacelike then M becomes timelike hypersurface of $\overline{M}$.

We note that the tension field of the isometric immersion $\Psi : M \to \overline{M}$ is

$$\tau(\Psi) = m\mu,$$

where $\mu = HN$ is the mean curvature vector field with the mean curvature function H.

Theorem 5.1. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be its timelike hypersurface. Then M is an f-biharmonic hypersurface of $\overline{M}$ if and only if*

$$\begin{cases} \frac{m}{2} f\, gradH^2 + 2A(grad f H) - 2f\, H(\overline{Q}(N)) = 0, \\ f\, H\ |A|^2 + \Delta\,(f\, H) - 2H\ (\Delta f) - f\, H\, \overline{S}(N,N) = 0. \end{cases} \tag{4.1}$$

where $\overline{S}$ is the Ricci curvature of the LP-Sasakian manifold $\overline{M}$, $\overline{Q}$ is the Ricci operator of $\overline{M}$ defined by $\overline{g}(\overline{Q}X, Y) = \overline{S}(X, Y)$ and A is the shape operator of the hypersurface with respect to the unit normal vector field N.

Proof. Assume that M is a timelike hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field N and $\Psi : M \to \overline{M}$ be an isometric immersion. Consider $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an local orthonormal basis for the hypersurface. Since the tension field of Ψ is $\tau(\Psi) = mHN$, we have

$$\begin{aligned} \tau_2(\psi) &= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}\tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}\tau(\Psi) - \overline{R}(d\Psi(e_i), \tau(\Psi))d\Psi(e_i)\} \\ &= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}(mHN) - \nabla^{\Psi}_{\nabla_{e_i}e_i}(mHN) - \overline{R}(d\Psi(e_i), mHN)d\Psi(e_i)\} \\ &= \sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(mHN) - \overline{\nabla}_{\nabla_{e_i}e_i}(mHN) - \overline{R}(d\Psi(e_i), mHN)d\Psi(e_i)\} \\ &= m\sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i}\left(e_i(H)N + H\overline{\nabla}_{e_i}N\right) - (\nabla_{e_i}e_i)(H)N - H\overline{\nabla}_{\nabla_{e_i}e_i}N \\ &\quad -H\overline{R}(d\Psi(e_i), N)d\Psi(e_i)\} \\ &= m\sum_{i=1}^{m} \varepsilon_i \{e_i e_i(H)N + 2e_i(H)\overline{\nabla}_{e_i}N + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N \\ &\quad - (\nabla_{e_i}e_i)(H)N - H\overline{\nabla}_{\nabla_{e_i}e_i}N - H\overline{R}(d\Psi(e_i), N)d\Psi(e_i)\} \\ &= -m(\Delta H)N - mH\Delta^{\Psi}N - 2mA(gradH) \\ &\quad -mH\left\{\sum_{i=1}^{m-1}\overline{R}(d\Psi(e_i), N)d\Psi(e_i) - \overline{R}(d\Psi(\xi), N)d\Psi(\xi)\right\}, \end{aligned} \tag{4.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$, $\overline{R}$ is the Riemannian curvature tensor of $\overline{M}$ and ∇^{Ψ} is the pull-back connection.

Now we shall compute the tangential and normal components of the $\Delta^{\Psi}N$ and the curvature term, respectively:

The tangential part of $\Delta^{\Psi}N$ can be calculated by

$$\begin{aligned}\left(\Delta^{\Psi}N\right)^{\top} &= -\sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N-\overline{\nabla}_{\nabla_{e_i}e_i}N,e_k)e_k \\ &= \sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}Ae_i-A(\nabla_{e_i}e_i),e_k)e_k \\ &= \sum_{i,k=1}^{m}\{e_i\overline{g}(Ae_i,e_k)-\overline{g}(Ae_i,\nabla_{e_i}e_k)-\overline{g}(A(\nabla_{e_i}e_i),e_k)\}e_k \\ &= \sum_{i,k=1}^{m}\{e_ib(e_i,e_k)-b(e_i,\nabla_{e_i}e_k)-b(\nabla_{e_i}e_i,e_k)\}e_k \\ &= \sum_{i,k=1}^{m}\{\nabla_{e_i}b(e_k,e_i)\}e_k, \qquad (4.3)\end{aligned}$$

where ∇ is the induced connection of the hypersurface and $b:\Gamma(TM)\times\Gamma(TM)\to C^{\infty}(M,R)$ is the function valued second fundamental form such that $B(X,Y)=b(X,Y)N$, for all vector fields X, Y on M. By Codazzi-Mainardi equation, we have

$$\begin{aligned}\sum_{i=1}^{m}(\nabla_{e_i}b(e_k,e_i)-\nabla_{e_k}b(e_i,e_i)) &= \sum_{i=1}^{m}\overline{g}\left(\overline{R}(e_i,e_k)e_i,N\right) \\ &= -\overline{S}(N,e_k). \qquad (4.4)\end{aligned}$$

which implies that

$$\sum_{i=1}^{m}\nabla_{e_i}b(e_k,e_i)=\sum_{i=1}^{m}\nabla_{e_k}b(e_i,e_i)-\overline{S}(N,e_k). \qquad (4.5)$$

By using (4.5) in (4.3), we obtain

$$\begin{aligned}\left(\Delta^{\Psi}N\right)^{\top} &= \sum_{i,k=1}^{m}\{\nabla_{e_k}b(e_i,e_i)-\overline{S}(N,e_k)\}e_k \\ &= m(gradH)-(\overline{Q}(N)). \qquad (4.6)\end{aligned}$$

By straightforward computations, the normal part of the $\Delta^{\Psi}N$ is

$$\begin{aligned}(\Delta^{\Psi}N)^{\perp} &= \overline{g}(\Delta^{\Psi}N,N)N \\ &= -\sum_{i=1}^{m}\{\varepsilon_i\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N-\overline{\nabla}_{\nabla_{e_i}e_i}N,N)\}N \\ &= \sum_{i=1}^{m}\{\varepsilon_i\overline{g}(\overline{\nabla}_{e_i}N,\overline{\nabla}_{e_i}N)\}N \\ &= |A|^2N, \qquad (4.7)\end{aligned}$$

where $\varepsilon_i=\overline{g}(e_i,e_i)$, $1\leq i\leq m$.

On the other hand, since

$$
\begin{aligned}
-\sum_{k=1}^{m-1}\overline{S}(N,e_k)e_k &= \sum_{i,k=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),e_k)e_k \\
&\quad -\sum_{i=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),\xi)\xi \\
&= (\overline{Q}(N))\xi \qquad (4.8)
\end{aligned}
$$

and

$$\sum_{i=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),N) = -\overline{S}(N,N)+1 \qquad (4.9)$$

then the tangential and normal components of the first curvature term in (4.2) are equal to $(\overline{Q}(N))$ and $-\overline{S}(N,N)+1$, respectively. Also, from (2.2.13) we have

$$\overline{R}(d\Psi(\xi),N)d\Psi(\xi) = N. \qquad (4.10)$$

Hence we get

$$
\begin{aligned}
(\tau_2(\psi))^{\top} &= -m\left[\frac{m}{2}(gradH^2)+2A(gradH)-2H(\overline{Q}(N))\right], \qquad (4.11)\\
(\tau_2(\psi))^{\perp} &= -m\left[(\Delta H)+H\,|A|^2-H\overline{S}(N,N)\right]N. \qquad (4.12)
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\nabla^{\Psi}_{gradf}\ (HN) &= \overline{\nabla}_{gradf}(HN) \\
&= gradf\,(H)N + H\overline{\nabla}_{gradf}\,N \\
&= g(gradf,gradH)N - HA(gradf), \qquad (4.13)
\end{aligned}
$$

which implies

$$\nabla^{\Psi}_{gradf}\ (HN) = div(fgradH)N-(f\,\Delta H)N-HA(gradf). \qquad (4.14)$$

Then the tangential and normal parts of $\nabla^{\Psi}_{gradf}\ (HN)$ are given by

$$
\begin{aligned}
\left[\nabla^{\Psi}_{gradf}\ (HN)\right]^{\top} &= -HA(gradf), \qquad (4.15)\\
\left[\nabla^{\Psi}_{gradf}\ (HN)\right]^{\perp} &= div(fgradH)N-(f\,\Delta H)N, \qquad (4.16)
\end{aligned}
$$

By reorganizing all the tangent and normal parts of the f-bitension field, we get

$$\begin{aligned}
(\tau_{2,f}(\psi))^{\top} &= f\left\{-m\left(\frac{m}{2}(gradH^2)+2A(gradH)-2H(\overline{Q}(N))\right)\right\}-2mHA(gradf)\\
&= -\frac{m^2}{2}f\,gradH^2-2mfA(gradH)+2mf\,H(\overline{Q}(N))-2mHA(gradf)\\
&= -m\left\{\frac{m}{2}f\,gradH^2+2A(gradfH)-2f\,H(\overline{Q}(N))\right\} \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
(\tau_{2,f}(\psi))^{\perp} &= f\left\{-m\left[(\Delta H)+H\,|A|^2-H\overline{S}(N,N)\right]N\right\}\\
&\quad +m\,(\Delta f)\,HN+2m\,\{g(gradf,gradH)N\}\\
&= -mf\,(\Delta H)N-mf\,H\,|A|^2\,N+mf\,H\,\overline{S}(N,N)N \qquad (4.18)\\
&\quad +m\,(\Delta f)\,HN+2mdiv(Hgradf)N-2mH\,(\Delta f)\,N\\
&= -m\left\{f\,H\,|A|^2+\Delta\,(f\,H)-2H\,(\Delta f)-f\,H\,\overline{S}(N,N)\right\}N,
\end{aligned}$$

which give

$$\begin{aligned}
\tau_{2,f}(\psi) &= -m\left\{\frac{m}{2}f\,gradH^2+2A(gradfH)-2f\,H(\overline{Q}(N))\right\} \qquad (4.19)\\
&\quad -m\left\{f\,H\,|A|^2+\Delta\,(f\,H)-2H\,(\Delta f)-f\,H\,\overline{S}(N,N)\right\}N.
\end{aligned}$$

This completes the proof.

Corollary 5.1. *Let M be a timelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with constant mean curvature. Then M is an f-biharmonic timelike hypersurface if and only if either it is minimal or*

$$\begin{cases} A(gradf)=f\,(\overline{Q}(N)),\\ \frac{\Delta f}{f}=|A|^2-\overline{S}(N,N). \end{cases} \qquad (4.20)$$

Corollary 5.2. *A timelike hypersurface of a Ricci flat LP-Sasakian manifold with a constant mean curvature is f-biharmonic if and only if*

$$\begin{cases} A(gradf)=0,\\ \frac{\Delta f}{f}=|A|^2. \end{cases} \qquad (4.21)$$

Theorem 5.2. *Let $\overline{M}$ be an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold and M be a timelike hypersurface of $\overline{M}$. Then M is f-biharmonic if and only if*

$$\begin{cases} A(grad\ln(fH))=-\frac{m}{2}\,(gradH)\,,\\ \frac{\Delta(f\,H)}{f\,H}-2\frac{(\Delta f)}{f}=-\,|A|^2+\frac{\overline{r}}{m}-1, \end{cases} \qquad (4.22)$$

where $\overline{r}$ is the scalar curvature of $\overline{M}$. Particularly, if M is a timelike hypersurface with $0 \neq H =$constant, then M is a non-minimal f-biharmonic timelike hypersurface if and only if

$$\begin{cases} A(gradf) = 0, \\ \frac{\Delta f}{f} = |A|^2 - \frac{\overline{r}}{m} + 1. \end{cases} \tag{4.23}$$

Proof. Assume that $\overline{M}$ be an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold. Then by using (2.2.9), we have

$$\overline{S}(N,N) = \frac{\overline{r}}{m} - 1. \tag{4.24}$$

On the other hand

$$(\overline{Q}(N)) = 0. \tag{4.25}$$

By using (4.24) and (4.25) in (4.1), we obtain the assertion of the theorem.

Theorem 5.3. *Let $\overline{M}$ be an $(m+1)$-dimensional LP-Sasakian space form and M be a timelike hypersurface of $\overline{M}$. Then M is f-biharmonic if and only if*

$$\begin{cases} A(grad\ln(fH)) = -\frac{m}{2}\left(gradH\right), \\ \frac{\Delta(fH)}{fH} - 2\frac{\Delta f}{f} = -\left|A\right|^2 + m, \end{cases} \tag{4.26}$$

In particular, M is a hypersurface of $\overline{M}$ with a constant mean curvature, then M is a non-minimal biharmonic timelike hypersurface if and only if $|A|^2 = \frac{\Delta f}{f} + m$.

Proof. In an $(m+1)$-dimensional LP-Sasakian space form $\overline{M}$, since

$$\overline{S}(X,Y) = m\overline{g}(X,Y),$$

for all vector fields X, Y , then $\overline{M}$ is an η-Einstein manifold with

$$\overline{r} = m(m+1). \tag{4.27}$$

Therefore, the f-biharmonic equation reduces to (4.26).

Theorem 5.4. *Let $\overline{M}$ be an $(m+1)$-dimensional (dim $\overline{M} > 2$) η-Einstein LP-Sasakian manifold. If M is a totally umbilical f-biharmonic timelike hypersurface of $\overline{M}$ then we have*

$$\begin{aligned} A(gradf) &= 2fgrad\lambda, \\ \frac{\Delta(f\lambda)}{f\lambda} - 2\frac{(\Delta f)}{f} &= -m\lambda^2 + \frac{\overline{r}}{m} - 1. \end{aligned}$$

Proof. Let $\{e_1, e_2, ..., e_{m-1}, e_m = \xi, N\}$ be a local orthonarmal basis of η-Einstein LP-Sasakian manifold $\overline{M}$ such that $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an orthonormal frame for the hypersurface M. Since M is totally umbilical, we have $A = \lambda I$, where λ is a smooth function. Then

$$
\begin{aligned}
H &= \frac{1}{m}\sum_{i=1}^{m}\varepsilon_i \overline{g}(B(e_i, e_i), N) \\
&= \frac{1}{m}\sum_{i=1}^{m-1}\overline{g}(Ae_i, e_i) \\
&= \frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i, e_i) \\
&= \frac{m-1}{m}\lambda. \qquad (4.28)
\end{aligned}
$$

From (4.28), we can write

$$A(gradH) = \frac{m-1}{2m} grad\lambda^2. \qquad (4.29)$$

On the other hand, by straightforward calculations one can easily see that

$$|A|^2 = m\lambda^2. \qquad (4.30)$$

By using (4.28), (4.29) and (4.30) in (4.22), we obtain

$$
\begin{aligned}
fA(gradH) + HA(gradf) &= -\frac{m}{4} f\left(gradH^2\right) \\
f\frac{m-1}{2m}grad\lambda^2 + \frac{m-1}{m}\lambda A(gradf) &= -\frac{m}{2} f\left(\frac{m-1}{m}\right)^2 grad\lambda^2 \\
\lambda A(gradf) &= f\left[\frac{1}{2} - m\right] grad\lambda^2 \\
A(gradf) &= 2fgrad\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\Delta(f\,H)}{f\,H} - 2\frac{(\Delta f)}{f} &= -|A|^2 + \frac{\overline{r}}{m} - 1, \\
\frac{\Delta(f\,\lambda)}{f\,\lambda} - 2\frac{(\Delta f)}{f} &= -m\lambda^2 + \frac{\overline{r}}{m} - 1,
\end{aligned}
$$

which complete the proof.

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Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, Adiyaman, TURKEY

E-mail address: sperktas@adiyaman.edu.tr

Adiyaman University, Faculty of Education, Department of Elementary Education, Adiyaman, TURKEY

E-mail address: ferdogan@adiyaman.edu.tr

Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, Adiyaman, TURKEY

E-mail address: eacet@adiyaman.edu.tr